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SoSe 2017

Probability Theory I - Exercise Sheet 6

Due date: **Friday, June 2, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 6.I (6 pts)

Let $\{(\Omega_i, \mathcal{F}_i, Q_i)\}_{i\in I}$ be a family of probability spaces. Show that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family $(X_i)_{i\in I}$ of independent random variables $X_i : \Omega \to \Omega_i$, s.t. $\mathbb{P}X_i^{-1} = Q_i$ for all $i \in I$.

Exercise 6.II [Random walk on $\mathbb{Z}$] (8 pts)

Let $p \in [0, 1]$ and

$$F := \big\{I\colon \mathbb{N}_0 \to \mathbb{Z} \;\big|\; I(k) - I(k-1) \in \{-1, +1\} \;\forall\, k \in \mathbb{N}, I(0) = 0\big\}.$$

a) For all $n \in \mathbb{N}$, $\omega = (\omega_1, \ldots, \omega_n) \in \{-1, 1\}^n$ and every subset $\{n_1, \ldots, n_k\} \subset \{1, \ldots, n\}$ let

$$\mathbb{P}^{(n)}\big[I(n_i) - I(n_i - 1) = \omega_{n_i} \;\forall\, i \in \{1, \ldots k\}\big] = p^{\sum_{i=1}^k \mathbb{1}_{\{1\}}(\omega_{n_i})}(1-p)^{\sum_{i=1}^k \mathbb{1}_{\{-1\}}(\omega_{n_i})}.$$

Show that this defines a consistent family of probability measures on $\big(F, \mathcal{P}(F)\big)$.

b) Show, without making use of Exercise 6.III or 6.IV, that there is a probability measure $\mathbb{P}$ on $\big(F, \mathcal{P}(F)\big)$ with

$$\mathbb{P}I^{-1}\big(A_1 \times A_2 \times \ldots \times A_n \times \mathbb{Z}^{\mathbb{N}}\big) = \mathbb{P}^{(n)}I^{-1}(A_1 \times A_2 \times \ldots \times A_n)$$

for all $I \in F$ and $A_i \subset \mathbb{Z}$, $i \le n$, and all $n \in \mathbb{N}$.

Exercise 6.III [Infinite coin toss - Part I] (8 pts)

In order to model the experiment of tossing a coin, we make use of the state space $\{0,1\}$ and identify one side of the coin with 1 and the other one with 0. Let $p \in (0,1)$ the probability that a single coin flip yields the result 1.

(i) Construct a probabilistic model $(\Omega, \mathcal{F}, \mathbb{P})$ suitable for describing an n-fold coin toss, $n \in \mathbb{N}$.

(ii) Construct a model for the ∞-fold coin toss, i.e. show the existence of a probability measure $\mathbb{P}$ on $\big(\{0,1\}^{\mathbb{N}}, \mathcal{P}(\{0,1\}^{\mathbb{N}})\big)$, s.t. for all $n \in \mathbb{N}$, the projection $\pi^{(n)}\colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^n$ yields an induced measure $\mathbb{P}(\pi^{(n)})^{-1}$ which is given by the distribution $\mathbb{P}^{(n)}$ of the n-fold coin toss.
Explain in detail, why the theorems from the lectures you use are applicable.

Exercise 6.IV [Infinite coin toss - Part II] (10 pts)

a) In order to construct a model of the ∞-fold *fair* coin toss, we do not necessarily need to resort to Kolmogorov's extension theorem: Look at the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ and define for all $n \in \mathbb{N}$ the random variables

$$X_n(\omega) = \begin{cases} 1 & \text{if } \lfloor 2^n \omega \rfloor \text{ odd}, \\ 0 & \text{if } \lfloor 2^n \omega \rfloor \text{ even}. \end{cases}$$

Show that the random variables $X_1, X_2, \ldots$ are independent with $\mathbb{P}[X_k = 1] = \mathbb{P}[X_k = 0] = 1/2$ for all $k \in \mathbb{N}$.

b) Deduce from exercise 6.II the existence of a probability measure $\mathbb{P}$ as in Exercise 6.III(ii). Do *not* make use of the results of 6.III or 6.IVa).