Scaler Products, Norms, and Metric Spaces

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Def: Let V be a complex vector space (C.V.S) (finite-dim)
A mapping $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$ is called a <u>scaler product</u> if $\forall\ f, g, h \in V,\ \alpha \in \mathbb{C}$,

i) $\langle f | g+h \rangle = \langle f | g \rangle + \langle f | h \rangle$

ii) $\langle f | \alpha g \rangle = \alpha \langle f | g \rangle$

iii) $\langle g | f \rangle = \overline{\langle f | g \rangle}$ (complex conjugation)

iv) $\langle f | f \rangle \geq 0$ w/ $\langle f | f \rangle = 0 \iff f = 0$

Then $(V, \langle \cdot | \cdot \rangle)$ is called an inner product space

Def: Let V be a C.V.S. A mapping $\| \cdot \| : V \to \mathbb{R}$ is called a <u>norm</u> if $\forall\ u, v \in V,\ \alpha \in \mathbb{C}$,

i) $\|v\| \geq 0$ w/ $\|v\| = 0 \iff v = 0$

ii) $\|\alpha v\| = |\alpha| \, \|v\|$

iii) $\|v + u\| \leq \|v\| + \|u\|$ (Δ-inequality)

Then $(V, \|\cdot\|)$ is called a normed vector space

Recall in $\mathbb{R}^n$, the scaler product is given by the dot product, i.e. w/ $\vec{u} = (u_1, u_2, \ldots u_n)$; $\vec{v} = (v_1, v_2, \ldots v_n)$,

$$\langle \vec{u} | \vec{v} \rangle = \vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n .$$

Moreover, the "standard" Euclidean norm $\|v\| = \left(\sum_{i=1}^{n} (v_i)^2 \right)^{\frac{1}{2}}$ is induced by the scaler product, that is

$$\|v\| = (v \cdot v)^{\frac{1}{2}} = (\langle v | v \rangle)^{\frac{1}{2}}$$

additional page

↓ We say that the scaler product <u>induces</u> (synonym: produces) the norm. This is true in general; if we are given a v.s. V w/ a scaler product $\langle \cdot | \cdot \rangle$, then we can define a norm by $\|v\| := (\langle v|v\rangle)^{\frac{1}{2}}$

$\Rightarrow$

<u>Post Script:</u> We have agreed to start at 10:00 precisely, and go to 11:15.

Examples: In $\mathbb{R}^n$ we have a few standard norms

i) $\|v\|_1 = |v_1| + |v_2| + \dots + |v_n|$

ii) $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$

iii) $\|v\|_\infty = \sup\{|v_1|, |v_2|, \dots |v_n|\} = \max\{|v_1| \dots, |v_n|\}$ in finite dim.

Take $v = (1, 2, 3)$, then, w/ $p = 3$

$\|v\|_1 = 6$; $\|v\|_p = (36)^{1/3} \simeq 3.302$; $\|v\|_\infty = 3$

Def: We say that two norms $\|\cdot\|_a, \|\cdot\|_b$ are equivalent if $\exists\ 0 < c_1 \leq c_2$ st $\forall v \in V$,

$$c_1 \|v\|_b \leq \|v\|_a \leq c_2 \|v\|_b$$

That is, you can work w/ your favourite norm, if they are equivalent.

Prop: In a (finite-dim) v.s., all norms are equivalent.

Idea of proof: Show any norm $\|\cdot\|_a$ is equivalent to $\|\cdot\|_1$.

↳ equivalence of norms is equivalence relation.

↳ "$\leq$" $\|v\|_a \leq |v_1| \|v_1\|_a + \dots + |v_n| \|v_n\|_a$

$\leq C \|v\|_1$ w/ $C := \max \|v_i\|_a$

↳ "$\geq$" By contradiction, w/ Bolzano-Weierstraß

$\square$

There are multiple equivalent proofs, see

Note: The proposition fails for infinite-dimensional v.s.

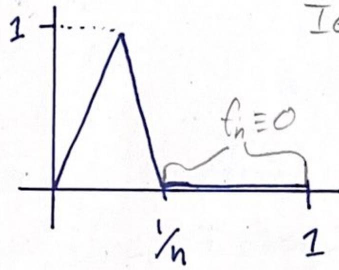
Take $V = C[0,1]$ over $\mathbb{R}$. $C[0,1] = \{f : [0,1] \to \mathbb{R} \mid \text{continuous}\}$.

We define $\|f\|_{max} = \max_{x \in [0,1]} |f(x)|$; $\|f\|_{L^2} = \int_0^1 |f(x)|\, dx$

Claim:
There does not exist $C > 0$ st $\|f\|_{max} \le C \|f\|_{L^2}$ $\forall f \in V$

Ie, the norms are not equivalent.

$\Rightarrow$ Take f_n as



$\int_0^1 |f_n(x)|\, dx$ = Area of triangle $= \dfrac{\text{height} \cdot \text{base}}{2} = \dfrac{1 \cdot 1/n}{2} = \dfrac{1}{2n}$

Then $\|f_n\|_{max} = 1$ and $\|f_n\|_{L^2} = \dfrac{1}{2n}$ $\forall n$.

$\therefore \neg\exists\, C > 0$ st $1 \le C \cdot \dfrac{1}{2n}$ $\forall n$. $\square$

$\therefore$ Not equivalent

Def: Let X be a set. A mapping $d : X \times X \to \mathbb{R}$ is called a metric if $\forall\, x, y, z$;

i) $d(x,y) \ge 0$ w/ $d(x,y) = 0$ iff $x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,y) \le d(x,z) + d(z,y)$ Δ-inequality

Then (X, d) is called a metric space.

Note: Given a norm, it induces a metric via $d(x,y) = \|x - y\|$, ie every normed v.s. is automatically a metric space.

Example: In $\mathbb{R}^n$; define $d(x,y) := \|x - y\|_1 = \sum_i^n |x_i - y_i|$

This is called the taxi-cab norm, or Manhattan metric.

Q: If every norm induces a metric, and (in finite dim) all norms are equivalent, are all metrics equivalent?

A: No. Take any metric not induced by a metric, for example the discrete metric

$$\delta(x,y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}.$$

Note: If two norms are equivalent, then their respectively induced metrics will obviously be equivalent.

Q: If we have a metric, do we automatically have a norm? (ie, is every metric space normed?)

A: No. In particular, a bounded metric can never be induced by a norm.

Suppose a bounded metric $d(x,y)$ is induced by a norm $\|\cdot\|$.

↳ bounded $\Rightarrow \exists r > 0$ st $d(x,y) < r \ \forall x, y$.

↳ induced $\Rightarrow d(x,y) = \|x - y\|$

Take $x \neq 0$ and $a := \frac{r+1}{\|x\|}$. Then as $d(x,0) = \|x - 0\| = \|x\|$,

$$r > d(ax, 0) = \|ax\| = |a| \cdot \|x\| = \frac{r+1}{\|x\|} \cdot \|x\| = r+1$$

a contradiction

Note: If a metric $d(x,y)$ is translation invariant $(d(x+z, y+z) = d(x,y))$ and scales. $(d(\alpha x, \alpha y) = |\alpha| \, d(x,y))$ then $d(x,0)$ defines a norm. Conversely, any metric induced by a norm has these properties.

"$\Leftarrow$" clear by definition of a norm and metric induced by a norm.

"$\Rightarrow$" We want to show $d(\cdot, 0)$ satisfies definition of norm. ie $\|\cdot\| := d(\cdot, 0)$ is a norm

" Non-negativity clear, Scaling holds by assumption.

Norm Δ-inequality: $$\underbrace{d(u+w, 0)}_{\|u+w\|} \underset{\substack{\uparrow \\ \text{metric } \Delta\text{-inequality}}}{\leq} d(u+w, w) + d(w,0) \underset{\substack{\uparrow \\ \text{translation} \\ \text{invariance}}}{=} \overbrace{d(u,0)}^{\|u\|} + \overbrace{d(w,0)}^{\|w\|}$$