ODE's and Vandermonde

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<u>Recall:</u> An ODE of the form $ax'' + bx' + cx = 0$ is called a second order, homogeneous, linear diff. eq w/ constant coefficients.

General Sol? → Convert to system of 1st-order

$\Rightarrow$ rewrite as $u = x \rightarrow u' = \underbrace{x'}_{=v}$; $u'' = x''$

$v = x' \rightarrow v' = x'' = -\frac{b}{a}x' + -\frac{c}{a}x$

$= -\frac{b}{a}v + -\frac{c}{a}u$

Gives system

$u' = v \qquad = 0u + 1v$

$v' = -\frac{c}{a}u + -\frac{b}{a}v \qquad = -\frac{c}{a}u + -\frac{b}{a}v$

$\Leftrightarrow \vec{x}' = \begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{\vec{x}} = A\vec{x}$

↳ Find solutions to $\vec{x} = A\vec{x}$

→ matrix exponential

↳ $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ → convergence / calculating A^n

↳ easier if A diagonalisable w/ $A = BDB^{-1}$ → $e^{BDB^{-1}} = Be^{D}B^{-1}$

→ characteristic poly / ev + $\vec{ev}$

Euler's ideeLansatz → consider if $\vec{x}(t) = \vec{\eta} e^{rt}$ solu.

$$\Rightarrow \quad r\vec{\eta} e^{rt} = A\vec{\eta} e^{rt} \Leftrightarrow (A\vec{\eta} - r\vec{\eta}) e^{rt} = \vec{0}$$
$$\Leftrightarrow (A - rI)\vec{\eta} e^{rt} = \vec{0}$$
$$\Leftrightarrow (A - rI)\vec{\eta} = 0$$

$\therefore$ $r = \lambda$ e.v, η = e.$\vec{v}$

Recall: $(A-\lambda I)\vec{\eta} = 0 \Leftrightarrow \det(A-\lambda I) = 0$
$\Leftrightarrow$ roots of char. poly exist

However, by construction, the char. eq. of A is

$$a\lambda^2 + b\lambda + c = 0$$

Q: Form of sol? A: Depends on e.v of A
→ roots of char. poly

i) Real + Distinct → $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

ii) Repeated → $x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

ii) Complex → ?

→ Roots of char. poly → $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $4ac > b^2$ → complex sol. $\lambda_1 = \alpha + i\beta$
$\lambda_2 = \alpha - i\beta$

$\alpha = \dfrac{-b}{2a}$ $\beta = \dfrac{\sqrt{4ac - b^2}}{2a}$

Exercise: $\eta_1 = \begin{pmatrix} 1 \\ (\alpha + i\beta) \end{pmatrix}$ e.$\vec{v}$ for λ_1

$$\Rightarrow \vec{x}(t) = \begin{pmatrix} 1 \\ (\alpha+i\beta) \end{pmatrix} e^{(\alpha+i\beta)t} = \begin{pmatrix} 1 \\ \alpha+i\beta \end{pmatrix} e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$$

$$= e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ \alpha\cos(\beta t) - \beta\sin(\beta t) \end{pmatrix} + i\, e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \alpha\sin(\beta t) + \beta\cos(\beta t) \end{pmatrix}$$

How to get real sol? Two ways

i) Recall • if complex solution, conjugate also sol.

• Linear comb of sol also sol.

$\therefore$ $Re(z) = \frac{z+\bar{z}}{2}$; $Im(z) = \frac{z-\bar{z}}{2i}$ also sol.

ii) $\underline{\vec{x}'_{Re}} + i\,\underline{\vec{x}'_{Im}} = \vec{x}' = A\vec{x} = A\underline{\vec{x}_{Re}} + i\,A\underline{\vec{x}_{Im}}$

$\therefore$ Real, imag. parts also sol.

Get back to sol to $ax'' + bx' + cx = 0$?

$\rightarrow \vec{x} := \begin{pmatrix} u \\ v \end{pmatrix}$ w/ $u = x$; $\vec{x} = e^{\alpha t}\begin{pmatrix} \cos\beta t \\ \sim \end{pmatrix}$ or $e^{\alpha t}\begin{pmatrix} \sin\beta t \\ \sim \end{pmatrix}$

$\therefore$ $x = c_1 e^{\alpha t}\cos(\beta t)$ or $x = c_2 e^{\alpha t}\sin(\beta t)$ solutions

$\therefore$ general sol is $\cancel{e^{\alpha t}(\cos(\beta t) + \sin(\beta t))}$

$$e^{\alpha t}\left(c_1\cos(\beta t) + c_2\sin(\beta t)\right)$$

Ex: Solve $x'' - 4x' + 9x = 0$, $x(0) = 0$, $x'(0) = -8$

Characteristic poly: $\lambda^2 - 4\lambda + 9 = 0$; $\lambda_{1,2} = 2 \pm i\sqrt{5}$

general sol: $x(t) = e^{2t}(c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t))$

initial conditions: $x(0) = c_1 \cdot 1 + 0 = 0 \rightarrow c_1 = 0$

$$\therefore x(t) = c_2 e^{2t} \sin(\sqrt{5}t)$$

$x'(t) = 2c_2 e^{2t} \sin(\sqrt{5}t) + c_2\sqrt{5} e^{2t} \cos(\sqrt{5}t)$

$x'(0) = -8 = c_2\sqrt{5} \Rightarrow c_2 = \frac{-8}{\sqrt{5}}$

$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -9 & 4 \end{pmatrix} \vec{x}$

$$\therefore x(t) = \frac{-8}{\sqrt{5}} e^{2t} \sin(\sqrt{5}t)$$

$\Diamond$

Vandermonde Determinant

Def: A Vandermonde matrix is one of the form

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & \ddots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{pmatrix}$$

Note: The discrete Fourier matrix is a Vandermonde matrix w/ x_i the n-th roots of unity.

(T) $\det(V) = \prod_{0 \le i < j \le n} (x_i - x_j)$ $V = n \times n$

Pf] Many $\rightarrow$ see online

Most elementary $\rightarrow$ induction

Applications

· Polynomial interpretation

(T) Given n points $(x_i, y_i)_{i=1}^{n}$, $x_i \neq x_j$, $\exists$ a polynomial $P(X)$ of degree $\leq n-1$ $P(X) = c_0 + c_1 X + \ldots + c_{n-1} X^{n-1}$ st $P(x_i) = y_i$ $\forall i = 1, \ldots, n$

$\boxed{\text{Pf}}$ As $P(x_i) = y_i$,

$$c_0 + c_1 x_1 + \ldots + c_{n-1} x_1^{n-1} = y_1$$

$$c_0 + c_1 x_2 + \ldots + c_{n-1} x_2^{n-1} = y_2$$

$$\vdots$$

$$c_0 + c_1 x_n + \ldots + c_{n-1} x_n^{n-1} = y$$

$$\Leftrightarrow \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

As (x_i) distinct, $\det(V) \neq 0 \Rightarrow \exists!$ unique sol. $\square$

Solve for c_i (Cramer's Rule) $\to$ recover Lagrange interpolation

$$P(X) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(X - x_j)}{x_i - x_j}$$