ODE's

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Def: A <u>differential equation</u> is an equation which describes how derivatives (space/time) depend on the original ~~equation~~ function.

The <u>order</u> of an diff. eq. is the highest level of derivative present. The ODE is called <u>linear</u> if it has the form

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \dots + a_2(x)\,y'' + a_1(x)\,y' + \overset{a_0(x)}{a_0}\,y = f(x)$$

If $f(x) = 0$, the ODE is called homogeneous.

↳ do not change scaling $y \to \lambda y$

<u>Ex:</u> $\dot{y} = ay$ is a first order linear ODE.

$$\Rightarrow \dot{y} = ay \iff y'(t) = a\,(y(t)) \iff \int \frac{\frac{dy(t)}{dt}}{y(t)}\,dt = \int a\,dt$$

$$\iff \int \frac{1}{y(t)}\,dt = at + c \iff \log(y(t)) = at + c$$

$$\iff y(t) = e^{at} \cdot \tilde{C}$$

<u>Note:</u> $\dot{y} = ay$ is also an example of a <u>separable</u> ODE, ie of the form $y' = f(x) \cdot g(y)$

↳ can "separate" RHS into product of two functions

Ex: Radioactive chains of decay

$N = \#$ of un-decayed radioactive nuclei

(1) $\frac{dN}{dt} = -kN$ w/ k = decay constant

Solution to (1) $\Rightarrow N(t) = N_0 e^{-kt}$

Recall half-life $\tau \Rightarrow N(\tau) \overset{\text{def of half life}}{=} \frac{N_0}{2} = N_0 e^{-k\tau}$

$$\Rightarrow e^{-k\tau} = \frac{1}{2} \Leftrightarrow -k\tau = \ln \frac{1}{2}$$

$$\Leftrightarrow k = \frac{\ln(2)}{\tau}$$

We look at a decay chain, ie original nucleus decays into another ~~?~~ radioactive nucleon, for example

protons + neutrons

232-Thorium → Radium → Actinium → ... a few more ...

→ Lead-208

For two nuclei, denoted ~~N_1, N_2~~ w/ 1, 2, we have

$$\frac{dN_1}{dt} = -k_1 N_1 \quad \text{and} \quad \frac{dN_2}{dt} = \underbrace{k_1 N_1}_{\text{activity of nucleus 1}} - k_2 N_2$$

~~...~~ This has the form

$$\frac{dy}{dt} = h(t) + -ky$$

$$\Leftrightarrow e^{kt}\frac{dy}{dt} + k e^{kt} y = h(t) e^{kt} \Leftrightarrow \frac{d(y e^{kt})}{dt} = h(t) e^{kt} \quad (*)$$

With $H(t) = \int h(t) e^{kt}\, dt$, $(*)$ has solution

$$y e^{kt} = H(t) + c \Leftrightarrow y = H(t) e^{-kt} + c e^{-kt}$$

With our variables we have, with $N_1 = N_0 e^{-k_1 t}$

$$\frac{dN_2}{dt} = k_1 N_1 - k_2 N_2 \iff e^{k_2 t} \cdot \frac{dN_2}{dt} + k_2 e^{k_2 t} N_2 = k_1 \overbrace{N_0 e^{-k_1 t}}^{N_1} e^{k_2 t}$$

$$\iff \frac{d(e^{k_2 t} \cdot N_2)}{dt} = k_1 N_0 e^{(k_2 - k_1)t}$$

$$\iff e^{k_2 t} \cdot N_2 = k_1 \underset{\text{constant}}{N_0} \cdot \overbrace{\frac{1}{k_2 - k_1} e^{(k_2-k_1)t}}^{= \int e^{(k_2-k_1)t} dt} + C$$

$$\iff e^{k_2 t} \cdot N_2 = N_0 \frac{k_1}{k_2 - k_1} e^{(k_2 - k_1)t} + C$$

$$\iff N_2 = N_0 \frac{k_1}{k_2 - k_1} e^{-k_1 t} + C e^{-k_2 t}$$

Initial conditions $N_2(0) = 0 \Rightarrow C = -N_0 \frac{k_1}{k_2 - k_1}$

$$\Updownarrow \quad N_2(t) = \frac{N_0 k_1 e^{-k_2 t}}{k_2 - k_1} \left(e^{(k_2 - k_1)t} - 1 \right)$$

$\diamond$

<u>Note:</u> First factor $N_0 \frac{k_1}{k_2 - k_1} e^{-k_1 t}$ = # of (1) nuclei decayed to (2) but not further

Second Factor $N_0 \frac{k_1}{k_2 - k_1} e^{-k_2 t}$ = law of decay for (2) nuclei

Application: Determine age of radioactive materials

↳ know k_1, k_2 and N_2/N_1

$$\Rightarrow \frac{N_2}{N_1} = \frac{k_1}{k_2 - k_1} \left(1 - e^{(k_1 - k_2)t}\right) \xrightarrow[k_2 > k_1]{t \to \infty} \frac{k_1}{k_2 - k_1}$$

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Ex) (Fourier transform w/ ODE's) Consider

$$\frac{d^2 y(t)}{dt^2} - y(t) = -g(t)$$

By linearity of FT,

$$\mathcal{F}\left\{\frac{d^2 y(t)}{dt^2}\right\} - \underbrace{\mathcal{F}\{y(t)\}}_{Y(f)} = \underbrace{\mathcal{F}\{-g(t)\}}_{-G(f)}$$

$$\Leftrightarrow \mathcal{F}\left\{\frac{d^2 y(t)}{dt^2}\right\} - Y(f) = -G(f) \qquad \mathcal{F}\left(\frac{d^n y(t)}{dt^n}\right) = (2\pi i f)^n Y(f)$$

Differentiation property of FT $\Rightarrow \mathcal{F}\left\{\frac{dy(t)}{dt}\right\} = (2\pi i f) Y(f)$

$$\Leftrightarrow (2\pi i f)^2 Y(f) - Y(f) = -G(f)$$

FT converts differential equations into algebraic

$$\Leftrightarrow Y(f) = \frac{-G(f)}{(i2\pi f)^2 - 1} = \frac{G(f)}{1 + 4\pi^2 f^2}$$

multiplic in freq dom $\Leftrightarrow$ conval. in time
multiplic in time dom $\Leftrightarrow$ conval in freq

Now we recover $y(t)$;

$$y(t) = \mathcal{F}^{-1}\{Y(f)\} = \mathcal{F}^{-1}\left\{\frac{1}{1+4\pi^2 f^2} \cdot G(f)\right\}$$

convolution

$$= \mathcal{F}^{-1}\left\{\frac{1}{1+4\pi^2 f^2}\right\} * \mathcal{F}^{-1}\{G(f)\}$$

$$= \frac{e^{-|t|}}{2} * g(t) = \frac{1}{2}\int_{\mathbb{R}} e^{-|t-\tau|} g(\tau)\, d\tau$$

Exercise: $\mathcal{F}\{e^{-|at|}\} = \dfrac{2|a|}{|a|^2 + (2\pi f)^2}$

Ex) (Logistic Equation)

Let $P(t)$ = population at time t, P_0 = initial pop

K = carrying capacity

r = growth rate

The Logistic equation is $\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$ $(**)$

↳ $\frac{P}{K}$ small ⇒ almost exponential growth
$1 - P/K > 0$

↳ $P < K$ w/ $\frac{P}{K} \to 1$ → slow growth $P = K \Rightarrow$ stable

↳ $\frac{P}{K} > 1$ ⇒ negative growth

Let $P_0 = 900{,}000$, $K = 1{,}072{,}764$, $r = \frac{\ln(2)}{3} \approx 0.2311$

$(**)$ $\frac{dP}{dt} = 0.2311 P\left(1 - \frac{P}{1{,}072{,}764}\right)$ Separation of variables

$\Leftrightarrow dP = 0.2311 P\left(\frac{1{,}072{,}764 - P}{1{,}072{,}764}\right) dt$

$\Leftrightarrow \int \frac{dP}{P(1{,}072{,}764 - P)} = \int \frac{0.2311}{1{,}072{,}764} dt$

Partial Fraction decomposition

$\Leftrightarrow \frac{1}{K} \int \frac{1}{P} + \frac{1}{K-P} \, dP = \frac{r}{K} t + C$

$\Leftrightarrow \ln|P| - \ln|K - P| = rt + C$

$\Leftrightarrow \ln\left|\frac{P}{K-P}\right| = rt + C \quad \Leftrightarrow \frac{P}{K-P} = e^{rt} \cdot C$

$\Leftrightarrow P = \frac{Ke^{rt}C}{1 + Ce^{rt}} \quad \Leftrightarrow P = \frac{(1{,}072{,}764)(25{,}000)e^{rt}}{4799 + 25000 e^{rt}} = \frac{1072764\, e^{rt}}{0.19196 + e^{rt}}$

↑ Can determine population at future times. ↑

Other examples!
→ Lotka-Volterra model
↳ (predator prey eqns)

→ Hodgkin-Huxley model
↳ (how nerves fire)

→ Epidemiology → SIR/SIS model
→ Allee effect → population bio

less pop ⇒ less success

$P(0) = 900{,}000$
$\Rightarrow C = \frac{25{,}000}{4{,}799}$
≈ 5.209