Review

08.01.2024

• Fourier - Series

Def: For a (2π) periodic function f, its Fourier series is defined

$$f(x) = \frac{1}{2} a_0 + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$$

w/ $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \qquad n \in \mathbb{N}_0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \qquad n \in \mathbb{N}$

alternatively / equivalently $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}$ w/ $c_n = \frac{a_n - ib_n}{2}$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ins} ds$

Motivation? -> Solutions to heat eq, approximation w/ "nice" functions

① Convergence of FS: If f is a (2π) periodic function on $\mathbb{R}$, with bounded variation, then

i) the FS converges to $s(x) = \frac{1}{2}(f(x^+) + f(x^-))$

ii) if f cont at x, then FS converges to $f(x)$

iii) if f continuous on $\mathbb{R}$, then FS converges uniformly

Ex: Find FS of $f(x) = x$ on $[-\pi, \pi]$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \overset{1}{\cos(0)}\, dx = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx = 0$

symmetry + $x\cos(nx)$ odd function.

$\Longrightarrow$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx)\, dx \right]$

$$= \left[\frac{-2\pi \cos(\pi n)}{n} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx)\, dx \right] \frac{1}{\pi}$$

$u = nx$

$$= \left[\frac{-2\pi \cos(\pi n)}{n} + \frac{1}{n^2} \int_{-n\pi}^{n\pi} \cos(u)\, du \right]$$

$$= \left[\frac{-2\pi \cos(\pi n)}{n} + \frac{2 \sin(\pi n)}{n^2} \right] \frac{1}{\pi} = \frac{2 \overbrace{\sin(\pi n)}^{=0 \; \forall n \in \mathbb{N}}}{\pi n^2} + - \frac{2 \cos(\pi n)}{n}$$

$$= \frac{-2 (-1)^n}{n}$$

$$\therefore f(x) = 0 + \sum_{n \geq 1} 0 \cdot \cos(nx) + \frac{-2(-1)^n}{n} \sin(nx)$$

Hilbert Spaces + FS

Def! A complex vector space w/ a scaler product that induces a norm, which is also complete, is called a Hilbert space

Recall: With $\langle f | g \rangle = \frac{1}{2\pi} \int_{\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$, $L^2([-\pi, \pi], \mathbb{C})$ is complete and so a Hilbert space.

(T) All 2π periodic continuous fktns have a FS, where convergence is in norm/average, w/ $\| \cdot \| = \sqrt{\langle \cdot | \cdot \rangle}$

Why ~~complete~~ Hilbert? => Sequences / limits stay in your space.

=> Maximal orthonormal system is a basis

↳ only 0-ele. $\perp \forall b_i \in ONS$ ↳ $\langle b_i | b_j \rangle = \delta_{ij}$

(T) 7.6

Gauss-Approx.-Q => FS best approxim. to f wrt L^2, norm

Recall: $f, g \in L^1([-\pi,\pi],\mathbb{C})$ w/ $\hat{f} = \hat{g}$ $\forall n \in \mathbb{N}$, then $f = g$ in L^1

Ⓣ 3.12 F.S. of a (periodically continued) L^2 fctn is pointwise convergent almost everywhere

! Ⓣ 3.6 $\not\Rightarrow$ Ⓣ 3.12

Convergence in $L^2([-\pi,\pi]) \subset L^1([-\pi,\pi])$ does not imply pointwise convergence almost everywhere

See 27.11.2023 notes

Def: $f, g \in L^1([-\pi,\pi])$; $f * g\,(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\tau)\, g(t-\tau)\, d\tau$

Short pf w/ Fubini

Ⓣ Convolution T: $\widehat{(f*g)}(n) = (\hat{f})(n) \cdot (\hat{g})(n)$. $\forall n \in \mathbb{Z}$

Fourier Transformation

Issue of FS: only for periodic fctns. see 4.1 / pg 26

For non-periodic, consider it having "period" of ∞

Sketch: ~~Generalise~~ General period $[-a,a] \Rightarrow f(x) = \sum_{n\in\mathbb{Z}} \tilde{c}_n e^{inx\frac{\pi}{a}}$

w/ convergence in L^2

$$\tilde{c}_n = \frac{1}{2a}\int_{-a}^{a} e^{-inx\frac{\pi}{a}} f(x)\,dx$$

- $a \to \infty \Rightarrow \tilde{c}_n = \Phi(n\frac{\pi}{a})$ = almost cont. fctn.

with respect to integration on $[-a,a]$ by periodicity of f, $\frac{1}{2a} f(x)$ not too changed

- $\frac{1}{2a} f(x) \approx g(x) \in L^1(\mathbb{R},\mathbb{C})$; $\Phi(n\frac{\pi}{a}) \approx h(y)$ w/

$$h(y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ixy} g(x)\,dx, \quad g(x) = \frac{1}{\sqrt{2\pi}}\int e^{iyx} h(y)\,dy$$

Def: $f \in L^1(\mathbb{R}^n) \Rightarrow \hat{f}(t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-itx} f(x)\, d^n(x)$

T Plancherel's: $\exists!\ F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ st

i) $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \Rightarrow F(f) = \hat{f}$

ii) F is an isometry

iii) — " — Hilbert-space isomorphism, in particular unitary

iv) FT on $[-a,a]$ converges in L^2 to $\hat{f}$ see prev. notes

Discrete FT

Idea: Numerical calculation of a FT

Def: $(Df)(n) = \frac{1}{\sqrt{N}} \sum_{0}^{N-1} f_k\, e^{-2\pi i kn/N}$

$C_N = \{0, 1, \dots N-1\}$

$f = \{f_0, f_1, \dots f_{N-1}\}$

$f_i = f(t_i)$

↳ data points

Matrix rep: (efficient calculation)

$$Df = \begin{pmatrix} (Df)(0) \\ \vdots \\ (Df)(N-1) \end{pmatrix} = M_N \begin{pmatrix} f_0 \\ \vdots \\ f_{N-1} \end{pmatrix} \quad \text{w/ } \omega = e^{\frac{2\pi i}{N}}$$

$$M_N := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & \bar{\omega} & & \bar{\omega}^2 & & \bar{\omega}^{N-1} \\ 1 & \bar{\omega}^2 & & \bar{\omega}^4 & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & \bar{\omega}^{N-1} & & \bar{\omega}^{2(N-1)} & & \bar{\omega}^{(N-1)^2} \end{pmatrix}$$

Note: - M_N is unitary $f = \overline{M_N}(Df)$

- Also have inverse DFT: $f(n) = \frac{1}{\sqrt{N}} \sum_{0}^{N-1} (Df)(k)\, \omega^{kn}$

- Discrete convolution: $(f*g)(N) = \frac{1}{\sqrt{N}} \sum_{0}^{N-1} f(j)\, g(k-j)$ arguments mod N if necessary

- Convolution T: $D(f*g)(n) = (Df)(n) \cdot (Dg)(n)$

- Can be used to interpolate functions
  ↳ See section 5.1 for discussion.

- Can be efficiently calculated via Fast Four. Trans. (Sect. 5.2)

Ex: FT of

i) $f(t) = e^{-a|t|} \quad a>0$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t}\, dt\right] = \int_{-\infty}^{0} e^{at - i\omega t}\, dt + \int_{0}^{\infty} e^{-(at + i\omega t)}\, dt$$

$$= \left[\frac{1}{a-i\omega} + \frac{1}{a+i\omega}\right]\frac{1}{\sqrt{2\pi}} = \frac{2a}{\sqrt{2\pi}(a^2+\omega^2)}$$

ii) $f(t) = \begin{cases} 1 & 1-T \le t \le 1+T \\ 0 & t < 1-T \text{ or } t > 1+T \end{cases}$ Shifted rectang. impulse

$$\hat{f}(\omega) = \int_{1-T}^{1+T} e^{-i\omega t}\, dt = \frac{-1}{i\omega}\, e^{-i\omega(1+T)} - e^{-i\omega(1-T)}$$

$$= \frac{-e^{-i\omega}}{i\omega}\left(e^{-i\omega T} - e^{i\omega T}\right) = \frac{2\sin(\omega T)}{\omega} \cdot e^{-i\omega}$$