Diagonalisation of Matrices

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Recall: For square matrix A, λ is <u>eigenvalue</u> of A if for $v \neq 0$, $Av = \lambda v$ (then v is eigenvector corresponding to λ)

To Find λ?

$\hookrightarrow$ Solutions to $Av = \lambda v$

$\iff (A - \lambda I)v = 0 \iff \det(A - \lambda I) = 0$

The polynomial $\det(A - \lambda I)$ is the <u>characteristic polynomial</u> of A, w/ $\det(A - \lambda I)$ the <u>characteristic equation</u>

Ex: $A = \begin{pmatrix} 7 & -15 \\ 2 & -4 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 7-\lambda & -15 \\ 2 & -4-\lambda \end{pmatrix}$

Characteristic polynomial $\Rightarrow (7-\lambda)(-4-\lambda) + 30 = \lambda^2 - 3\lambda + 2$

— " — equation $\Rightarrow \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$

$\rightsquigarrow$ 1, 2 eigenvalues

$\left(\begin{array}{cc|c} 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \frac{5}{2} v_2$

Find corresponding eigenvectors:

$\lambda = 1 \Rightarrow \begin{pmatrix} 6 & -15 \\ 2 & -5 \end{pmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{pmatrix} 1 & -\frac{5}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow v = \alpha \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ $\alpha \in \mathbb{R}$

$\lambda = 2 \Rightarrow \begin{pmatrix} 5 & -15 \\ 2 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow v = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\alpha \in \mathbb{R}$

$\triangleleft$

Def: For $A \in M_{n \times n}(\mathbb{R})$, λ eig. val, the <u>eigenspace</u> of λ is the nullspace $N(A - \lambda I) = \{v \in \mathbb{R} \mid (A - \lambda I)v = 0\}$

Exercise: Check this is a subspace

Note: For $A \in M_{n\times n}(\mathbb{R})$

- $|A-\lambda I| = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0)$ -> degree n
- $|A-\lambda I| = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_n)$ -> n roots
  ↳ maybe repeated

Prop: For $A \in M_{n\times n}(\mathbb{R})$, $\det(A) = \prod_{i=1}^{n} \lambda_i$

Pf] In the second point, w/ $\lambda = 0$

$$(-1)^n(0-\lambda_1)(0-\lambda_2)\dots(0-\lambda_n) = (-1)^n(-1)^n\prod_{i=1}^n \lambda_i = \prod_{i=1}^n \lambda_i \quad \square$$

Def: $A, B \in M_{n\times n}(\mathbb{R})$, $A \sim B$ (similar) iff $\exists$ P invertible st

$$P^{-1}AP = B$$

$A \in M_{n\times n}(\mathbb{R})$ is diagonalisable iff $A \sim D$, D a diagonal matrix.

Q: When diagonalisable? How to find P?

- Assume A diagonalisable, ie $P^{-1}AP = D = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

column vectors

$$\Rightarrow AP = PD \;\to\; AP = A[v_1, v_2 \dots v_n] = [Av_1, \dots Av_n]$$

$$\to PD = [v_1, \dots, v_n]\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix} = [\lambda_1 v_1, \dots \lambda_n v_n]$$

$\therefore Av_1 = \lambda_1 v_1, \dots Av_n = \lambda_n v_n$

As P^{-1} exists, cannot have 0 as solution (else P zero column).

Equivalently, λ_i, v_i eigenvalues and linearly independent eigenvectors.

- Converse also holds -> A n lin ind eigenvectors, let P matrix w/ eigenvectors as columns ($\therefore$ invertible). Then

$$Av = \lambda v \Rightarrow AP = PD \Rightarrow P^{-1}AP = D$$

① $A \in M_{n\times n}(\mathbb{R})$ diagonalisable iff it has n linearly independent eigenvectors

Note: This implies A diagonalisable if it has n distinct eigenvalues, equivalently the dimension of each eigenspace is one.

↳ ie geometric multiplicity = algebraic multiplicity

Ex: $A = \begin{pmatrix} 7 & -15 \\ 2 & -4 \end{pmatrix}$ (first example) ↙ diagonalisable

$\lambda_1 = 1,\ \lambda_2 = 2,\quad v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix},\ v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$

$\Rightarrow P^{-1} = \frac{1}{5-6} \cdot \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}$

$\Rightarrow P^{-1}AP = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 7 & -15 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\Updownarrow$

Geometric Interpretation:

Consider A as the matrix representing a linear transformation in standard coordinates, ie $T_A(x) = Ax$

Assume A has set of lin. ind. vectors $\{v_i\}_i^n = B$ ← B a basis of $\mathbb{R}^n$

$$\begin{tikzcd} & \mathbb{R}^n \arrow[r, "A"'] & \mathbb{R}^n \arrow[dr, "V^{-1}"'] & \\ \mathbb{R}^n_{\{v_i\}} \arrow[ur, "V"'] \arrow[rrr, "V^{-1}AV"'] & & & \mathbb{R}^n_{\{v_i\}} \end{tikzcd}$$

Q: How to represent T_A wrt B?

A: $A_{[B,B]} = P^{-1}AP$ w/ $P = [v_1, \dots v_n]$ (change of basis)

↳ similar matrices represent the same transformation, in different bases.

As $A_{[B,B]} = [\,[T_A(v_1)]_B, \dots [T_A(v_n)]_B\,] = [\,[\lambda_1 v_1]_B, \dots [\lambda_n v_n]_B\,]$

↳ $A_{[B,B]}$ diagonal, thus for $x \in \mathbb{R}^n$, $[x]_B = [b_1, \dots, b_n]^t$

$[T_A(x)]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} \lambda_1 b_1 \\ \vdots \\ \lambda_n b_n \end{bmatrix}$ ↳ fixes lines w/ a stretch

① Similar matrices have the same eigenvalues and same corresponding eigenvectors, w/ respect to their bases

Sketch: $A \sim B$

$\to |B-\lambda I| = |P^{-1}AP-\lambda I| = |P^{-1}AP - \lambda P^{-1} I P|$ ↑ same polynomial

$= |P^{-1}(A-\lambda I)P| = |P^{-1}|\,|A-\lambda I|\,|P| = |A-\lambda I|$

↑ same e.v.

$\to$ P transition matrix from bases S to standard coord. $\to v = P[v]_S$

$\to Av = \lambda v \Rightarrow B[v]_S = P^{-1}AP[v]_S = P^{-1}Av = P^{-1}\lambda v$

$= \lambda P^{-1} v = \lambda [v]_S$

↑ same e.$\vec{v}$ □

Def: For e.v λ_0 of A has

i) algebraic multiplicity k if $k \in \mathbb{N}$ maximal st $(\lambda-\lambda_0)^k$ factor of characteristic polynomial

ii) geometric multiplicity l if l is the dimension of the eigenspace of λ_0.

Fact: $1 \leq$ geometric multip $\leq$ algebraic multip

① (Rephrased) $A \in M_{n\times n}(\mathbb{R})$ diagonalisable iff all $\lambda \in \mathbb{R}$ and for all λ, alg. multip = geo multip.

Applications

$\to$ Powers of matrices: write $D = P^{-1}AP \iff A = PDP^{-1}$

$\Rightarrow A^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = P\overbrace{D\cdots D}^{n\text{-times}}P^{-1} = PD^nP^{-1}$

For $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \to D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$

→ Markov chains

↳ A closed system at fixed population, distributed into n different states, transitioning b/w them over time

Ex: Suppose two stores, w/ 20,000 customers. Each customer goes at most one store in a week.

Probability that person changes from one to other store

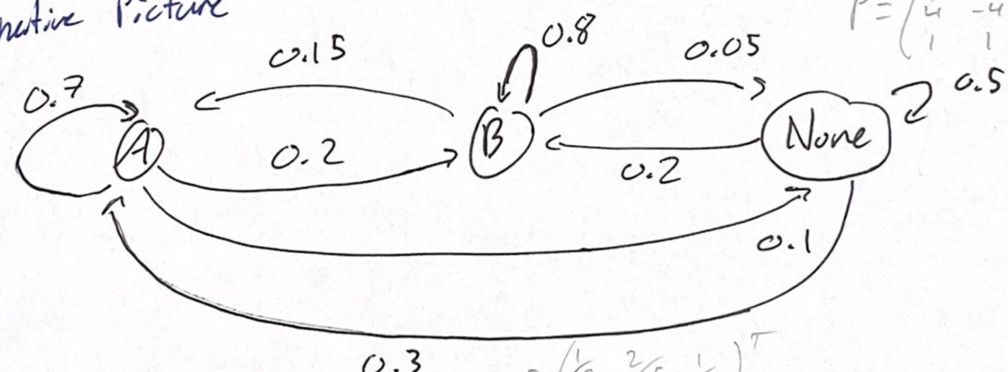
$$\begin{array}{c|c} & \begin{array}{ccc}\text{From A} & \text{From B} & \text{From None}\end{array} \\ \hline \begin{array}{c}\text{To A}\\ \text{To B}\\ \text{To None}\end{array} & \begin{pmatrix} 0.7 & 0.15 & 0.3 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.05 & 0.5 \end{pmatrix} = A \end{array}$$

transition probabilities.

$D = \begin{pmatrix} 1 & & 0 \\ & 3/5 & \\ 0 & & 2/5 \end{pmatrix}$

$P = \begin{pmatrix} 3 & 3 & -1 \\ 4 & -4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

Alternative Picture



$x_0 = (1/2, 2/5, 1/10)^T$

Assume initially; 10,000 went to A, 8000 to B, 2000 to None. What is the long-term distribution? (time unit = week)

$\Rightarrow x_n = A x_{n-1}$ $(x_n = [w_n, y_n, z_n])$ has solution $x_n = A^n x_0 = P D^n P^{-1} x_0$

Write $x_0 = (1/2, 2/5, 1/10)^T$ in eigenvector basis $\Rightarrow (1/8, 1/40, -1/20) = (b_1, b_2, b_3) = P^{-1} x_0$

$x_n = A^n(b_1 v_1 + b_2 v_2 + b_3 v_3) = b_1 \lambda_1^n v_1 + b_2 \lambda_2^n v_2 + b_3 \lambda_3^n v_3$

$\lim_n \lambda_2^n = \lim_n (3/5)^n = 0 = \lim_n (\lambda_3^n)$, $\lambda_1 = 1$

$\therefore \lim_{n \to \infty} x_n = b_1 \lambda_1^n v_1 = \frac{1}{8} \cdot \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = [3/8, 1/2, 1/8]^T$