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SoSe 2017

Probability Theory I - Exercise Sheet 9

Due date: **Friday, June 23, 11:00 h**

Solutions to the assigned problems must be deposited in your tutors mailbox (Katharina von der Lühe: 186, Peter Kuchling: 197, Timo Krause: 59) located in V3-128 no later than 11:00 h on the due date.
Solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Exercise 9.I (8 pts)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X_i)_{i\in\mathbb{N}}$ a family of independent random variables. Let

$$S_{m,n} = \sum_{i=m+1}^{n} X_i \quad \text{für alle } m, n \in \mathbb{N}, m < n.$$

Show that for all $a, b \in \mathbb{R}$ and $m, n \in \mathbb{N}$ with $m < n$ we have:

$$\mathbb{P}\left[\max_{i\in\{m,\dots,n\}} |S_{m,i}| > a + b\right] \min_{i\in\{m,\dots,n\}} \mathbb{P}\left[|S_{i,n}| \le b\right] \le \mathbb{P}\left[|S_{m,n}| > a\right].$$

Exercise 9.II (8 pts)

Let $(X_i)_{i\in\mathbb{N}}$ be a family of independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $S_n := \sum_{i=1}^{n} X_i$, $n \in \mathbb{N}$. Assume that S_n converges for $n \to \infty$ in probability $\mathbb{P}$. Show that $(S_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s.

Exercise 9.III (8 pts)

Let $(X_i)_{i\in\mathbb{N}}$ be independent random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that for all $i \in \mathbb{N}$ we have $\mathbb{E}(X_i) = 0$, as well as $\text{var}(X_n) = \sigma_n^2$.

a) Let $\sum_{i=1}^{\infty} \sigma_i^2/i^2 < \infty$. Show that

$$\text{(i)} \quad \sum_{i=1}^{n} \frac{X_i}{i} \qquad \text{and} \qquad \text{(ii)} \quad \frac{1}{n}\sum_{i=1}^{n} X_i$$

converge $\mathbb{P}$-a.s., as $n \to \infty$.

b) Let $\sum_{i=1}^{\infty} \sigma_i^2/i^2 = \infty$. Show the existence of random variables $(Y_i)_{i\in\mathbb{N}}$ with $\mathbb{E}(Y_i) = 0$ and $\text{var}(Y_i) \leq \sigma_i^2$, for all $i \in \mathbb{N}$, s.t. neither Y_n/n nor $\frac{1}{n}\sum_{i=1}^{n} Y_i$ converge $\mathbb{P}$-a.s., as $n \to \infty$.

Exercise 9.IV (8 pts)

Let $\mathcal{F}_n' := \sigma(X_n, X_{n+1}, ...)$ and $\mathcal{T} := \cap_n \mathcal{F}_n'$. Let furthermore $X_1, X_2, ...$ be real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define $S_n := X_1 + X_2 + ... + X_n$ for all $n \in \mathbb{N}$. Show that

a) $\{\omega \in \Omega \mid \lim_{n\to\infty} S_n(\omega) \text{ exists}\} \in \mathcal{T}$,

b) $\{\omega \in \Omega \mid \limsup_{n\to\infty} S_n(\omega) > 0\} \notin \mathcal{T}$,

c) $\left\{\omega \in \Omega \mid \limsup_{n\to\infty} \frac{S_n(\omega)}{c_n} > x\right\} \in \mathcal{T} \quad \forall x \in \mathbb{R}, \quad \text{if } c_n \to \infty.$