Markov Chains

29.01.2024.

Discrete time: time steps $\{t_i\}_{i=0}$

Def: A seq. of random variables $(X_n)_{n\geq 0}$ w/ values in finite set E is called a Markov-Chain w/ state space E and transition matrix $P = (P_{xy})_{x,y \in E}$ if

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_0 = x_0, \ldots X_n = n) = P_{x_n x_{n+1}}$$

↳ Markov property / memory-less

A chain can be represented by a graph and matrix

Def: A matrix $M \in M_n(\mathbb{R})$ is called a Markov matrix if

i) $M_{ij} \geq 0 \quad \forall\, 1 \leq i,j \leq n$

ii) $\sum_j M_{ij} = 1 \quad \forall\, 1 \leq i,j \leq n$

Three ~~common / important properties~~ classes

a) irreducible: $\forall i,j : \exists m \in \mathbb{N}$ st $(M^m)_{ij} > 0$ → can reach every state from every state

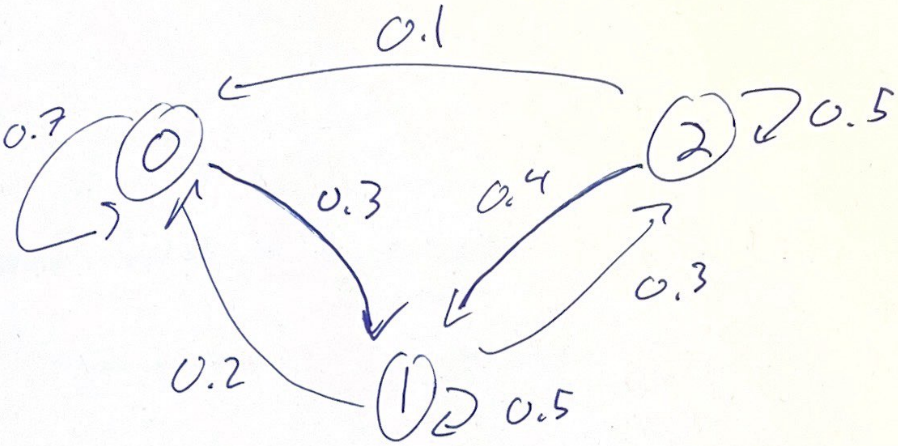
b) primitive: $\exists m : \forall i,j,\ (M^m)_{ij} > 0$ → can go $i \to j$, any pair, in n steps

c) aperiodic: $\gcd(n \in \mathbb{N} \mid M^n_{ii} > 0) = 1$ → "multiple cycles"

(T) primitive $\Longleftrightarrow$ irreducible + aperiodic

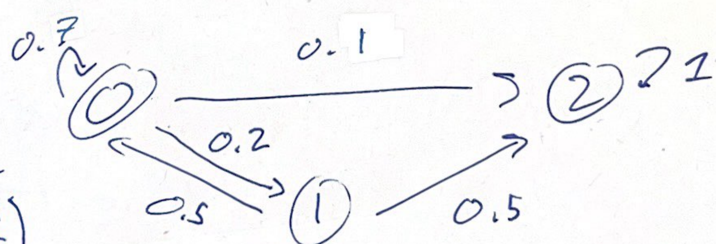
We examine what these properties look like.

a) Irreducible: $M = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}$



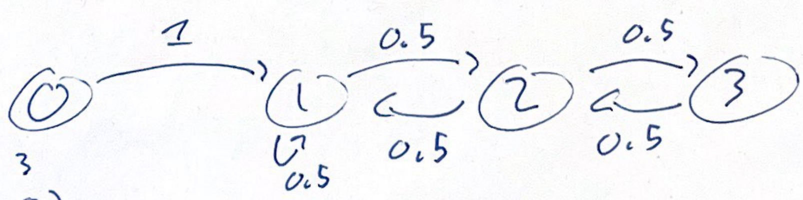
Reducible → negation of def → ∃ state from which you don't leave → absorbing
or/ once leave never return → repelling

absorbing:



$$M = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

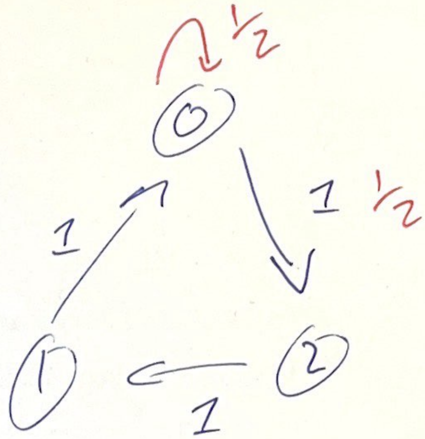
Repelling:



$$M = \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} \end{array}$$

6c) Aperiodic.

Chain w/ period:



→ period/length of cycle $0 \to 0 \implies 3$ → periodic

→ ———— '' ————→ $\gcd(1,3) = 1$

<u>Proposition:</u> If a Markov chain is irreducible, then all states have the same period

Pf] Let state i have period d, and j have period d'. We show $d = d'$

As irreducible, $\exists\, m,n \geq 0$ st $(M^m)_{ij} > 0$, $(M^n)_{ji} > 0$

Thus $(M^{m+n})_{ii} \geq (M^m)_{ij}(M^n)_{ji} > 0 \quad \therefore d \mid m+n$

Let $(M^k)_{jj} > 0$ some k (irred). As

$$(M^{m+n+k})_{ii} \geq (M^m)_{ij}(M^k)_{jj}(M^n)_{ji} > 0 \quad \therefore d \mid m+n+k$$

$\therefore d \mid k$

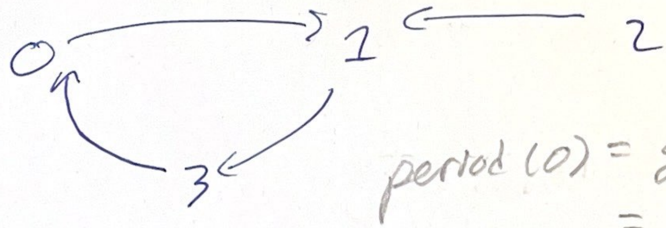
$d \mid$ every k w/ $(M^k)_{jj} > 0$ → divides gcd of them $\quad \therefore d \mid d'$ ← gcd

Similar argument shows $d' \leq d \implies d = d'$. $\square$

Exercises: Construct chains w/ given period + irred. etc

- Aperiodic + ¬ Irred   - Primitive
- Irred + ¬ Aperiodic

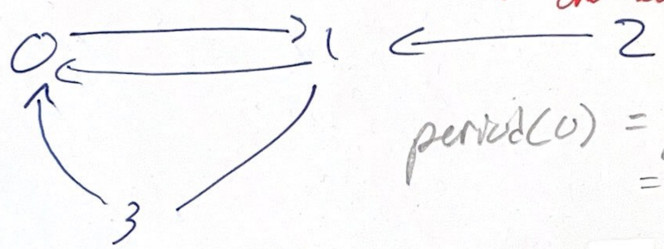
- Period 3 + reducible



period(0) = gcd{3, 6, 9, ...}
= 3

- Aperiodic + reducible

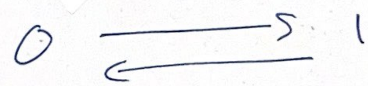
Note: loop => period = 1
① implies if irreducible has at least one loop, then aperiodic



period(0) = gcd{2, 3, 4, 5, ...}
= 1

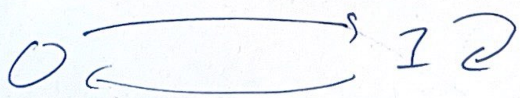
- Irreducible + periodic

period(0) = gcd{2, 4, 6, 8, ...}
= 2

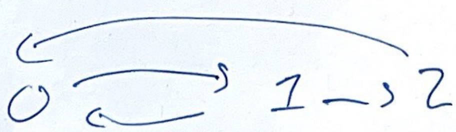


∃ vertex w/ period > 1 → periodic.

- Primitive ⟺ Aperiodic + Irreducible



period(0) = gcd{2, 3, 4, 5, ...} = gcd{1, 2, 3, 4, ...}



period(0) = gcd{2, 3, ...} = 1
period(1) = gcd{2, 3, ...} = 1

period(2) = gcd{3, 5, 7, ... 2k+1, ...}
= 1