Exchanging Operations with Integrals

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Motivation: When can we interchange an operation, such as limits, sums, derivatives, with the Lebesgue Integral?

Example: Not true in general, take $f_n = \begin{cases} 1 & n < x < n+1 \\ 0 & \text{else} \end{cases}$

Then $f_n \to f$ pointwise: $\forall x \in \mathbb{R}, \varepsilon > 0$, take $N \in \mathbb{N}$ s.t. $N > x$.
Then $\forall n > N$, $|f_n(x) - 0| = |0 - 0| = 0 < \varepsilon$.

However, $\int_{\mathbb{R}} f_n \, d\mu = \int_n^{n+1} f_n \, d\mu = 1 \cdot \mu((n, n+1)) = 1$,

thus $0 = \int_{\mathbb{R}} \lim_n f_n \, d\mu \neq \lim_n \int_{\mathbb{R}} f_n \, d\mu = 1$ ⚡

We introduce some theorems which allow us to interchange limits and integrals, if we impose restrictions on the functions.

Θ1) Monotone Convergence Theorem (MCT)

Let $f_n : X \to \mathbb{R}^+ \cup \infty$ sequence of measurable functions, on a measurable set X, such that $0 \leq f_1(x) \leq f_2(x) \leq \dots \; \forall x \in X$ and $f_n \to f$ pointwise. Then

$$\lim_n \int_X f_n \, d\mu = \int_X f \, d\mu = \int_X \lim_n f_n \, d\mu.$$

Examples: i) Let $(r_i)_{i=1}^{\infty}$ an enumeration of rationals in $[0,1]$ → sequence contains only the rationals, and each rational exactly once

and define $f_n(x) = \begin{cases} 1 & x \in \{r_1, \dots, r_n\} \\ 0 & \text{else} \end{cases}$. Then (f_n) satisfy requirements of Θ1, as they are Riemann integrable, and ~~$\lim f_n = 1_{\mathbb{Q}}$~~

$\lim_n f_n = 1_{\mathbb{Q} \cap [0,1]}$ pointwise. Thus by MCT,

$$\int_0^1 1_{\mathbb{Q}} \, d\mu = \int_0^1 \lim_n f_n \, d\mu \overset{MCT}{=} \lim_n \int_0^1 f_n \, d\mu = \lim_n 0 = 0.$$

ii) Let $f : X \to \mathbb{R}^+ \cup \infty$ measurable. Calculate

$$\lim_n \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu.$$

<u>Solution</u>: Define $f_n(x) = n \log\left(1 + \frac{f(x)}{n}\right) = \log\left(1 + \frac{f(x)}{n}\right)$

(f_n) are non-negative and measurable (composition of measurable function with a continuous fnctn). They are monotonically increasing as log is an increasing fnctn. Thus $0 \leq f_1(x) \leq f_2(x) \leq - - -$

Now as $\lim_n \left(1 + \frac{x}{n}\right)^n = e^x \; \forall x \in \mathbb{R}$ (standard calculus/analysis question)

we have

$$\lim_n f_n(x) = \lim_n \log\left(1 + \frac{f(x)}{n}\right)^n = \log\left(e^{f(x)}\right) = f(x)$$

Therefore by MCT, $\lim_n \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu$

$$= \lim_n \int_X f_n \, d\mu \overset{MCT}{=} \int_X f \, d\mu$$

Known as f measurable

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<u>Note</u>: MCT implies for $(f_n)_{n \geq 1}$ non-negative measurable, fnctns from $X \to \mathbb{R}^+ \cup \infty$,

$$\int_X \sum_{n \geq 1} f_n \, d\mu = \sum_{n \geq 1} \int_X f_n \, d\mu$$

Just apply MCT to $g_N := \sum_{n=1}^{N} f_n$

<u>Ex</u>: Calculate $\int_0^1 \frac{x^{1/3}}{1-x} \log(1/x) dx$

Note $\frac{1}{1-x} = \sum_{n \geq 0} x^n$, thus

$$\int_0^1 \frac{x^{1/3}}{1-x} \log(1/x) dx$$

$$= \int_0^1 \sum_{n \geq 0} x^{n+1/3} \log(1/x) dx$$

$$\underset{MCT}{=} \sum_{n \geq 0} \int_0^1 x^{n+1/3} \log(1/x) dx$$

Integration by parts $\to$

$$\underset{IBP}{=} 9 \sum_{n \geq 0} \frac{1}{(3n+1)^2}$$

(I) 2) Dominated Convergence Theorem (DCT)

$\boxed{\overline{\mathbb{R}} = \mathbb{R} \cup \infty}$

Let $\{f_n : X \to \overline{\mathbb{R}}\}$ sequence of measurable fktns w/ $f_n \to f$ pointwise. If $\exists$ integrable function g st $|f_n(x)| \le g(x)$ $\forall n, x$, then

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu .$$

Examples: i) Compute $\displaystyle \lim_{n \to \infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(x^2+1)} \, dx$

Let $x \in \mathbb{R}$ and define $f_n(x) = \dfrac{n \sin(\frac{x}{n})}{x(x^2+1)} = \overbrace{\dfrac{\sin(x/n)}{x/n}}^{\to 1 \text{ fixed } n} \cdot \dfrac{1}{(x^2+1)}$

Each f_n is measurable and converges to $\dfrac{1}{x^2+1}$ pointwise.

Then $g(x) = \dfrac{1}{x^2+1}$ is a suitable dominating function.;

$|\sin(x)| \le |x| \ \forall x$

$$|f_n(x)| = \left| \frac{\sin(x/n)}{x/n} \right| \cdot \left| \frac{1}{1+x^2} \right| = \frac{|\sin(x/n)|}{|x/n|} \cdot \frac{1}{x^2+1}$$

$$\le 1 \cdot \frac{1}{x^2+1} = g(x)$$

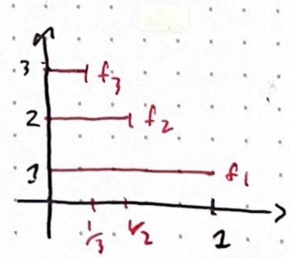
Thus; $\displaystyle \lim_n \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(x^2+1)} \, dx = \lim_n \int_{-\infty}^{\infty} \frac{n \sin(x/n)}{x(x^2+1)} \, dx$

$$\overset{DCT}{=} \int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx = \arctan(x) \Big|_{-\infty}^{\infty} = \pi .$$

ii) Why domination? Take $f_n(x) = \begin{cases} n & 0 < x \le 1/n \\ 0 & \text{else} \end{cases}$

Then $f_n \to 0$ pointwise, but $\neg\exists$ dominating function

$1 = \lim_n \int_0^1 f_n(x) \, dx \ne \int_0^1 0 \, dx = 0$



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Next a theorem which relaxes the assumptions on the fctns.

(T)3 Fatou's Lemma

i.e., no way integral of limiting function larger than limit of original integrals

Let $\{f_n : E \to \overline{\mathbb{R}}\}$ sequence of non-negative measurable fctns.

Then the function $\liminf_n f_n$ is measurable and

$$\int_E \liminf_n f_n \, d\mu \leq \liminf_n \int_E f_n \, d\mu$$

Ex: Let $f_n = \begin{cases} \mathbb{1}_{(1,2]} & n \text{ even} \\ \mathbb{1}_{[0,1]} & n \text{ odd} \end{cases}$ $\Rightarrow \forall n, \int_0^2 f_n = 1$ but $\liminf_n f_n = 0$

Thus $0 = \int_0^2 \liminf f_n \, d\mu < \liminf \int_0^2 f_n \, d\mu = 1$ ◇

When can we exchange differentiation?

(T)4 Leibniz's Integral Rule

Let $f(x,t)$, $a(x)$, $b(x)$ continuously differentiable on some region R in the (x,t) plane. Then $\forall (x,t) \in R$

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t)\, dt = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt.$$

If $a(x) = a$, $b(x) = b$ (constants) we have

$$\frac{d}{dx} \left(\int_a^b f(x,t)\, dt \right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt$$

Example: i) Compute $\int_0^1 \frac{t^3-1}{\ln(t)}\, dt$

Recall: $\frac{d}{dx} t^x = t^x \ln(t)$ via chain rule w/ $t^x = e^{x\log t}$

Define $g(x) = \int_0^1 \frac{t^x-1}{\ln(t)}\, dt$. Then by Leibniz

$$g'(x) = \frac{d}{dx}\int_0^1 \frac{t^x-1}{\ln(t)}\, dt \overset{\text{Leibniz}}{=} \int_0^1 \frac{t^x \ln(t)}{\ln(t)} = \frac{t^{x+1}}{x+1}\Big|_0^1 = \frac{1}{x+1}$$

$\rightsquigarrow g(x) = \ln|x+1| + c$

But $g(0) = \int_0^1 \frac{0}{\ln(t)}\, dt = 0 \Rightarrow \ln|1| + c = c = 0$

$\therefore g(3) = \int_0^1 \frac{t^3-1}{\ln(t)}\, dt = \ln|4|$.

ii) Computing the Gaussian $\int_0^\infty e^{-x^2/2}\, dx$; $g(t) := \left(\int_0^t e^{-\frac{x^2}{2}}\right)^2 dx$

Want to find $g(\infty)$ then take square root

$\leftarrow$ Fundamental Theorem of Calculus.

$$g'(t) = 2\int_0^t e^{-\frac{x^2}{2}}\, dx \cdot \overbrace{\frac{d}{dt}\int_0^t e^{-x^2/2}\, dx}^{\text{FTC} = e^{-\frac{t^2}{2}}} = 2e^{-\frac{t^2}{2}}\int_0^t e^{-\frac{x^2}{2}}\, dx$$

$$= 2\int_0^t e^{-\frac{(t^2+x^2)}{2}}\, dx \overset{u = x/t}{=} 2\int_0^1 t\exp\left(-\frac{t^2(1+u^2)}{2}\right) du$$

$$\rightsquigarrow g'(t) = -2\int_0^1 \frac{\frac{\partial}{\partial t}\exp\left(-\frac{t^2(1+u^2)}{2}\right)}{1+u^2}\, du \overset{\text{Leibniz}}{=} -2\frac{d}{dt}\overbrace{\int_0^1 \frac{\exp\left(-\frac{t^2(1+u^2)}{2}\right)}{1+u^2}\, du}^{h(t)}$$

$g(t) = -2h(t) + c$. Take $t \to 0$, we know $g(0) = 0$ thus

$h(0) = \int_0^1 \frac{1}{x^2+1}\, dx = \arctan(x)\Big|_0^1 = \frac{\pi}{4} \Rightarrow c = \frac{\pi}{2}$

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Take $t \to \infty \Rightarrow g(\infty) = -2\overbrace{h(\infty)}^{\to 0} + \frac{\pi}{2} \Rightarrow \int_0^\infty e^{-\frac{x^2}{2}}\, dx = \sqrt{\frac{\pi}{2}}$ ∎