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SoSe 2017

Probability Theory I - Exercise Sheet 5

Due date: **Friday, May 26, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 5.I (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let for all $i \in \{1, ..., n\}$ $(E_i, \mathcal{E}_i)$ be a measurable space and $X_i : \Omega \to E_i$ a random variable.

Show that $X_1, \dots, X_n$ are independent random variables if and only if for all $\mathcal{E}_i$-$\mathcal{B}(\mathbb{R})$-measurable functions $f_i : E_i \to \mathbb{R}$, satisfying $\mathbb{E}\big[|f_i(X_i)|\big] < \infty$ and $\mathbb{E}\big[\prod_{i=1}^n |f_i(X_i)|\big] < \infty$, we have

$$\mathbb{E}\left[\prod_{i=1}^n |f_i(X_i)|\right] = \prod_{i=1}^n \mathbb{E}\left[|f_i(X_i)|\right].$$

Exercise 5.II (8 pts)

Let $\tau, X_1, X_2, ...$ be independent, real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{P}(\tau \in \mathbb{N}_0) = 1$ with and $X_1, X_2, ...$ be identically distributed. Furthermore assume that $\mathbb{E}[|\tau|] < \infty$ as well as $\mathbb{E}[|X_1|] < \infty$. Define

$$S_\tau := \sum_{i=1}^{\tau} X_i$$

i.e.

$$S_\tau(\omega) := \sum_{i=1}^{\tau(\omega)} X_i(\omega), \quad \text{for all } \omega \in \Omega$$

and show that:

a) $\mathbb{E}[|S_\tau|] < \infty$,

b) $\mathbb{E}[S_\tau] = \mathbb{E}[\tau]\,\mathbb{E}[X_1]$.

Exercise 5.III (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X and Y be independent, exponentially distributed random variables with parameter λ_X and λ_Y respectively, i.e. the probability densities of X and Y are given by $\rho_X(x) = \lambda_X e^{-\lambda_X x} \mathbb{1}_{[0,\infty)}(x)$ and $\rho_Y(x) = \lambda_Y e^{-\lambda_Y x} \mathbb{1}_{[0,\infty)}(x)$. Let $Z := \frac{X}{Y}$.

a) Calculate the distribution function $F_Z(z)$ of the random variable Z.
Hint: Note that $\mathbb{P}(\frac{X}{Y} \le z) = \mathbb{P}(X \le Y z)$ *for all* $z \in \mathbb{R}$.

b) Show that $\mathbb{P}(X < Y) = \frac{\lambda_X}{\lambda_X + \lambda_Y}$.

c) Calculate the probability density $f_Z(z)$ of Z.

Exercise 5.IV (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a real valued random variable, s.t. $\mathbb{E}[|X|^2] < \infty$. The *variance* of X is given by $\mathrm{var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2]$. Show that:

a) var(X)=0 if and only if $X = \mathbb{E}(X)$ $\mathbb{P}-$a.s.

b) The map $f : \mathbb{R} \to \mathbb{R}, x \to \mathbb{E}\left[(X - x)^2\right]$ has a unique minimum in $x_0 = \mathbb{E}\left[X\right]$.

Let $X_1, ..., X_n$ be real valued random variables, s.t. $\mathbb{E}[|X_i|^2] < \infty$ for all $i = 1, ..., n$. and let

$$\Sigma := (\mathrm{cov}(X_i, X_j))_{ij}$$

denote their *covariance matrix*, where the *covariance* is given by

$$\mathrm{cov}(X_i, X_j) := \mathbb{E}[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])].$$

Show that

i) If the covariance matrix Σ is a diagonal ($X_1, ..., X_n$ are called *uncorrelated* in this case), then it follows that:

$$\mathrm{var}(X_1 + ... + X_n) = \mathrm{var}(X_1) + ... + \mathrm{var}(X_n).$$

ii) $\det(\Sigma) = 0$ if and only if there are $a_1, ..., a_n, b \in \mathbb{R}$, s.t. $\mathbb{P}(a_1 X_1 + ... + a_n X_n = b) = 1$.