Prof. Dr. Barbara Gentz
Christian Wiesel
Faculty of Mathematics
Bielefeld University

SoSe 2017

Probability Theory I - Exercise Sheet 8

Due date: **Friday, June 16, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Julian Femmer: 237, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date. Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 8.I (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ real valued random variables.

a) Assume that the random variables $(X_n)_{n\in\mathbb{N}}$ are i.i.d. exponentially distributed with $\mathbb{P}[X_1 > x] = \exp(-x)$ for $x \geq 0$ and let $M_n = \max\{X_1, \ldots, X_n\}$, $n \in \mathbb{N}$. Show that

$$\text{(i)} \quad \limsup_{n\to\infty} \frac{X_n}{\log(n)} = 1 \quad \mathbb{P}\text{-a.s.} \qquad\qquad \text{(ii)} \quad \frac{M_n}{\log(n)} \xrightarrow[n\to\infty]{} 1 \quad \mathbb{P}\text{-a.s.}$$

b) Let the $(X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables with a distribution function $F(x) = \mathbb{P}[X_1 \leq x], x \in \mathbb{R}$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be an increasing sequence and define

$$A_n := \big\{ \max\{X_1, \ldots, X_n\} > \lambda_n \big\} \text{ for all } n \in \mathbb{N}.$$

Show that

$$\mathbb{P}[A_n \text{ infinitely often}] = \begin{cases} 1 & \text{if } \sum_{n=1}^{\infty} \big(1 - F(\lambda_n)\big) = \infty, \\ 0 & \text{if } \sum_{n=1}^{\infty} \big(1 - F(\lambda_n)\big) < \infty. \end{cases}$$

Exercise 8.II (8 pts)

Let $(Y_i)_{i\in\mathbb{N}}$ be i.i.d., $\mathbb{R}^2$-valued, random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let Y_1 be uniformly distributed on the ball $B_1(0) := \big\{x \in \mathbb{R}^2 \mid ||x|| < 1\big\}$. Let $X_0 = (1, 0)$ and define $X_{n+1} = \|X_n\| Y_{n+1}$ for all $n \in \{0, 1, \ldots\}$, i.e. X_{n+1} is chosen with a uniform distribution on the ball $B_{\|X_n\|}(0) := \big\{x \in \mathbb{R}^2 \mid \|x\| < \|X_n\|\big\}$. Show that

$$\frac{1}{n} \log(\|X_n\|) \xrightarrow[n\to\infty]{} c \quad \mathbb{P}\text{-a.s.}$$

for some constant $c \in \mathbb{R}$ and calculate the value of c.

Exercise 8.III (8 pts)

Let $(X_i)_{i\in\mathbb{N}}$, $(Y_i)_{i\in\mathbb{N}}$ be independent families of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which satisfy $\mathbb{E}[X_1] < \infty$ and $\mathbb{E}[Y_1] < \infty$ respectively.
Let for all $i \in \mathbb{N}$, X_i denote the life-span of the i-th lightbulb, i.e. the amount of time that passes between it being switched on and the moment it stops working. It is not switched off at any time - we leave it on until it breaks down.
Let Y_i denote the time it takes to replace the i-th lightbulb by the $(i+1)$-th lightbulb.
For $t \geq 0$ let R_t denote the amount of time the lamp was able to shine, up to time t. (For instance if we look at time t prior to the breakdown of our first lightbulb, then R_t simply coincides with t, since the lamp was able to shine constantly, without any time being lost for replacing a lightbulb.) Define R_t and show that

$$\frac{R_t}{t} \xrightarrow[t\to\infty]{} \frac{\mathbb{E}[X_i]}{\mathbb{E}[X_i] + \mathbb{E}[Y_i]} \qquad \mathbb{P}\text{-a.s.}$$

Exercise 8.IV (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

a) Show that

$$d(X,Y) = \mathbb{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$

defines a metric on the space of random variables, i.e. show that arbitrary random variables X, Y, Z we have:

(i) $d(X,Y) = 0$ if and only if $X = Y$ $\mathbb{P}$-a.s.,

(ii) $d(X,Y) = d(Y,X)$,

(iii) $d(X,Z) \leq d(X,Y) + d(Y,Z)$.

b) Show that for a sequence $(X_n)_{n\in\mathbb{N}}$ it follows that:

$$d(X_n, X) \xrightarrow[n\to\infty]{} 0 \quad \Leftrightarrow \quad X_n \xrightarrow[n\to\infty]{} X \text{ in probability.}$$

c) Show that the space of random variables is complete under the metric $d(\cdot,\cdot)$, defined in part a), i.e., if $d(X_n, X_m) \xrightarrow[n,m\to\infty]{} 0$ then there is a random variable X_∞ s.t. $X_n \xrightarrow[n\to\infty]{} X_\infty$ in probability.