Image Compression and Logistic model

15.01.2024

Recall: (Discrete Fourier Transform) $f : C_N \to \mathbb{C}$

$$(Df)(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} f_k \, e^{-2\pi i k \frac{n}{N}}$$

where C_N = cyclic group, $f(i) = f_i := f(t_i)$, $\{t_i\}$ data points.

For images, need 2-dim version

Def: (2-d. DFT) $f : (C_{N_1} \times C_{N_2}) \to \mathbb{C}$

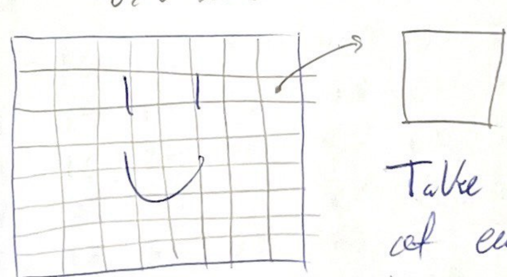
$$(Df)(n_1, n_2) = \frac{1}{\sqrt{N_1}\sqrt{N_2}} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} f(k_1, k_2)\, e^{-2\pi i k_1 \frac{n_1}{N_1}} e^{-2\pi i k_2 \frac{n_2}{N_2}}$$

Image Compression:

- Take a picture (colour) => ? MB of data
  → want to reduce

8x8 subdiv.

- JPEG:



Take 2-d FT (discrete) of each square
↳ e.g.: gray value at ith pixel = f_i

compression step

Quantize

→ Discard / truncate small values of freq

→ "Natural" / Non-static / noise images are tiny % of all image space
↳ Ex → 10×10 pixels only w/ black & white?
→ $2^{100} \simeq 10^{3}$ possible images

→ Left only w/ signif. values → Inverse transform back

Depending on how you transform + truncate, there may be varying levels of loss

-> Throw away more data -> more compression + more loss
-> chunkier image

Note: that the 2-dim DFT is equivalent (for sake of compression) to doing the DFT on rows/colums; both return an 8x8 matrix

- The truncation is done w/rt a quantization matrix.

Ex: Say we get the following freq. matrix

$$A = \begin{pmatrix} -415 & -30 & -61 & 27 & 56 & -20 & -2 & 0 \\ 4 & -22 & -61 & 10 & 13 & -7 & -9 & 5 \\ 47 & 7 & 77 & 25 & 29 & 10 & 5 & 6 \\ 49 & 12 & 34 & -15 & -10 & 6 & 2 & 2 \\ 12 & -7 & -13 & -4 & -2 & 2 & -3 & 3 \\ -8 & 3 & 2 & -6 & -2 & 1 & 4 & 2 \\ -1 & 0 & 0 & -2 & -1 & -3 & 4 & -1 \\ 0 & 0 & -1 & -4 & -1 & 0 & 1 & 2 \end{pmatrix}$$

truncate wrt Q
$\hookrightarrow$ divide each A entry by resp. Q then round to integer

$$Q = \begin{pmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{pmatrix}$$

$\Rightarrow$

gives

$$\tilde{A} = \begin{pmatrix} -26 & -3 & -6 & 2 & 2 & -1 & 0 & 0 \\ 0 & -2 & -4 & 1 & 1 & 0 & 0 & 0 \\ -3 & 1 & 5 & -1 & -1 & 0 & 0 & 0 \\ -4 & 1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & & & & & & \\ 0 & 0 & & & 0 & & & \\ 0 & 0 & & & & & & \\ 0 & 0 & & & & & & \end{pmatrix}$$

Other transformation options exist, eg FFT, discrete cosine transform, etc. Lossless ~~tr~~ compression also possible.

<u>More info:</u> Dr. Dirk Frettlöh

<u>Logistic Growth</u>

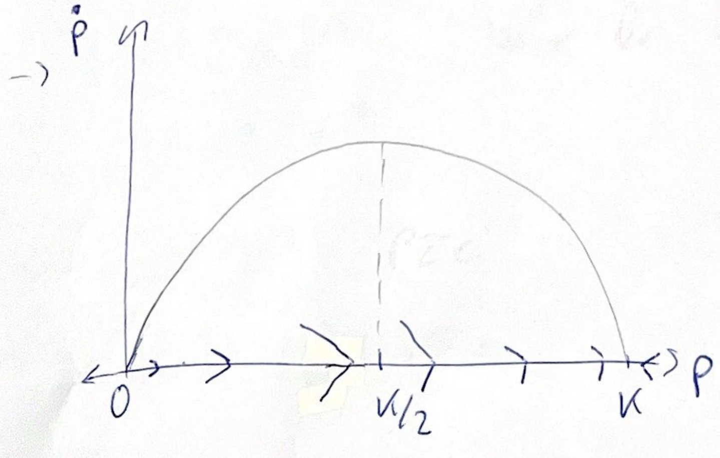
Let $P(t)$ = population @ time t, P_0 = initial pop

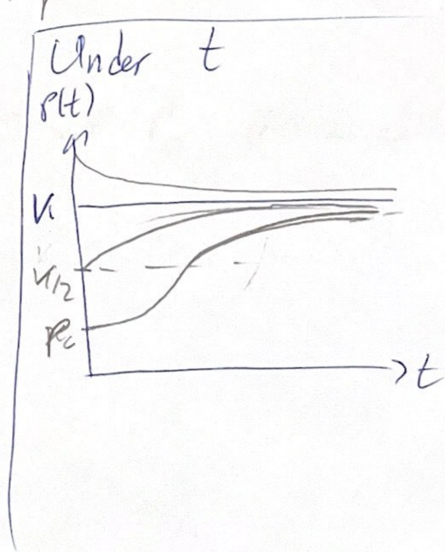
K = carrying capacity ; r = growth rate

<u>Def!</u> The logistic growth model is $\frac{dP}{dt} = rP\left(\frac{K-P}{K}\right)$ (*)

$\hookrightarrow \dot{P}$

Examine behaviour under P





- Solution to (*)? See page *
  - → Done on 11.12.2023 → $P(t) = \frac{cKe^{rt}}{1+ce^{rt}} = \frac{K}{1+be^{-rt}}$ w/ $b = \frac{1}{c}$

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}$$

$$\Leftrightarrow \int \frac{K}{P(K-P)}\, dP = \int r\, dt \qquad \frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{(K-P)}$$

$$\Leftrightarrow \int \frac{1}{P}\, dP + \int \frac{1}{(K-P)}\, dP = rt + c$$

$$\overset{*}{\Leftrightarrow} \ln|P| - \ln|K-P| = rt + c$$

$$* \int \frac{1}{K-P}\, dP = -\int \frac{1}{(P-K)}\, dP = -\ln|P-K| = -\ln|K-P|$$

$$\Leftrightarrow \ln\left|\frac{P}{K-P}\right| = rt + c$$

$$\Leftrightarrow e^{\ln\left|\frac{P}{K-P}\right|} = \tilde{c}\, e^{rt}$$

$$\Leftrightarrow \frac{P}{K-P} = \tilde{c}\, e^{rt}$$

$$\Leftrightarrow P = \frac{\tilde{c} K e^{rt}}{1+\tilde{c} e^{rt}}$$ Solve for $\tilde{c}$ w/ P_0 initial data

$$\hookrightarrow P = K\tilde{c}e^{rt} + -P\tilde{c}e^{rt}$$

$$\Leftrightarrow P(1+\tilde{c}e^{rt}) = K\tilde{c}e^{rt} \Leftrightarrow P = \frac{K\tilde{c}e^{rt}}{1+\tilde{c}e^{rt}}$$

$*$

Stability?

Def: An equilibrium point $\bar{x}$ ($\rightarrow x(t) = \bar{x} \ \forall t > 0$) for the diff. eq

$\boxed{\dot{x} = g(x)}$ is called

i) Locally stable if $\forall$ neighbourhoods U of $\bar{x}$ $\exists \, U' \subseteq U$ st $\forall x_0 \in U'$, $x(t) \in U \ \forall t \geq 0$

↳ $x(t)$ sol to $x_0 = x(0)$

ii) Asymptotically stable (attractor) if $\exists \varepsilon > 0$ st $\forall x_0 \in U_\varepsilon(\bar{x})$ gives $\lim_{t \to \infty} x(t) = \bar{x}$

↳ by def, also locally stable

If not asymptotically stable, then neutrally stable.

If not locally stable, then unstable (repulsive)

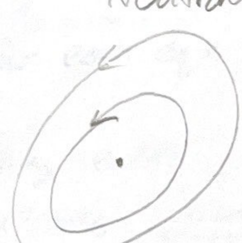
In $\mathbb{R}$, sufficient criteria $\rightarrow$ $g'(\bar{x}) < 0 \rightarrow$ attractive (↳ decreasing to x-axis $\rightarrow$)

$g'(\bar{x}) > 0 \rightarrow$ repulsive (↳ increasing to x-axis.)

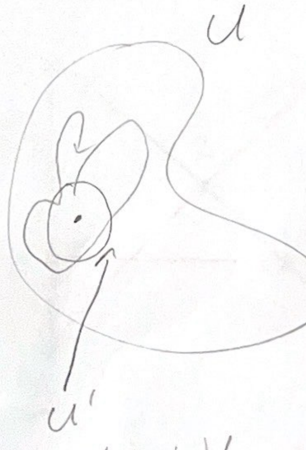
↷ $P = 0$ is an unstable equilibrium

$P = K$ is a stable — '' —

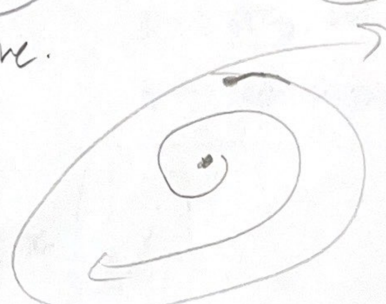
Neutrally stable



Attractive.



Locally stable



Repulsive.

System of Eqns → Predator-Prey

- $\dot{x} = g(x,y)$, $x(0) = x_0$ ← population of lynx's
  $\dot{y} = h(x,y)$, $y(0) = y_0$ ← and hares

Start w/ logistic growth, set λ_x = birth rate of prey
λ_y = death rate of pred
"r/k" → r_x, r_y = competition w/in species

$\Rightarrow \dot{x} = \lambda_x x - r_x x^2$ $\quad r = \lambda_x,\ r_x = r/k \Rightarrow rx - \frac{r}{k}x^2 = rx\left(1 - \frac{x}{k}\right)$

- $\dot{y} = -\lambda_y y - r_y y^2$ ← w/out prey, pred only die.

Add interaction b/w prey and pred.

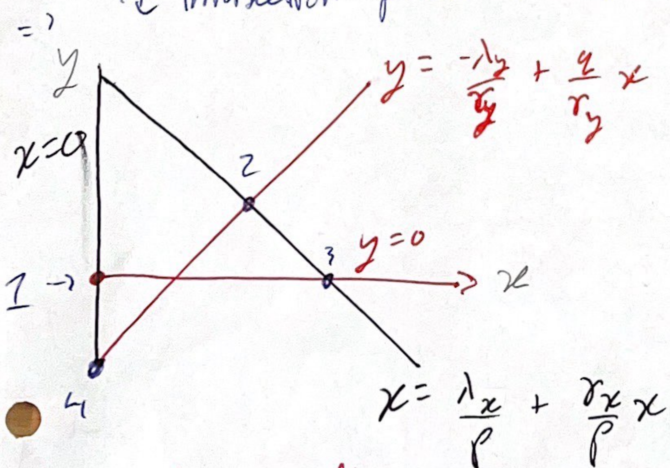
$\Rightarrow \dot{x} = \lambda_x x - r_x x^2 - p x y =: g$ $\quad p$ = loss of prey when encounter pred

$\dot{y} = -\lambda_y y - r_y y^2 + q x y =: h$ $\quad q$ = growth of pred when encounter prey

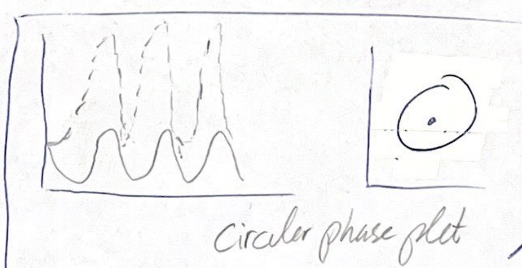
xy → dead prey proport. to both pop (assume each lynx can access all hares)

- → Equilibrium points ⇒ solve $\dot{x} = g = 0$; $\dot{y} = h = 0$
  2 intersection points of "null-isoclines" for each eqn

⇒



1 → nothing exists
2 → coexistence → neutral
3 → prey. w/out pre.
4 → non-sense.



Circular phase plot

More info: Dr. Ellen Baake