Freezing Method with Comsol Multiphysics

Nagumo Equation: Traveling 1-front

Denny Otten[1]
Department of Mathematics
Bielefeld University
33501 Bielefeld
Germany

Date: 26. Oktober 2015

1. Freezing Traveling Front of Nagumo Equation

Consider the **Nagumo equation**

$$u_t = u_{xx} + u(1-u)(u-b), \quad x \in \mathbb{R},\, t \geqslant 0$$

for some $b \in (0,1)$, where $u = u(x,t) \in \mathbb{R}$.

a) We first solve the **nonfrozen Nagumo-equation**

$$\begin{aligned} u_t &= u_{xx} + u(1-u)(u-b) && ,\, x \in \Omega,\, t \in (0,T_1], \\ u(0) &= u_0 && ,\, x \in \bar{\Omega},\, t = 0, \\ \partial u_x &= 0 && ,\, x \in \partial\Omega,\, t \in [0,T_1], \end{aligned} \tag{1.1}$$

on the spatial domain $\Omega = [-50, 50]$ for end time $T_1 = 150$, initial data $u_0(x) = \frac{\tanh(x)+1}{2}$ and parameter $b = \frac{1}{4}$. For the space discretization we use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-3}$ and absolute tolerance atol $= 10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

[1]e-mail: dotten@math.uni-bielefeld.de, phone: +49 (0)521 106 4784,
fax: +49 (0)521 106 6498, homepage: `http://www.math.uni-bielefeld.de/~dotten/`.

b) We then solve the **frozen Nagumo-equation**

$$
\begin{aligned}
v_t &= v_{\xi\xi} + \mu v_\xi + v(1-v)(v-b) &&, \xi \in \Omega,\ t \in (0, T_2],\\
v(0) &= v_0 &&, \xi \in \bar{\Omega},\ t = 0,\\
v_\xi &= 0 &&, \xi \in \partial\Omega,\ t \in [0, T_2],\\
0 &= (v - \hat{v}, \hat{v}_x)_{L^2(\Omega,\mathbb{R})} &&, t \in [0, T_2],\\
\gamma_t &= \mu &&, t \in (0, T_2],\\
\gamma(0) &= 0 &&, t = 0
\end{aligned} \tag{1.2}
$$

on the spatial domain $\Omega = [-50, 50]$ for end time $T_2 = 150$, initial data $v_0(\xi) = u_0(\xi)$, reference function $\hat{v}(\xi) = u_0(\xi)$ and parameter $b = \frac{1}{4}$. For the space discretization we use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-3}$ and absolute tolerance atol $= 10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method (automatic (Newton)).

c) Finally, we solve the **eigenvalue problem** for the linearization of the Nagumo equation

$$
\begin{aligned}
\lambda w &= w_{\xi\xi} + \mu_\star w_\xi + f'(v_\star)w &&, \xi \in \Omega,\\
w_\xi &= 0 &&, \xi \in \partial\Omega
\end{aligned} \tag{1.3}
$$

on the spatial domain $\Omega = [-50, 50]$, where

$$f(v) = v(1-v)(v-b), \quad f'(v) = (1-v)(v-b) - v(v-b) + v(1-v).$$

For $v_\star$ and $\mu_\star$ we use the solutions v and μ of (1.2) at the end time $T_2 = 150$. We determine neigs $= 400$ eigenvalues λ and the correspondig eigenfunctions w. The eigenvalues are choosen such that they are closest in absolute value around the shift $-b$.

2. Model Wizard

Start Comsol Multiphysics.
To start Comsol Multiphysics 5.1 open the **Terminal** and enter

- **comsol**

If you are using the classroom licence enter

- **comsol -ckl**

Model Wizard.
Space dimension

- In the **New** window, click **Model Wizard**.
- In the **Model Wizard** window, click **1D** in the **Select Space Dimension** menu.

Equation for the u-component

- In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Coefficient Form PDE (c)**.
- Click **Add**.
- Next, locate the **Dependent Variables** section.
- In the **Field name** text field, type **u**.
- In the **Dependent variables** text field, type also **u**.

Equation for the v-component

- In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Coefficient Form PDE (c)**.
- Click **Add**.
- Next, locate the **Dependent Variables** section.
- In the **Field name** text field, type **v**.
- In the **Dependent variables** text field, type also **v**.

Equation for the μ-component

- In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Lower Dimensions>Weak Form Boundary PDE (wb)**.
- Click **Add**.
- Next, locate the **Dependent Variables** section.
- In the **Field name** text field, type **mu**.
- In the **Dependent variables** text field, type also **mu1**.

Equation for the γ-component

- In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Lower Dimensions>Weak Form Boundary PDE (wb)**.
- Click **Add**.
- Next, locate the **Dependent Variables** section.
- In the **Field name** text field, type **g**.
- In the **Dependent variables** text field, type also **g1**.
- Click **Study**.

Study settings

- In the **Select Study** tree, select **Preset Studies for Selected Physics Interfaces>Time Dependent**.
- Click **Done**.

Some Advanced Settings.
Hint: In the **Model Builder** window you should click on the **Show** icon and enable everything that is possible from the menu: **Equation Sections** (**Equation View**, **Override and Contribution**, **Discretization**, **Stabilization**, **Advanced Physics Options**, **Advanced Study Options** and **Advanced Results Options**). Done this, click **Expand All** icon.

3. Geometry

- In the **Model Builder** tree, expand the **Component 1 (comp1)** node, right-click **Geometry** and select **Interval**.
- In the **Settings** window for Interval, locate the **Interval** section.
- In the **Left endpoint** text field, type **-50**.
- In the **Right endpoint** text field, type **50**.
- In the **Model Builder** tree, right-click on the **Component 1 *(comp1)*→Geometry 1** node and select **Build all**. (Alternatively, press the short cut **F8**.)

4. Partial differential equation for the u-component (Nonfrozen PDE)

General Settings.

- Click on **Component 1** ***(comp1)***→**Coefficient Form PDE** ***(c)***.
- Locate the **Settings** window for Coefficient Form PDE.
- In the **Label** text field, type **Nonfrozen Equation**.
- In the **Name** text field, type **PDE1**.
- In the **Discretization** section choose
  - **Shape function type**: **Lagrange**,
  - **Element order**: **Linear**.

Partial differential equation. We define the PDE for the u-component:

- Switch to **Component 1** ***(comp1)***→**Nonfrozen Equation** ***(PDE1)***→**Coefficient Form PDE 1**

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f$$

  with $\nabla = \frac{\partial}{\partial x}$, and enter the following values
  - **Mass Coefficient** e_a: **0**,
  - **Damping or Mass Coefficient** d_a: **1**,
  - **Diffusion coefficient** c: **1**,
  - **Conservative Flux Convection Coefficient** α: **0**,
  - **Conservative Flux Source** γ: **0**,
  - **Convection Coefficient** β: **0**,
  - **Absorption Coefficient** a: **0**,
  - **Source Term** f: **fu**.

The quantity **fu** will be defined later in Section 8.

Boundary Conditions. Since the PDE of the u-component requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. Hint: By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

Initial Values. We define the initial value $u(\cdot, 0) = u_0$ for the partial differential equation:

- Click on **Component 1** ***(comp1)***→**Nonfrozen Equation** ***(PDE1)***→**Initial Values 1**.
- In the **Initial Values** section enter
  - **Initial value for u**: **u0**,
  - **Initial time derivative of u**: **0**.

The quantity **u0** will be defined later in Section 8. This completes the implementation of the initial boundary value problem for the u-component.

5. Partial differential equation for the v-component (Frozen PDE)

General Settings.

- Click on **Component 1** ***(comp1)***→**Coefficient Form PDE 2** ***(c2)***.
- Locate the **Settings** window for Coefficient Form PDE.

- In the **Label** text field, type **Frozen Equation**.
- In the **Name** text field, type **PDE2**.
- In the **Discretization** section choose
  - **Shape function type**: **Lagrange**,
  - **Element order**: **Linear**.

Partial differential equation. We define the PDE for the v-component:

- Switch to **Component 1** ***(comp1)***→**Frozen Equation** ***(PDE2)***→**Coefficient Form PDE 1**

$$e_a \frac{\partial^2 v}{\partial t^2} + d_a \frac{\partial v}{\partial t} + \nabla \cdot (-c\nabla v - \alpha v + \gamma) + \beta \cdot \nabla v + av = f$$

  with $\nabla = \frac{\partial}{\partial x}$, and enter the following values
  - **Mass Coefficient** e_a: **0**,
  - **Damping or Mass Coefficient** d_a: **1**,
  - **Diffusion coefficient** c: **1**,
  - **Conservative Flux Convection Coefficient** α: **0**,
  - **Conservative Flux Source** γ: **0**,
  - **Convection Coefficient** β: **0**,
  - **Absorption Coefficient** a: **0**,
  - **Source Term** f: **Fv**.

The quantity **Fv** will be defined later in Section 8.

Boundary Conditions. Since the PDE of the v-component requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. Hint: By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

Initial Values. We define the initial value $u(\cdot, 0) = u_0$ for the partial differential equation:

- Click on **Component 1** ***(comp1)***→**Frozen Equation** ***(PDE2)***→**Initial Values 1**.
- In the **Initial Values** section enter
  - **Initial value for u**: **v0**,
  - **Initial time derivative of u**: **0**.

The quantity **v0** will be defined later in Section 8. This completes the implementation of the initial boundary value problem for the v-component.

6. Algebraic constraint for the μ-component (Velocity)

In the following we define the phase condition for μ_1:

- Click on **Component 1** ***(comp1)***→**Weak Form Boundary PDE** ***(wb)***
- Locate the **Settings** window for Weak Form Boundary PDE.
- In the **Label** text field, type **Phase Condition**.
- In the **Name** text field, type **AC1** for **a**lgebraic **c**onstraint.
- In the **Boundary Selection** section choose
  - **Selection**: **Manual**.

  Now, click on the boundary point **2** and click on the minus sign **-** to remove the point from the selection list.
- In the **Discretization** section choose

- **Shape function type**: **Lagrange**,
- **Element order**: **Linear**.

Weak Form PDE. We define the phase condition:

- Click on **Component 1** ***(comp1)***→**Phase Condition** ***(AC1)***→**Weak Form PDE 1**.
- In the **Weak Expression** section enter
  - **weak**: $test(mu1) * pc1$

The quantity **pc1** will be defined later in Section 8.

Initial Values. Finally, we define the initial value:

- Click on **Component 1** ***(comp1)***→**Phase Condition** ***(AC1)***→**Initial Values 1**.
- In the **Initial Values** section enter
  - **Initial value for mu1**: **0**,
  - **Initial time derivative of mu1**: **0**.

This completes the implementation of the phase condition for μ.

7. Ordinary differential equation for the γ-component (Position)

In the following we implement the reconstruction equation for the position γ:

- Click on **Component 1** ***(comp1)***→**Weak Form Boundary PDE 2** ***(wb2)***
- Locate the **Settings** window for Weak Form Boundary PDE.
- In the **Label** text field, type **Reconstruction Equation**.
- In the **Name** text field, type **ODE1**.
- In the **Boundary Selection** section choose
  - **Selection**: **Manual**.

  Now, click on the boundary point **2** and click on the minus sign **-** to remove the point from the selection list.
- In the **Discretization** section choose
  - **Shape function type**: **Lagrange**,
  - **Element order**: **Linear**.
- In the **Model Builder** tree, right-click **Component 1** ***(comp1)***→**Reconstruction Equation** ***(ODE1)*** and select **Weak Form PDE**.

Weak Form PDE 1. We define the ordinary differential equation for γ in a weak form:

- Click on **Component 1** ***(comp1)***→**Reconstruction Equation** ***(ODE1)***→**Weak Form PDE 1**.
- In the **Weak Expression** section enter
  - **weak**: $test(g1) * mu1$

Initial Values. Finally, we define the initial value:

- Click on **Component 1** ***(comp1)***→**Reconstruction Equation** ***(ODE1)***→**Initial Values 1**.
- In the **Initial Values** section enter
  - **Initial value for g1**: **0**,
  - **Initial time derivative of g1**: **0**.

Weak Form PDE 2.

- Click on **Component 1** ***(comp1)***→**Reconstruction Equation** ***(ODE1)***→**Weak Form PDE 2**.
- In the **Boundary Selection** section switch to **Selection All boundaries**, then back to **Selection Manual**, since otherwise there is no boundary point contained in the selection list.
- In the **Weak Expression** section enter
  - **weak**: $-test(g1) * g1\_time$

This completes the implementation of the reconstruction equation for γ.

8. Parameters, Variables and Integration Coupling Variables

Parameters. We first define the parameters and constants arising in our model as 'global parameters':

- In the **Model Builder** tree, right-click on the **Global Definitions** node and select **Parameters**. (Alternatively: On the **Model** toolbar, click **Parameters**.)
- In the **Settings** window for Parameters, locate the **Parameters** section.
- In the table add the following entry:

Name	Expression	Value	Description
b	1/4	0.25	constant of Nagumo equation

Variables 1. We now define all functions which appear in our model as 'local variables'.

- In the **Model Builder** tree, right-click on the **Component 1** ***(comp1)***→**Definitions** node and select **Variables**.
- In the **Settings** window for Variables, locate the **Variables** section.
- In the table add the following entries:

Name	Expression	Unit	Description
fu	u*(1-u)*(u-b)		
u0	(tanh(x)+1)/2		
fv	v*(1-v)*(v-b)		
Fv	mu1cpl*vx+fv		
v0	u0		
vh	v0		
pc1_fix	d(vh,x)*(v-vh)		

Integration Coupling Variables 1. We next define an integration operator, that integrates a function over the whole spatial domain.

- In the **Model Builder** tree, right-click on the **Component 1** ***(comp1)***→**Definitions** node and select **Component Couplings>Integration**.
- In the **Settings** window for Integration, locate the **Source Selection** section and choose
  - **Geometric entity level**: **Domain**,
  - **Selection**: **Manual**.
- Next, locate the **Advanced** section and choose
  - **Method**: **Integration**,
  - **Integration order**: **2**,
  - **Frame**: **Spatial (x,y,z)**.

Integration Coupling Variables 2. We next define an integration operator, that integrates a function over the boundary of the spatial domain.

- In the **Model Builder** tree, right-click on the **Component 1** ***(comp1)***→**Definitions** node and select **Component Couplings**>**Integration**.
- In the **Settings** window for Integration, locate the **Source Selection** section and choose
  - **Geometric entity level**: **Boundary**,
  - **Selection**: **Manual**.

  Now, click on the boundary point **2** and click on the minus sign **-** to remove the point from the selection list.
- Next, locate the **Advanced** section and choose
  - **Method**: **Integration**,
  - **Integration order**: **1**,
  - **Frame**: **Spatial (x,y,z)**.

Variables 2. We next define the functions, that are integrated over the whole spatial domain:

- In the **Model Builder** tree, right-click on the **Component 1** ***(comp1)***→**Definitions** node and select **Variables**.
- In the **Settings** window for Variables, locate the **Geometric Entity Selection** section and choose
  - **Geometric entity level**: **Domain**,
  - **Selection**: **Manual**.
- Next, locate the **Variables** section and add the following entries into the table:

Name	Expression	Unit	Description
intcpl_source_pc1	pc1_fix		
intcpl_source_sqr_vt	vt^2	$1/s^2$	

Variables 3. We next define the functions, that are integrated over the boundary of the spatial domain:

- In the **Model Builder** tree, right-click on the **Component 1** ***(comp1)***→**Definitions** node and select **Variables**.
- In the **Settings** window for Variables, locate the **Geometric Entity Selection** section and choose
  - **Geometric entity level**: **Boundary**,
  - **Selection**: **Manual**.

  Now, click on the boundary point **2** and click on the minus sign **-** to remove the point from the selection list.
- Next, locate the **Variables** section and add the following entries into the table:

Name	Expression	Unit	Description
intcpl_source_mu1cpl	mu1		

Variables 4. Finally, we define variables that contain the results of the integration

- In the **Model Builder** tree, right-click on the **Global Definitions** node and select **Variables**.
- In the **Settings** window for Variables, locate the **Variables** section.
- In the table add the following entries:

Name	Expression	Unit	Description
pc1	comp1.intop1(intcpl_source_pc1)		
sqr_vt	comp1.intop1(intcpl_source_sqr_vt)	m/s^2	
mu1cpl	comp1.intop2(intcpl_source_mu1cpl)		

9. Mesh

- In the **Model Builder** tree, click on **Component 1 *(comp1)*→Mesh**.
- In the **Settings** window for Mesh, locate the **Mesh Settings** section.
- Set the **Sequence type** on **User-controlled mesh**.
- In the **Model Builder** tree, switch to **Component 1 *(comp1)*→Mesh→Size**.
- In the **Settings** window for Size, locate the **Element Size Parameters** section.
- In the **Maximum element size** text field, type **0.1**.
- In the **Model Builder** tree, right-click on **Component 1 *(comp1)*→Mesh** and select **Build All**.

10. Studies and Computation

Study 1. Study 1

- Click on **Study 1**.
- Locate the **Settings** window for Study.
- In the **Label** text field, type **Study 1: Nonfrozen Equation**.

Step 1

- Click on **Study 1: Nonfrozen Equation→Step 1: Time Dependent**.
- Locate the **Settings** window for Time Dependent.
- In the **Study Settings** section enter
  - **Time unit**: s,
  - **Times**: **range(0,0.1,150)**,
  - **Relative tolerance**: **0.001**.

  The last input requires to enable the corresponding checkbox.
- In the **Physics and Variables Selection** section disable the checkboxes for **Frozen Equation (PDE2)**, **Phase Condition (AC1)** and **Reconstruction Equation (ODE1)**.

Solver Configurations

- Right-click on **Study 1: Nonfrozen Equation→Solver Configurations** and select **Show Default Solver**.
- Click on **Study 1: Nonfrozen Equation→Solver Configurations→Solution 1→Time-Dependent Solver 1**.
- Locate the **Settings** window for Time Dependent Solver.
- In the **Absolute Tolerance** section enter
  - **Global method**: **Unscaled**,
  - **Tolerance**: **0.0001**.
- In the **Time Stepping** section enter
  - **Method**: **BDF**,
  - **Steps taken by solver**: **intermediate**,
  - **Maximum BDF order**: **2**.

- Click on **Study 1: Nonfrozen Equation**→**Solver Configurations**→**Solution 1**→**Time-Dependent Solver 1**→**Fully Coupled 1**.
- Locate the **Settings** window for Fully Coupled.
- In the **Method and Termination** section, choose
  - **Nonlinear Method**: **Automatic (Newton)**,

Solution Store

- Right-click on **Study 1: Nonfrozen Equation**→**Solver Configurations**→**Solution 1** and select **Other**>**Solution Store** from the list.
- Click on **Study 1: Nonfrozen Equation**→**Solver Configurations**→**Solution 1**→**Solution Store 1**.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Nonfrozen Solution**.

Study 2.

- Right-click on **unknown.mph** ***(root)*** and select **Add Study**.
- In the **Add Study** window, select **Time Dependent** and confirm by click on **Add Study**.

Study 2

- Click on **Study 2**.
- Locate the **Settings** window for Study.
- In the **Label** text field, type **Study 2: Frozen Equation**.

Study 2 Step 1

- Click on **Study 2: Frozen Equation**→**Step 1: Time Dependent**.
- Locate the **Settings** window for Time Dependent.
- In the **Study Settings** section enter
  - **Time unit**: **s**,
  - **Times**: **range(0,0.1,150)**,
  - **Relative tolerance**: **0.001**.

  The last input requires to enable the corresponding checkbox.
- In the **Physics and Variables Selection** section disable the checkbox for **Nonfrozen Equation (PDE1)**.

Solver Configurations

- Right-click on **Study 2: Frozen Equation**→**Solver Configurations** and select **Show Default Solver**.
- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Time-Dependent Solver 1**.
- Locate the **Settings** window for Time Dependent Solver.
- In the **Absolute Tolerance** section enter
  - **Global method**: **Unscaled**,
  - **Tolerance**: **0.0001**.
- In the **Time Stepping** section enter
  - **Method**: **BDF**,
  - **Steps taken by solver**: **intermediate**,
  - **Maximum BDF order**: **2**.

- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Time-Dependent Solver 1**→**Fully Coupled 1**.
- Locate the **Settings** window for Fully Coupled.
- In the **Method and Termination** section, choose
  - **Nonlinear Method**: **Automatic (Newton)**,

Solution Store

- Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Other**>**Solution Store** from the list.
- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Solution Store 1**.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Frozen Solution**.

Study 2 Step 2 First, we generate the solver and study step.

- Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Compile Equations**.
- Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Dependent Variables**.
- Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Solvers**>**Eigenvalue Solver**.
- Right-click on **Study 2: Frozen Equation** and select **Study Steps**>**Eigenfrequency**>**Eigenvalue**.
- Click on **Study 2: Frozen Equation**→**Step 2: Eigenvalue**.
- Locate the **Settings** window for Eigenvalue.
- In the **Study Settings** section enter
  - **Eigenvalue search method**: **Manual**,
  - **Desired number of eigenvalues**: **400**,
  - **Search for eigenvalues around**: **-b**,
  - **Eigenvalue search method around shift**: **Closest in absolute value**.
- In the **Physics and Variables Selection** section enable only the checkbox for **Frozen Equation (PDE2)**.
- In the **Values of dependent Variables** section enable the checkbox **Values of variables not solved for** and enter
  - **Method**: **Solution**,
  - **Study**: **Study 2: Frozen equation, Time Dependent**,
  - **Solution**: **Solution 3**,
  - **Use**: **Frozen Solution**,
  - **Selection**: **Last**.

Solver Configurations

- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Compile Equations**.
- Locate the **Settings** window for Compile Equations.
- In the **Study and Step** section enter
  - **Use study**: **Study 2: Frozen Equation**,
  - **Use study step**: **Step 2: Eigenvalue**.

- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Dependent Variables 2**.
- Locate the **Settings** window for Dependent Variables.
- In the **General** section enter
  - **Defined by study step**: **Step 2: Eigenvalue**.
- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Eigenvalue Solver 1**.
- Locate the **Settings** window for Dependent Variables.
- In the **General** section enter
  - **Defined by study step**: **Step 2: Eigenvalue**,
  - **Relative Tolerance**: **1E-7**.
- In the **Values of Linearization Point** section enter
  - **Prescribed by**: **Solution**,
  - **Solution**: **Solution 3**,
  - **Use**: **Frozen Solution**,
  - **Selection**: **Last**.

  Finally, activate the checkbox **Store linearization point and deviation in output**.

Solution Store

- Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Other**>**Solution Store** from the list.
- Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Solution Store 1**.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Eigenvalues and Eigenfunctions**.

10.1. Computation.

- Right-click on **Study 1: Nonfrozen Equation** and select **compute** from the list.
- Right-click on **Study 2: Frozen Equation** and select **compute** from the list.

11. Postprocessing and graphical output

In this section we generate 10 Plot groups for visualizing our results.

11.1. Results for nonfrozen equation.

Plot Group 1: Traveling Front, View 1

- Click on **Results**→**1D Plot Group 1**. Hint: If **1D Plot Group 1** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 1**.
- In the **Data** section select **Data set** **Study 1: Nonfrozen Equation/Nonfrozen Solution**, **Time selection** **Interpolated** and **Times (s)** **0 40 80 120**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **u(x,t)**.

- Click on **Results→Traveling Front, View 1→Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results→Traveling Front, View 1** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set From parent**.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression u**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
- In the **Coloring and Style** section select **Line Solid**, **Color Cycle** and **Width 2** in the **Line style** subsection.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends Manual** and enter the legends **t=0**, **t=40**, **t=80** and **t=120**.

Plot Group 2: Traveling Front, View 2

- Click on **Results→1D Plot Group 2**. Hint: If **1D Plot Group 2** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 2**.
- In the **Data** section select **Data set Study 1: Nonfrozen Equation/Nonfrozen Solution** and **Time selection All**.
- In the **Title** section select **Title type None**.
- In the **Plot Settings** section select **x-axis label x** and **y-axis label t**.
- Click on **Results→Traveling Front, View 2→Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results→Traveling Front, View 2** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set From parent**.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression t**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
- Right-click on **Results→Traveling Front, View 2→Line Graph 1** and select **Color Expression**.
- Click on **Results→Traveling Front, View 2→Line Graph 1→Color Expression 1**.
- Locate the **Settings** window for Color Expression.
- In the **Expression** section select **Expression u**.

11.2. Results for frozen equation.

Plot Group 3: Profile, View 1

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Profile, View 1**.
- In the **Data** section select **Data set Study 2: Frozen Equation/Frozen Solution** and **Time selection Last**.

- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **v(x,150)**.
- Right-click on **Results→Profile, View 1** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All domains**.
- In the **y-Axis Data** section select **Expression** **v**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **x**.
- In the **Coloring and Style** section select **Line** **Solid**, **Color** **Blue** and **Width** **2** in the **Line style** subsection.

Plot Group 4: Profile, View 2

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Profile, View 2**.
- In the **Data** section select **Data set** **Study 1: Frozen Equation/Frozen Solution** and **Time selection** **All**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **t**.
- Right-click on **Results→Profile, View 2** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All domains**.
- In the **y-Axis Data** section select **Expression** **t**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **x**.
- Right-click on **Results→Profile, View 2→Line Graph 1** and select **Color Expression**.
- Click on **Results→Profile, View 2→Line Graph 1→Color Expression 1**.
- Locate the **Settings** window for Color Expression.
- In the **Expression** section select **Expression** **v**.

Plot Group 5: Velocities

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Velocities**.
- In the **Data** section select **Data set** **Study 2: Frozen Equation/Frozen Solution** and **Time selection** **All**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **t** and **y-axis label** **mu(t)**.
- Right-click on **Results→Velocities** and select **Point Graph** from the list.
- Locate the **Settings** window for Point Graph.
- In the **Data** section select **Data set** **From parent**.

- In the **Selection** section select **Selection All boundaries**. Now, select boundary point **2** and click on the minus sign **-** to remove the point from selection.
- In the **y-Axis Data** section select **Expression mu1**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression t**.
- In the **Coloring and Style** section select **Line Solid**, **Color Blue** and **Width 2** in the **Line style** subsection.

Plot Group 6: Positions

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Positions**.
- In the **Data** section select **Data set Study 2: Frozen Equation/Frozen Solution** and **Time selection All**.
- In the **Title** section select **Title type None**.
- In the **Plot Settings** section select **x-axis label t** and **y-axis label gamma(t)**.
- Right-click on **Results→Positions** and select **Point Graph** from the list.
- Locate the **Settings** window for Point Graph.
- In the **Data** section select **Data set From parent**.
- In the **Selection** section select **Selection All boundaries**. Now, select boundary point **2** and click on the minus sign **-** to remove the point from selection.
- In the **y-Axis Data** section select **Expression g1**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression t**.
- In the **Coloring and Style** section select **Line Solid**, **Color Blue** and **Width 2** in the **Line style** subsection.

Plot Group 7: Reference function

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Reference Function**.
- In the **Data** section select **Data set Study 2: Frozen Equation/Frozen Solution** and **Time selection First**.
- In the **Title** section select **Title type None**.
- In the **Plot Settings** section select **x-axis label x** and **y-axis label vh(x)**.
- Right-click on **Results→Reference Function** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set From parent**.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression vh**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
- In the **Coloring and Style** section select **Line Solid**, **Color Blue** and **Width 2** in the **Line style** subsection.

Plot Group 8: $\|v_t\|_{L^2}$ and $\mu_{1,t}$

- Right-click on **Results** and select **1D Plot Group** from the list.

- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **L2-Norm of vt and abs of mu1t**.
- In the **Data** section select **Data set** **Study 2: Frozen Equation/Frozen Solution** and **Time selection** **All**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **t**. The **y-axis label** remains disabled.
- In the **Axis** section enable the checkbox for **y-axis log scale**.
- Right-click on **Results→L2-Norm of vt and abs of mu1t** and select **Point Graph** from the list.
- Locate the **Settings** window for Point Graph.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All boundaries**. Now, select boundary point **2** and click on the minus sign **-** to remove the point from selection.
- In the **y-Axis Data** section select **Expression** **sqrt(sqr_vt)**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **t**.
- In the **Coloring and Style** section select **Line** **Solid**, **Color** **Blue** and **Width** **2**.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends** **Manual** and enter the legend **||v_t||_L2̂**.
- Right-click on **Results→L2-Norm of vt and abs of mu1t** and select **Point Graph** from the list.
- Locate the **Settings** window for Point Graph.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All boundaries**. Now, select boundary point **2** and click on the minus sign **-** to remove the point from selection.
- In the **y-Axis Data** section select **Expression** **abs(mu1t)**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **t**.
- In the **Coloring and Style** section select **Line** **Dashed**, **Color** **Red** and **Width** **2**.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends** **Manual** and enter the legend **|mu1_t|** in the **Line style** subsection.

11.3. Results for eigenvalue study.

Plot Group 9: Eigenvalues and Spectrum

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Eigenvalues and Spectrum**.
- In the **Data** section select **Data set** **Study 2: Frozen Equation/Eigenvalues and Eigenfunctions** and **Eigenvalue selection** **All**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **Re lambda** and **y-axis label** **Im lambda**.
- In the **Axis** section enable the **Manual axis limits** checkbox and select **x minimum** **-6**, **x maximum** **2**, **y minimum** **-1** and **y maximum** **1**.

- Right-click on **Results→Eigenvalues and Spectrum** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set Study 2: Frozen Equation/Eigenvalues and Eigenfunctions**, **Eigenvalue selection From list** and select all eigenvalues except the zero eigenvalue.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression imag(lambda)**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression -real(lambda)**.
- In the **Coloring and Style** section select **Line Solid**, **Color Red** and **Width 1** in the **Line style** subsection, and **Marker Circle** as well es **Positioning in data points** in the **Line markers** subsection.
- Right-click on **Results→Eigenvalues and Spectrum** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set Study 2: Frozen Equation/Eigenvalues and Eigenfunctions**, **Eigenvalue selection From list** and select the zero eigenvalue.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression imag(lambda)**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression -real(lambda)**.
- In the **Coloring and Style** section select **Line None**, **Color Blue** and **Width 1** in the **Line style** subsection, and **Marker Circle** as well es **Positioning in data points** in the **Line markers** subsection.

Plot Group 10: Eigenfunctions

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Eigenfunctions**.
- In the **Data** section select **Data set Study 2: Frozen Equation/Eigenvalues and Eigenfunctions**, **Eigenvalue selection From list** and select all eigenvalues except the zero eigenvalue.
- In the **Title** section select **Title type None**.
- In the **Plot Settings** section select **x-axis label x** and **y-axis label v(x)**.
- Right-click on **Results→Eigenfunctions** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph.
- In the **Data** section select **Data set From parent**.
- In the **Selection** section select **Selection All domains**.
- In the **y-Axis Data** section select **Expression v**.
- In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
- In the **Coloring and Style** section select **Line Solid**, **Color Cycle** and **Width 2** in the **Line style** subsection.