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SoSe 2017

Probability Theory I - Exercise Sheet 7

Due date: **Friday, June 9, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 7.I (8 pts)

Let $\big(X_i\big)_{i\in\mathbb{N}}$ be a sequence of random variables $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}(X_i) = 0$ for all $i \in \mathbb{N}$. Let $r : \{0, 1, \ldots\} \to \mathbb{R}$ satisfy $r(n) \xrightarrow[n\to\infty]{} 0$.
Show that if $\mathbb{E}[X_n X_m] \le r(n-m)$ for all $m \le n$, then we have

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} 0 \quad \text{in probability.}$$

Exercise 7.II (8 pts)

(a) Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $h \ge 0$ be a $\mathcal{F}$-$\mathcal{B}(\mathbb{R})$-measurable mapping and $H(x) := \int_{-\infty}^{x} h(y)dy$.
Show that

$$\mathbb{E}\left[H(X)\right] = \int_{-\infty}^{\infty} h(y)\mathbb{P}\big[X \ge y\big]dy.$$

(b) Let $\big(X_i\big)_{i\in\mathbb{N}}$ be independent, identically distributed (i.i.d.) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that

$$\mathbb{P}\big[X_1 > x\big] = \begin{cases} 1 & \text{for } x \le e, \\ \dfrac{e}{x\log(x)} & \text{for } x > e. \end{cases}$$

i) Show that X_1 is *not* integrable, i.e. $\mathbb{E}[X_1] = \infty$.

ii) Show that there is a sequence of constants $\mu_1, \mu_2, \ldots$, s.t. $\mu_i \xrightarrow[i\to\infty]{} \infty$, and

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mu_i \xrightarrow[n\to\infty]{} 0 \quad \text{in probability.}$$

Exercise 7.III (Monte-Carlo-Integration) (8 pts)

Let f a measurable function $f\colon [0,1] \to \mathbb{R}$ with $\int_0^1 |f(x)|dx < \infty$ and let $\left(U_i\right)_{i\in\mathbb{N}}$ be a family of i.i.d random variables, uniformly distributed on $[0,1]$. Define

$$I_n := \frac{1}{n}\sum_{i=1}^{n} f\left(U_i\right) \quad \text{for all } i \in \mathbb{N}.$$

(i) Show that

$$I_n \xrightarrow[n\to\infty]{} I := \int_0^1 f dx \quad \text{probability.}$$

(ii) Assume that $\int_0^1 |f(x)|^2 < \infty$. For $a > 0$, give a meaningful estimate on the probability

$$\mathbb{P}\left[|I_n - I| > \frac{a}{\sqrt{n}}\right].$$

Exercise 7.IV (*Preparation for a mini-presentation*) (8 pts)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Durrett: Example 2.2.5. An occupancy problem

Suppose we put r_n balls at random in n boxes, i.e., all n^{r_n} assignments of balls to boxes have equal probability. Let $N_n =$ the number of empty boxes.

Show that if $\frac{r_n}{n} \to c \geq 0$ as $n \to \infty$, then it follows that

$$\frac{N_n}{n} \to e^{-c} \quad \text{in probability, as } n \to \infty.$$