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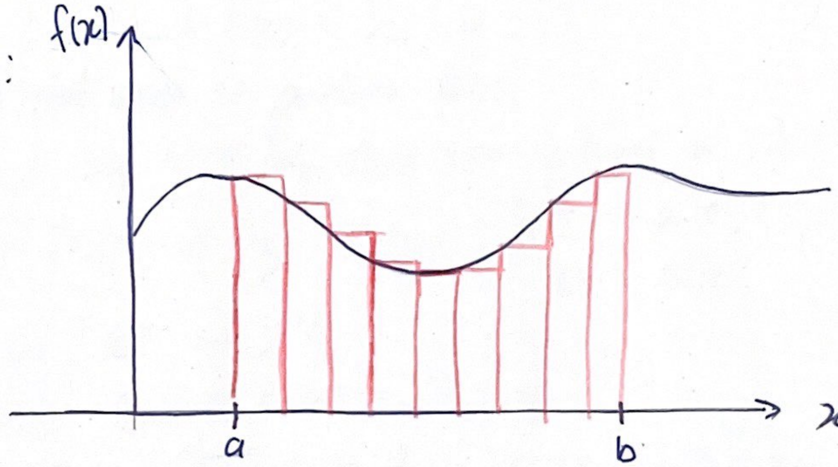
Riemann and Lebesgue Integration

Recall: (Riemann Integral) For f a real valued function on $[a,b]$ the Riemann sum, w.r.t the partition $(x_i)_{i=0}^{n}$ and test values $(t_i)_0^n$ is $\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$

The limit (if it exists) of the Riemann sums, as the partition becomes finer, is the Riemann integral.
In this case, the function f is said to integrable.

Geometrically:



Q: When is a function Riemann-Integrable? (On compact intervals)

A: - Continuous functions
- Monotonic (w/ possibly countably infinite # of jumps)
↳ measure zero
- Lebesgue's Characterization
↳ bounded function on $[a,b]$ Riem.-Integrable iff continuous almost everywhere
↳ discontinuities form measure zero set.

Issues:

i) RI cannot handle too many discontinuities

$\Rightarrow$ Take $f(x) = 1_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ Dirichlet function

Jumps b/w 0, 1

By density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, this function is discontinuous on both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$

ii) RI does not work w/ unbounded functions

$\hookrightarrow f(x) = \frac{1}{x}$ on $[0,1]$

iii) RI does not work w/ pointwise limits

$\Rightarrow (r_i)_{i \geq 1}$ seq that only includes each rational in $[0,1]$ exactly once

$f_k(x) : [0,1] \to \mathbb{R}$ by $f_k(x) = \begin{cases} 1 & x \in \{r_1, r_2, \dots r_k\} \\ 0 & \text{else} \end{cases}$

$\Rightarrow$ Each one RI w/ $\int^1 f_k = 0$

clearly $\lim_{k \to \infty} f_k(x) = f(x) = 1_{\mathbb{Q}}(x)$

$\Rightarrow$ Pointwise limit of RI functions not RI.

$\Rightarrow$ Need better idea of integration.

$\hookrightarrow$ Introduce new idea of measuring size of sets.

Recall: Definitions: a) Given set X, a collection of subsets (A) is a sigma-algebra if

i) $\emptyset \in A$

ii) $a \in A \Rightarrow a^c \in A$

iii) $\bigcup_k a_k \in A$

Then (X, A) is called a measurable space.

b) Given (X, A) a function $\mu : A \to \mathbb{R} \cup \{\infty\}$ is called a measure if

i) $\mu(a) \geq 0 \quad \forall a \in A$

ii) $\mu(\emptyset) = 0$

iii) $\mu(\bigcup_{k\geq 1} a_k) = \sum_{k\geq 1} \mu(a_k)$ (countable additivity) a_k pairwise disjoint

↳ Measures generalize concept of size/volume. There are many measures, we care about the Lebesgue measure.

Def: For any interval $I \subset \mathbb{R}$, denote length by $\ell(I)$. The Lebesgue outer measure for $E \subseteq \mathbb{R}$, $\mu^*(E)$, is

$$\mu^*(E) = \inf \{ \sum_{k\geq 1} \ell(I_k) \}$$

where $(I_k)_k$ sequence of open intervals w/ $E \subset \bigcup_{k\geq 1} I_k$.

Def: A set $E \subseteq \mathbb{R}$ is Lebesgue measurable if $\forall \varepsilon > 0$ $\exists$ open set G st $E \subseteq G$ and

$$\mu^*(G \setminus E) < \varepsilon$$

Then the Lebesgue measure of E is defined $\mu(E) = \mu^*(E)$

Note: Not every subset of $\mathbb{R}$ is Lebesgue measurable, such as a Vitali set.

Caratheodory's Measurability criterion gives a way to check for measurability.

Ex: i) $\mu(\{a\}) = 0 \quad a \in \mathbb{R}$

ii) $\mu(\mathbb{Q}) = 0$ iii) $\mu((a,b)) = b-a$ $\ell(a,b) = b-a$

$$\mu(\mathbb{Q}) = \mu(\bigcup_{n\geq 1} \{q_n\}) = \sum_{n\geq 1} \mu(\{q_n\}) = \sum 0 = 0.$$

We have a new way of measuring sets, next we need functions to integrate.

Def: Let $f : [a,b] \to \mathbb{R}$. We say f is Lebesgue measurable on $[a,b]$ if $\forall s \in \mathbb{R}$, the set $\{x \in [a,b] \mid f(x) > s\}$ is a Lebesgue measurable set. Can be extended to $(-\infty, \infty)$, $[a, \infty)$ etc.

Ex: Let $f(x) = x^2$ on $\underbrace{[-1,5]}_{I}$, $s \in \mathbb{R}$. We show f Lebesgue meas.

i) $s \geq 25 \Rightarrow \{x \in I \mid f(x) > s\} = \emptyset$ Lebesgue measurable.

ii) $s < 0 \Rightarrow$ —— " —— $= [-1, 5]$ L-meas $= 5 - -1 = 6$

iii) $0 \leq s < 1 \Rightarrow$ —— " —— $= [-1, -\sqrt{s}) \cup (\sqrt{s}, 5]$ L-meas

iv) $1 \leq s < 25 \Rightarrow$ —— " —— $= (\sqrt{s}, 5]$ L-meas.

$\Rightarrow$ f Lebesgue measurable function on $[-1, 5]$.

Note: • TFAE

i) f Lebesgue meas ii) $\forall s \in \mathbb{R}$, $\{x \in I \mid f(x) \leq s\}$ L-measurable

iii) $\forall s \in \mathbb{R}$, $\{x \in I \mid f(x) < s\}$ L-meas, iv) $\forall s \in \mathbb{R}$, $\{x \in I, f(x) \geq s\}$ L-meas

• Can be extended to a set X, ie $f: X \to \mathbb{R}$ Lebesgue measurable if the pre-image of every interval is measurable.

$\hookrightarrow f^{-1}(E) = \{x \in X \mid f(x) \in E\} \in$ Sigma-Algebra

Note: • Increasing functions and continuous functions are measurable.

• f meas $\Rightarrow$ $f(x) + c$, $c \cdot f(x)$ both meas

• f, g meas $\Rightarrow$ $f + g$; $f \cdot g$, $\frac{f(x)}{g(x)}$ (w/ $g(x) \neq 0$) all measurable.

Def: For f a measurable function on a finite interval I, w/ $\alpha, \beta \in \mathbb{R}$ st $\alpha < f(x) < \beta \ \forall x \in I$, let $P = \{[y_{i-1}, y_i]\}_{i=1}^{n}$ be a partition of $[\alpha, \beta]$.

Set $A_i = f^{-1}((y_{i-1}, y_i])$. The lower Lebesgue sum of f wrt P is

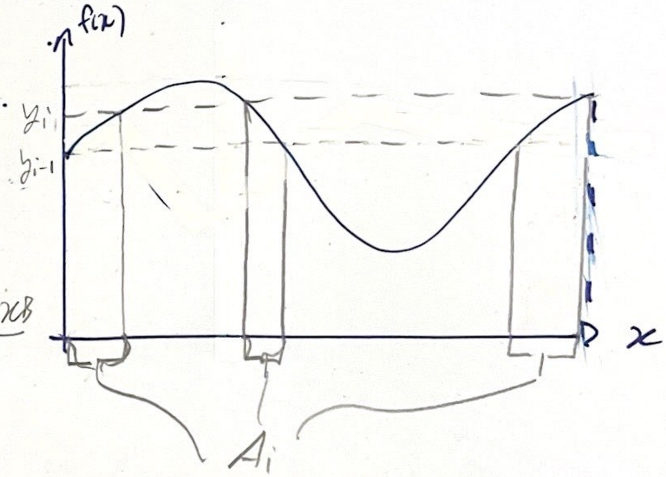
$$\underline{S}_L(f,P) = \sum_{i=1}^{n} y_{i-1} \cdot \mu(A_i)$$

· Similarly, the upper Lebesgue sum is

$$\overline{S}_L(f,P) = \sum_{i=1}^{n} y_i \cdot \mu(A_i).$$

Compare: Geometric picture.

↳ Better approximation
Same idea as Riemann, but
now we are partitioning the y-axis



Def: The Lebesgue integrals for the lower/upper sums are given by

i) Upper integral: $_a\overline{\int}^b f = \inf_{\text{partitions}} \overline{S}_L(f,P)$

ii) Lower integral: $_a\underline{\int}^b f = \sup_{\text{partitions}} \underline{S}_L(f,P)$

iii) If $_a\overline{\int}^b f = {_a\underline{\int}^b} f$ then f is Lebesgue integrable and write it $_a\int^b f(x)\,dx$ (or sometimes $_a\int^b f(x)\,d\mu$ if you want to specify the measure)

Note: · Linearity as in Riemann

- f bd + meas ⇒ Integral exists
- $f=g$ a.e ⇒ Integrals are the same.
- f Riem.-integ ⇒ f Leb-integ, and integrals are equal
- Easier to take limits (ie Monotone convergence, Dominated convergence)
- $p \geq 2$. L^p spaces are complete

(T) Monotone Convergence

Let (f_n) be a sequence of non-negative measurable functions on a meas. set A, st

i) $0 \leq f_1(x) \leq f_2(x) \leq \ldots \leq f_n(x) \leq \ldots \leq f(x) \quad \forall x \in A$

ii) $f_n \xrightarrow{\text{pointwise}} f$ on A

Then $\lim_{n \to \infty} \int_A f_n = \int_A f$

(T) Dominated convergence.

Let (f_n) a sequence of measurable functions on a meas set A. st

i) $f_n \xrightarrow{\text{pointwise}} f$ on A

ii) $\exists$ integrable function g st $|f_n(x)| \leq |g(x)| \quad \forall x \in A$

Then $\lim_n \int_A f_n = \int_A f$.

Examples

i) We show $f(x) = \mathbb{1}_{\mathbb{Q}}$ is <u>not</u> Riemann integrable on $[0,1]$, but is Lebesgue integrable.

<u>Riemann Integrability:</u> We use Lebesgue's Criterion for Riemann integrability.

[Lebesgue's Criterion: A bounded function is Riemann integrable iff the set of discontinuities has Lebesgue measure 0.]

Let $x \in [0,1] \cap \mathbb{Q}$. Then $f(x) = 1$. Pick a sequence of <u>irrational</u> numbers $(x_n)_n$ w/ $x_n \to x$, which is possible by the density of $\mathbb{R} \setminus \mathbb{Q}$. Then $f(x_n) = 0$ which does not converge to $f(x) = 1$, so f is discontinuous on the rationals.

Let $x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$. Then $f(x) = 0$. Pick a sequence of <u>rational</u> numbers $(x_n)_n$ w/ $x_n \to x$, again possible by the density of $\mathbb{Q}$. Then $f(x_n) = 1$, which does not converge to $f(x) = 0$, so f is discontinuous on $\mathbb{R} \setminus \mathbb{Q}$.

Thus, the set of discontinuities for f in $[0,1]$ is <u>all</u> of $[0,1]$, which has Lebesgue measure $1 - 0 = 1 \neq 0$.

Therefore, by Lebesgue's criterion, f is not Riemann integrable.

$\Longrightarrow$

Next, we show $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is Lebesgue integrable on $[0,1]$.

Note that $-1 < f(x) < 2$, so we partition $[-1,2]$, denote it $P = \{[y_{i-1}, y_i]\}_{i=1}^{n}$. Now, there exists unique integers i_0, i_1 st

$$y_{i_0-1} < 0 \le y_{i_0} \quad \text{and} \quad y_{i_1-1} < 1 \le y_{i_1}$$

These are the only two horizontal segments worth considering, as f takes only the values $0, 1$. In particular, the pre-image of any other value in $(0,1)$ is the empty set, and thus of measure 0. Therefore

$$ {}_0\!\int^1 f(x)\,d\mu = \inf_{\text{partitions}} \overline{S}_L(f,P)$$

$$= \inf_{\text{partition}} \sum_{i=1}^{n} y_i \cdot \mu(\overset{\text{pre-image}}{A_i})$$

$$= \inf_{\text{part.}} \Big(\underbrace{0}_{=y_{i0}} \cdot \overbrace{\mu\big([0,1] \cap (\mathbb{R}\setminus\mathbb{Q})\big)}^{\text{measure}=1} + \underbrace{1}_{y_{i_1}} \cdot \overbrace{\mu\big(\mathbb{Q} \cap [0,1]\big)}^{=0\ (\text{countable})} \Big)$$

$$= 0 \cdot 1 + 1 \cdot 0 = 0$$

$\diamond$

ii) The Lebesgue of a step fktn: Consider $f(x) = \begin{cases} 1 & -1 \le x < 2 \\ 2 & 2 \le x < 4 \\ 3 & 4 \le x \le 8 \\ 0 & \text{else} \end{cases}$

Then,

$$ {}_{-1}\!\int^8 f(x)\,d\mu = 1\mu[-1,2] + 2\mu[2,4] + 3\mu([4,8] + 0$$

$$= 19$$

ii) Next we show that $f = g$ almost everywhere, with one (and thus both) Lebesgue measurable, then

$$\int_a^b f = \int_a^b g$$

Recall that $f = g$ a.e means, outside some set of measure zero set, call it M, they give the same value. For $x \in M$, we have $f(x) \neq g(x)$!

both parts ≥ 0.

$$\text{As any function can be written as } f = \underbrace{f_+}_{\text{non-negative part}} - \underbrace{f_-}_{\text{negative part}}, \text{ it}$$

is sufficient to consider it only for non-negative and measurable functions.

Moreover, as $f = g$ a.e, we can consider the claim for $h := f - g \equiv 0$ a.e. That is, if $h = 0$ a.e then $\int_a^b h \, d\mu = 0$; the claim then follows by linearity of the integral.

Without loss of generality, let $f \geq g$ on M, ie $h > 0$ on M.

<u>Claim</u>: $\int h \, d\mu = 0$ iff $h = 0$ a.e.

Pf] As h measurable, the set $N := \{h \neq 0\} = \{h > 0\}$ is measurable

We show $\int h \, d\mu = 0 \iff \mu(N) = 0$.

"$\Rightarrow$" Assume $\int h \, d\mu = 0$, and define $A_n := \{h \geq 1/n\}$ measurable $\forall n$, w/ $A_n \nearrow N$. Obviously $h > 1/n \cdot 1_{A_n}$, thus

Note: $\int 1_{A_n} \, d\mu = \mu(A_n)$

$$0 = \int h \, d\mu \geq \int 1/n \, 1_{A_n} \, d\mu = 1/n \, \mu(A_n) \quad \forall n$$

$$\Rightarrow \mu(N) = 0.$$

"$\Leftarrow$" $\mu(N) = 0$, define $a_n := n 1_N$ and $j := \sup_{n \in \mathbb{N}} a_n$; note $f \leq j$.

Then j measurable w/ $a_n \nearrow j$ and

$$\int j \, d\mu = \sup_{n \in \mathbb{N}} \int a_n \, d\mu = 0.$$

as $a_n = n \cdot 1_N$ and $\mu(N) = 0$

Thus $0 \leq \int f \, d\mu \leq \int j \, d\mu = 0 \Rightarrow \int f \, d\mu = 0$. $\Diamond$ $\Longrightarrow$

Note: Some technical details omitted.

Example: i) $1_{\mathbb{Q}} = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} = 0$ a.e.

as the set where $1_{\mathbb{Q}} \neq 0$ is $\mathbb{Q}$ which has measure zero.

ii) $f(x) = \lceil x \rceil$ (ceiling function)

$g(x) = \lfloor x \rfloor + 1$ (floor function)

Then $f = g$ a.e, w/ the zero set being the integers, obviously measure zero. $\diamond$

If one wishes to see more, see "A User-Friendly Introduction to Lebesgue measure and integration" by Gail S. Nelson.