Prop $0 \to A \to B \to C \to 0$ weakly exact sequence of Hilbert modules, then

$$\dim_{W(G)} B = \dim_{W(G)} A + \dim_{W(G)} C .$$

How to prove the proposition?

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

weakly exact.

$p^*: C \to B$ satisfies $\ker p^{\perp} = \overline{\operatorname{im} p^*}$

$\ker p^{*\perp} = \overline{\operatorname{im} p}$

Consider $i \oplus p^*: A \oplus C \to B$.

Since $\overline{\operatorname{im} p} = C$, we know $\ker p^* = 0$.

Also, $\ker i = 0$.

$\therefore$ $i \oplus p^*$ is injective.

$$\overline{\mathrm{im}(i \oplus p^*)} = \overline{\mathrm{im}\, i} \oplus \overline{\mathrm{im}\, p^*} =$$

$$= \ker p \oplus \ker p^\perp = B$$

So $i \oplus p^* : A \oplus C \longrightarrow B$ is a weak isomorphism.

Polar decomposition:

$\forall\ x : X \longrightarrow Y$ bounded operator on Hilbert spaces,

$\exists\ q : X \to X$ positive and $u : X \to Y$ partial isometry

st. $\ker u = \overline{\mathrm{im}\, q}^{\perp}$ and

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr, "q"'] & & Y \\ & X \arrow[ur, "u"'] & \end{tikzcd}$$

commutes.

In our case, take $i \oplus p^* = uq$

$$\mathrm{im}(i \oplus p^*) \leq \mathrm{im}\, u \quad \therefore \quad \overline{\mathrm{im}\, u} = Y.$$

u is a partial isometry, so u^*u is a projection.

$$\therefore \quad u^*u = (u^*u)^2 = u^*uu^*u$$

$$(u - uu^*u)^*(u - uu^*u) = u^*u - u^*uu^*u - u^*uu^*u + u^*uu^*uu^*u = 0$$

$$\therefore \quad u - uu^*u = 0$$

$u = uu^*u$

$uu^* = uu^*uu^* = (uu^*)^2$

$\therefore$ uu^* is also a projection.

uu^* is a projection.

Now $\overline{\operatorname{im} u^*} = \ker u^\perp$

$\therefore$ $\overline{\operatorname{im} uu^*} = \overline{\operatorname{im} u} = B$.

But projection with dense image is an isometry!

$\therefore$ $u : A \oplus C \to B$ is an isometry.

So $A \oplus C \xrightarrow{\cong} B$.

We still need to show that $\dim A \oplus C =$
$\dim A + \dim C$.

$A \leq L^2(G)^n$, $C \leq L^2(G)^m$,

closed subspaces.

Let p, q be the corresponding projections.

Now $p \oplus q : L^2(G)^{n+m} \twoheadrightarrow A \oplus C$, a projection.

Pick a basis $e_1, \dots, e_{n+m}$ for $L^2(G)^{n+m}$,

respecting $L^2(G)^{n+m} = L^2(G)^n \oplus L^2(G)^m$.

Now $f_v(p+q) = \sum \langle p \oplus q(e_i), e_i \rangle =$

$$= \sum_{i=1}^{n} \langle p(e_i), e_i \rangle + \sum_{i=n+1}^{m} \langle q(e_i), e_i \rangle =$$

$$= f_v(p) + f_v(q) = \dim_{\mathcal{N}} A + \dim_{\mathcal{N}} C. \quad \square$$

Back to L^2-Betti numbers.

Let $f: X \to Y$ be a (true) G-equivariant cellular map between G-CW complexes of finite type.

Fix n; let $e_1, \dots, e_m$ be a choice of representatives of G-n-cells of X. Similarly, let $f_1, \dots, f_k$ be such a choice in Y.

On the level of n-chains, we have

$$f_*: C_n(X) \longrightarrow C_n(Y).$$

$$f_*(e_i) = \sum \lambda_{ij} f_j, \quad \lambda_{ij} \in \mathbb{Z}G$$

Now $\forall g \in G$: $f_*(g.e_i) = g.f_*(e_i) =$

$$= g \sum \lambda_{ij} f_j = \sum \lambda_{ij} (g.f_j)$$

So we also have $f_* : \mathbb{Z}G^m \xrightarrow[(\lambda_{ij})]{} \mathbb{Z}G^m$.

We also have $\partial f_* = f_* \partial$ and so f_* induces $f_* : H_n(X) \longrightarrow H_n(Y)$.

In the L^2-world, the situation is the same:

$f_* : e_i \mapsto \sum \lambda_{ij} f_j$ induces

$$f_*^{(2)} : \begin{array}{ccc} C_n^{(2)}(X) & \longrightarrow & C_n^{(2)}(Y) \\ L^2(G)^m & \xrightarrow[(\lambda_{ij})]{} & L^2(G)^k \end{array}.$$

We still have $f_* \partial = \partial f_*$, and so

$f_* : H_n^{(2)}(X) \longrightarrow H_n^{(2)}(Y)$ is defined.

Def (Restriction)
Let X be a free G-CW complex; let $H \le G$ be a subgroup of finite index. Then $H \curvearrowright X$, and the resulting free H-CW cplx is denoted by $\mathrm{res}_H^G(X)$.

$\mathrm{res}^G_H(X)$ is $\left\{\begin{array}{l}\text{finite} \\ \text{of finite type}\end{array}\right.$ iff so is X.

<u>Thm</u> i) Homotopy invariance:

$f, f' : X \to Y$ as above, $f \simeq_G f'$, i.e.

f and f' are G-equivariantly homotopic.

Then $f_*^{(2)} = f'^{(2)}_* : H_n^{(2)}(X) \to H_n^{(2)}(Y)$.

2) Restriction: X a free G-CW cplx of finite type, $H \leq G$ subgroup of finite index.

$$\beta_n^{(2)}(\mathrm{res}_H^G(X)) = [G:H] \cdot \beta_n^{(2)}(X) \quad \forall n.$$

<u>Proof</u> We have $f_*, f_*' : C_n(X) \to C_n(Y)$.

Pick representatives $e_1, \dots, e_m, f_1, \dots, f_n$ as before.

We have $\forall i \; \exists x_i \in C_{n+1}(Y)$:

$$f_*(e_i) = f_*'(e_i) + \partial x_i .$$

Every L^2-n-chain σ of the form $\sum \mu_i e_i$, $\mu_i \in L^2(G)$.

We have $(f_* - f_*')(\sum \mu_i e_i) = \partial(\sum \mu_i x_i)$

and $\sum \mu_i x_i \in C_{n+1}^{(2)}(Y)$.

<u>So</u> $f_*^{(2)} = f_*'^{(2)}$.

2) Forgetting the group action, chains and L^2-chains in X and $\mathrm{res}_H^G X$ are identical.

$\therefore$ $H_n^{(2)}(X)$ and $H_n^{(2)}(\mathrm{res}_H^G X)$ are the same Hilbert space.

We have $H_n^{(2)}(X) \leq L^2(G)$, giving the G-Hilbert module structure.

The H-Hilbert module structure is given by restriction:

choose $g_1, \dots, g_N$ coset representatives of H in G.

the map $\mathbb{Z}H^N \longrightarrow \mathbb{Z}G$

$$\begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix} \longmapsto \sum h_i g_i$$

is a ring isomorphism, and so is the analogous map $\mathbb{C}H^N \to \mathbb{C}G$ and $L^2(H)^N \longrightarrow L^2(G)$.

These are also (left) H-module isomorphisms.

Now $\dim_{W(G)} H_n^{(2)}(X) = \mathrm{tr}_{W(G)}\, p$

where $p: L^2(G)^n \twoheadrightarrow H_n^{(2)}(X)$ is a projection.

Now consider $\rho_H : L^2(H)^{n\cdot N} \cong L^2(G)^N \xrightarrow{\rho} \mathcal{H}_n^{(1)}(X) \cong \mathcal{H}_n^{(2)}(\ldots)$

We can view $\rho : L^2(G)^N \to L^2(G)^N$, and since ρ is G-equiv, it is a matrix

$$(\rho_{ij})_{i,j \in N} \quad , \rho_{ij} \in L^2(G).$$

ρ_H is a matrix $(\rho'_{ij})_{i,j \in nN}$, $\rho'_{ij} \in L^2(H)$, where

$$\rho_{ij}\, g_k = \sum_{l=1}^{N} \rho'_{((i-1)N+k)((j-1)N+l)}\, g_l$$

Now $\operatorname{tr} \rho_{ii} = \langle \rho_{ii}(1), 1 \rangle = \frac{1}{N} \sum_{k=1}^{N} \langle \rho_{ii}(1), 1 \rangle$

$$= \frac{1}{N} \sum \langle \rho_{ii}(g_k), g_k \rangle =$$

$$= \frac{1}{N} \sum_k \langle \sum_{l=1}^{N} \rho'_{((i-1)N+k)((i-1)N+l)}\, g_l, g_k \rangle$$

$$= \frac{1}{N} \sum_k \langle \rho'_{((i-1)N+k)((i-1)N+k)}(1), 1 \rangle$$

$$= \frac{1}{N} \sum_k \operatorname{tr}_{\nu(H)} \rho'_{((i-1)N+k)((i-1)N+k)}$$

$$\therefore \quad \operatorname{tr} \rho = \sum \operatorname{tr} \rho_{ii} = \frac{1}{N} \sum \operatorname{tr} \rho'_{ii} = \frac{1}{N} \operatorname{tr} \rho' .$$

$$\therefore \quad \dim_{N(H)} H_n^{(2)}(\operatorname{res}_H^G X) = N \dim_{N(G)} H_n^{(2)}(X) .$$

$\square$