1. Freezing Traveling Front of Nagumo Equation

Consider the Nagumo equation

\[ u_t = u_{xx} + u(1-u)(u-b), \quad x \in \mathbb{R}, \ t \geq 0 \]

for some \( b \in (0,1) \), where \( u = u(x,t) \in \mathbb{R} \).

a) We first solve the nonfrozen Nagumo-equation

\[
\begin{align*}
    u_t &= u_{xx} + u(1-u)(u-b), \quad x \in \Omega, \ t \in (0,T_1], \\
    u(0) &= u_0, \quad x \in \bar{\Omega}, \ t = 0, \\
    \partial u_x &= 0, \quad x \in \partial \Omega, \ t \in [0,T_1],
\end{align*}
\]

on the spatial domain \( \Omega = [-50, 50] \) for end time \( T_1 = 150 \), initial data \( u_0(x) = \frac{\tanh(x) + 1}{2} \) and parameter \( b = \frac{1}{4} \). For the space discretization we use linear Lagrange elements with maximal element size \( \Delta x = 0.1 \). For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize \( \Delta t = 0.1 \), relative tolerance \( rtol = 10^{-3} \) and absolute tolerance \( atol = 10^{-4} \) with global method set to be unscaled. The nonlinear equations should be solved by the Newton method, i.e. automatic (Newton).
b) We then solve the frozen Nagumo-equation
\[ v_t = v_{\xi\xi} + \nu v_{\xi} + v(1-v)(v-b), \quad \xi \in \Omega, \ t \in (0, T_2], \]
\[ v(0) = v_0, \quad \xi \in \Omega, \ t = 0, \]
\[ v_{\xi} = 0, \quad \xi \in \partial\Omega, \ t \in [0, T_2], \]
\[ 0 = (v - \hat{v}, \hat{w})_{L^2(\Omega, \mathbb{R})}, \quad t \in [0, T_2], \]
\[ \gamma_{\xi} = \mu, \quad t \in (0, T_2], \]
\[ \gamma(0) = 0, \quad t = 0. \]

(1.2)

on the spatial domain \( \Omega = [-50, 50] \) for end time \( T_2 = 150 \), initial data \( v_0(\xi) = u_0(\xi) \), reference function \( \hat{v}(\xi) = u_0(\xi) \) and parameter \( b = \frac{1}{4} \). For the space discretization we use linear Lagrange elements with maximal element size \( \Delta x = 0.1 \). For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize \( \Delta t = 0.1 \), relative tolerance \( \text{rtol} = 10^{-3} \) and absolute tolerance \( \text{atol} = 10^{-4} \) with global method set to be unscaled. The nonlinear equations should be solved by the Newton method (automatic (Newton)).

c) Finally, we solve the eigenvalue problem for the linearization of the Nagumo equation
\[ \lambda w = w_{\xi\xi} + \mu w_{\xi} + f'(v_*) w, \quad \xi \in \Omega, \]
\[ w_{\xi} = 0, \quad \xi \in \partial\Omega \]

(1.3)

on the spatial domain \( \Omega = [-50, 50] \), where
\[ f(v) = v(1-v)(v-b), \quad f'(v) = (1-v)(v-b) - v(v-b) + v(1-v). \]

For \( v_* \) and \( \mu_* \) we use the solutions \( v \) and \( \mu \) of (1.2) at the end time \( T_2 = 150 \). We determine \( \text{neigs} = 400 \) eigenvalues \( \lambda \) and the correspondig eigenfunctions \( w \). The eigenvalues are choosen such that they are closest in absolute value around the shift \(-b\).

2. Model Wizard

Start Comsol Multiphysics.

To start Comsol Multiphysics 5.1 open the Terminal and enter

- \texttt{comsol}
- \texttt{comsol -ckl}

Model Wizard.

Space dimension

- In the New window, click Model Wizard.
- In the Model Wizard window, click 1D in the Select Space Dimension menu.

Equation for the \( u \)-component

- In the Select Physics tree, select Mathematics\( > \)PDE Interfaces\( > \)Coefficient Form PDE (c).
- Click Add.
- Next, locate the Dependent Variables section.
- In the Field name text field, type \( u \).

Equation for the \( v \)-component
In the Select Physics tree, select Mathematics>PDE Interfaces>Coefficient Form PDE (c).

• Click Add.

• Next, locate the Dependent Variables section.

• In the Field name text field, type v.

• In the Dependent variables text field, type also v.

Equation for the μ-component

• In the Select Physics tree, select Mathematics>PDE Interfaces>Lower Dimensions>Weak Form Boundary PDE (wb).

• Click Add.

• Next, locate the Dependent Variables section.

• In the Field name text field, type mu.

• In the Dependent variables text field, type also mu1.

Equation for the γ-component

• In the Select Physics tree, select Mathematics>PDE Interfaces>Lower Dimensions>Weak Form Boundary PDE (wb).

• Click Add.

• Next, locate the Dependent Variables section.

• In the Field name text field, type g.

• In the Dependent variables text field, type also g1.

• Click Study.

Study settings

• In the Select Study tree, select Preset Studies for Selected Physics Interfaces>Time Dependent.

• Click Done.

Some Advanced Settings,

Hint: In the Model Builder window you should click on the Show icon and enable everything that is possible from the menu: Equation Sections (Equation View, Override and Contribution, Discretization, Stabilization, Advanced Physics Options, Advanced Study Options and Advanced Results Options). Done this, click Expand All icon.

3. Geometry

• In the Model Builder tree, expand the Component 1 (comp1) node, right-click Geometry and select Interval.

• In the Settings window for Interval, locate the Interval section.

• In the Left endpoint text field, type -50.

• In the Right endpoint text field, type 50.

• In the Model Builder tree, right-click on the Component 1 (comp1)→Geometry 1 node and select Build all. (Alternatively, press the short cut F8.)
4. Partial differential equation for the $u$-component (Nonfrozen PDE)

General Settings.
- Click on **Component 1 (comp1)→Coefficient Form PDE (c)**.
- Locate the **Settings** window for Coefficient Form PDE.
- In the **Label** text field, type **Nonfrozen Equation**.
- In the **Name** text field, type **PDE1**.
- In the **Discretization** section choose
  - Shape function type: **Lagrange**.
  - Element order: **Linear**.

Partial differential equation. We define the PDE for the $u$-component:
- Switch to **Component 1 (comp1)→Nonfrozen Equation (PDE1)→Coefficient Form PDE 1**

\[
e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} + \nabla \cdot (-c \nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f
\]

with $\nabla = \frac{\partial}{\partial x}$, and enter the following values
- Mass Coefficient $e_a$: 0.
- Damping or Mass Coefficient $d_a$: 1.
- Diffusion coefficient $c$: 1.
- Conservative Flux Convection Coefficient $\alpha$: 0.
- Conservative Flux Source $\gamma$: 0.
- Convection Coefficient $\beta$: 0.
- Absorption Coefficient $a$: 0.
- Source Term $f$: $fu$.

The quantity $fu$ will be defined later in Section 8.

Boundary Conditions. Since the PDE of the $u$-component requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. **Hint:** By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

Initial Values. We define the initial value $u(\cdot, 0) = u_0$ for the partial differential equation:
- Click on **Component 1 (comp1)→Nonfrozen Equation (PDE1)→Initial Values 1**.
- In the **Initial Values** section enter
  - Initial value for $u$: $u_0$.
  - Initial time derivative of $u$: 0.

The quantity $u_0$ will be defined later in Section 8. This completes the implementation of the initial boundary value problem for the $u$-component.

5. Partial differential equation for the $v$-component (Frozen PDE)

General Settings.
- Click on **Component 1 (comp1)→Coefficient Form PDE 2 (c2)**.
- Locate the **Settings** window for Coefficient Form PDE.
• In the **Label** text field, type *Frozen Equation*.

• In the **Name** text field, type *PDE2*.

• In the **Discretization** section choose
  - **Shape function type**: Lagrange.
  - **Element order**: Linear.

**Partial differential equation.** We define the PDE for the $v$-component:

- Switch to **Component 1 (comp1)**→**Frozen Equation (PDE2)**→**Coefficient Form PDE**

\[ e_a \frac{\partial^2 v}{\partial t^2} + d_a \frac{\partial v}{\partial t} + \nabla \cdot \left( -c \nabla v - \alpha v + \gamma \right) + \beta \cdot \nabla v + av = f \]

with $\nabla = \frac{\partial}{\partial x}$, and enter the following values
  - **Mass Coefficient** $e_a$: 0,
  - **Damping or Mass Coefficient** $d_a$: 1,
  - **Diffusion coefficient** $c$: 1,
  - **Conservative Flux Convection Coefficient** $\alpha$: 0,
  - **Conservative Flux Source** $\gamma$: 0,
  - **Convection Coefficient** $\beta$: 0,
  - **Absorption Coefficient** $a$: 0,
  - **Source Term** $f$: Fv.

The quantity $Fv$ will be defined later in Section 8.

**Boundary Conditions.** Since the PDE of the $v$-component requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. **Hint**: By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

**Initial Values.** We define the initial value $u(\cdot, 0) = u_0$ for the partial differential equation:

- Click on **Component 1 (comp1)**→**Frozen Equation (PDE2)**→**Initial Values 1**.

- In the **Initial Values** section enter
  - **Initial value for u**: v0,
  - **Initial time derivative of u**: 0.

The quantity $v0$ will be defined later in Section 8. This completes the implementation of the initial boundary value problem for the $v$-component.

---

6. **Algebraic constraint for the $\mu$-component (Velocity)**

In the following we define the phase condition for $\mu_1$:

- Click on **Component 1 (comp1)**→**Weak Form Boundary PDE (wb)**

- Locate the **Settings** window for Weak Form Boundary PDE.

- In the **Label** text field, type **Phase Condition**.

- In the **Name** text field, type **AC1** for algebraic constraint.

- In the **Boundary Selection** section choose
  - **Selection**: Manual.

Now, click on the boundary point 2 and click on the minus sign - to remove the point from the selection list.

- In the **Discretization** section choose
– Shape function type: Lagrange,
– Element order: Linear.

**Weak Form PDE.** We define the phase condition:
• Click on **Component 1 (comp1)**→**Phase Condition (AC1)**→Weak Form PDE 1.
• In the **Weak Expression** section enter
  – weak: test(mu1) * pc1
The quantity pc1 will be defined later in Section 8.

**Initial Values.** Finally, we define the initial value:
• Click on **Component 1 (comp1)**→**Phase Condition (AC1)**→Initial Values 1.
• In the **Initial Values** section enter
  – Initial value for mu1: 0.
  – Initial time derivative of mu1: 0.
This completes the implementation of the phase condition for μ.

7. Ordinary differential equation for the γ-component (Position)
In the following we implement the reconstruction equation for the position γ:
• Click on **Component 1 (comp1)**→**Weak Form Boundary PDE 2 (wb2)**
• Locate the **Settings** window for Weak Form Boundary PDE.
• In the **Label** text field, type **Reconstruction Equation**.
• In the **Name** text field, type **ODE1**.
• In the **Boundary Selection** section choose
  – **Selection**: Manual.
  Now, click on the boundary point 2 and click on the minus sign - to remove the point from the selection list.
• In the **Discretization** section choose
  – Shape function type: Lagrange,
  – Element order: Linear.
• In the **Model Builder** tree, right-click **Component 1 (comp1)**→**Reconstruction Equation (ODE1)** and select **Weak Form PDE**.

**Weak Form PDE 1.** We define the ordinary differential equation for γ in a weak form:
• Click on **Component 1 (comp1)**→**Reconstruction Equation (ODE1)**→Weak Form PDE 1.
• In the **Weak Expression** section enter
  – weak: test(g1) * mu1

**Initial Values.** Finally, we define the initial value:
• Click on **Component 1 (comp1)**→**Reconstruction Equation (ODE1)**→Initial Values 1.
• In the **Initial Values** section enter
  – Initial value for g1: 0.
  – Initial time derivative of g1: 0.
Weak Form PDE 2.

- Click on Component 1 (comp1) → Reconstruction Equation (ODE1) → Weak Form PDE 2.
- In the Boundary Selection section switch to Selection All boundaries, then back to Selection Manual, since otherwise there is no boundary point contained in the selection list.
- In the Weak Expression section enter
  \[\text{weak: } -\text{test}(g_1) \times g_1 \times \text{time}\]

This completes the implementation of the reconstruction equation for \(\gamma\).

8. Parameters, Variables and Integration Coupling Variables

Parameters. We first define the parameters and constants arising in our model as 'global parameters':

- In the Model Builder tree, right-click on the Global Definitions node and select Parameters. (Alternatively: On the Model toolbar, click Parameters.)
- In the Settings window for Parameters, locate the Parameters section.
- In the table add the following entry:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>(1/4)</td>
<td>0.25</td>
<td>constant of Nagumo equation</td>
</tr>
</tbody>
</table>

Variables 1. We now define all functions which appear in our model as 'local variables'.

- In the Model Builder tree, right-click on the Component 1 (comp1) → Definitions node and select Variables.
- In the Settings window for Variables, locate the Variables section.
- In the table add the following entries:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_u)</td>
<td>(u^3(1-u)^4(u-b))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(u_0)</td>
<td>((\tanh(x)+1)/2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(f_v)</td>
<td>(v^4(1-v)^3(v-b))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(F_v)</td>
<td>(mulcpl*vx+fv)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v_0)</td>
<td>(u_0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v_h)</td>
<td>(v_0)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(pc1_fix)</td>
<td>(d(v_h,x)^2(v-v_h))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Integration Coupling Variables 1. We next define an integration operator, that integrates a function over the whole spatial domain.

- In the Model Builder tree, right-click on the Component 1 (comp1) → Definitions node and select Component Couplings > Integration.
- In the Settings window for Integration, locate the Source Selection section and choose
  - Geometric entity level: Domain,
- Next, locate the Advanced section and choose
  - Method: Integration,
  - Integration order: 2,
  - Frame: Spatial (x,y,z).
Integration Coupling Variables 2. We next define an integration operator, that integrates a function over the boundary of the spatial domain.

- In the Model Builder tree, right-click on the Component 1 (comp1) ➔ Definitions node and select Component Couplings ➔ Integration.
- In the Settings window for Integration, locate the Source Selection section and choose
  - Geometric entity level: Boundary,

Now, click on the boundary point 2 and click on the minus sign - to remove the point from the selection list.

- Next, locate the Advanced section and choose
  - Method: Integration,
  - Integration order: 1,
  - Frame: Spatial (x,y,z).

Variables 2. We next define the functions, that are integrated over the whole spatial domain:

- In the Model Builder tree, right-click on the Component 1 (comp1) ➔ Definitions node and select Variables.
- In the Settings window for Variables, locate the Geometric Entity Selection section and choose
  - Geometric entity level: Domain,

Now, click on the boundary point 2 and click on the minus sign - to remove the point from the selection list.

- Next, locate the Variables section and add the following entries into the table:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>intcpl_source_pcl</td>
<td>pc1_fix</td>
<td></td>
<td></td>
</tr>
<tr>
<td>intcpl_source_sqr_vt</td>
<td>vt^2</td>
<td>1/s^2</td>
<td></td>
</tr>
</tbody>
</table>

Variables 3. We next define the functions, that are integrated over the boundary of the spatial domain:

- In the Model Builder tree, right-click on the Component 1 (comp1) ➔ Definitions node and select Variables.
- In the Settings window for Variables, locate the Geometric Entity Selection section and choose
  - Geometric entity level: Boundary,

Now, click on the boundary point 2 and click on the minus sign - to remove the point from the selection list.

- Next, locate the Variables section and add the following entries into the table:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>intcpl_source_mulcpl</td>
<td>mul</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Variables 4. Finally, we define variables that contain the results of the integration

- In the Model Builder tree, right-click on the Global Definitions node and select Variables.
- In the Settings window for Variables, locate the Variables section.
- In the table add the following entries:
<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>pc1</td>
<td>comp1.intop1(intcpl_source_pc1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>sqr_vt</td>
<td>comp1.intop1(intcpl_source_sqr_vt)</td>
<td>m/s²</td>
<td></td>
</tr>
<tr>
<td>mu1cpl</td>
<td>comp1.intop2(intcpl_source_mu1cpl)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. Mesh

- In the Model Builder tree, click on Component 1 (comp1)→Mesh.
- In the Settings window for Mesh, locate the Mesh Settings section.
- Set the Sequence type on User-controlled mesh.
- In the Model Builder tree, switch to Component 1 (comp1)→Mesh→Size.
- In the Settings window for Size, locate the Element Size Parameters section.
- In the Maximum element size text field, type 0.1.
- In the Model Builder tree, right-click on Component 1 (comp1)→Mesh and select Build All.

10. Studies and Computation

Study 1. Study 1
- Click on Study 1.
- Locate the Settings window for Study.
- In the Label text field, type Study 1: Nonfrozen Equation.

Step 1
- Click on Study 1: Nonfrozen Equation→Step 1: Time Dependent.
- Locate the Settings window for Time Dependent.
- In the Study Settings section enter
  - Time unit: s,
  - Times: range(0,0.1,150),
  - Relative tolerance: 0.001.
  The last input requires to enable the corresponding checkbox.
- In the Physics and Variables Selection section disable the checkboxes for Frozen Equation (PDE2), Phase Condition (AC1) and Reconstruction Equation (ODE1).

Solver Configurations
- Right-click on Study 1: Nonfrozen Equation→Solver Configurations and select Show Default Solver.
- Click on Study 1: Nonfrozen Equation→Solver Configurations→Solution 1→Time-Dependent Solver 1.
- Locate the Settings window for Time Dependent Solver.
- In the Absolute Tolerance section enter
  - Global method: Unscaled,
  - Tolerance: 0.0001.
- In the Time Stepping section enter
  - Method: BDF,
  - Steps taken by solver: intermediate,
  - Maximum BDF order: 2.
Click on **Study 1: Nonfrozen Equation** → **Solver Configurations** → **Solution 1** → **Time-Dependent Solver 1** → **Fully Coupled 1**.

- Locate the **Settings** window for Fully Coupled.
- In the **Method and Termination** section, choose
  - **Nonlinear Method**: Automatic (Newton).

**Solution Store**

- Right-click on **Study 1: Nonfrozen Equation** → **Solver Configurations** → **Solution 1** and select **Other** → **Solution Store** from the list.
- Click on **Study 1: Nonfrozen Equation** → **Solver Configurations** → **Solution 1** and then select **Other** → **Solution Store 1**.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Nonfrozen Solution**.

**Study 2.**

- Right-click on *unknown.mph (root)* and select **Add Study**.
- In the **Add Study** window, select **Time Dependent** and confirm by click on **Add Study**.

**Study 2**

- Click on **Study 2**.
- Locate the **Settings** window for Study.
- In the **Label** text field, type **Study 2: Frozen Equation**.

**Study 2 Step 1**

- Click on **Study 2: Frozen Equation** → **Step 1: Time Dependent**.

- Locate the **Settings** window for Time Dependent.

- In the **Study Settings** section enter
  - **Time unit**: s.
  - **Times**: range(0,0.1,150).
  - **Relative tolerance**: 0.001.

  The last input requires to enable the corresponding checkbox.

- In the **Physics and Variables Selection** section disable the checkbox for **Nonfrozen Equation (PDE1)**.

**Solver Configurations**

- Right-click on **Study 2: Frozen Equation** → **Solver Configurations** and select **Show Default Solver**.
- Click on **Study 2: Frozen Equation** → **Solver Configurations** → **Solution 3** → **Time-Dependent Solver 1**.
- Locate the **Settings** window for Time Dependent Solver.

- In the **Absolute Tolerance** section enter
  - **Global method**: Unscaled,
  - **Tolerance**: 0.0001.

- In the **Time Stepping** section enter
  - **Method**: BDF,
  - **Steps taken by solver**: intermediate,
  - **Maximum BDF order**: 2.
• Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Time-Dependent Solver 1**→**Fully Coupled 1**.

• Locate the **Settings** window for Fully Coupled.

• In the **Method and Termination** section, choose
  – **Nonlinear Method**: Automatic (Newton).

**Solution Store**

• Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Other>Solution Store** from the list.

• Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3**→**Solution Store 1**.

• Locate the **Settings** window for Solution Store.

• In the **Label** text field, type **Frozen Solution**.

**Study 2 Step 2** First, we generate the solver and study step.

• Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Compile Equations**.

• Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Dependent Variables**.

• Right-click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Solvers>Eigenvalue Solver**.

• Right-click on **Study 2: Frozen Equation** and select **Study Steps>Eigenfrequency>Eigenvalue**.

• Click on **Study 2: Frozen Equation**→**Step 2: Eigenvalue**.

• Locate the **Settings** window for Eigenvalue.

• In the **Study Settings** section enter
  – **Eigenvalue search method**: Manual,
  – **Desired number of eigenvalues**: 400,
  – **Search for eigenvalues around**: -b,
  – **Eigenvalue search method around shift**: Closest in absolute value.

• In the **Physics and Variables Selection** section enable only the checkbox for **Frozen Equation (PDE2)**.

• In the **Values of dependent Variables** section enable the checkbox **Values of variables not solved for** and enter
  – **Method**: Solution,
  – **Study**: Study 2: Frozen equation, Time Dependent,
  – **Solution**: Solution 3,
  – **Use**: Frozen Solution,
  – **Selection**: Last.

**Solver Configurations**

• Click on **Study 2: Frozen Equation**→**Solver Configurations**→**Solution 3** and select **Compile Equations**.

• Locate the **Settings** window for Compile Equations.

• In the **Study and Step** section enter
  – **Use study**: Study 2: Frozen Equation,
  – **Use study step**: Step 2: Eigenvalue.
• Click on Study 2: Frozen Equation→Solver Configurations→Solution 3 and select Dependent Variables 2.
• Locate the Settings window for Dependent Variables.
• In the General section enter
• Click on Study 2: Frozen Equation→Solver Configurations→Solution 3 and select Eigenvalue Solver 1.
• Locate the Settings window for Dependent Variables.
• In the General section enter
  – Defined by study step: Step 2: Eigenvalue,
• In the Values of Linearization Point section enter
  – Prescribed by: Solution,
  – Solution: Solution 3,
  – Use: Frozen Solution,
  – Selection: Last.
Finally, activate the checkbox Store linearization point and deviation in output.

Solution Store
• Right-click on Study 2: Frozen Equation→Solver Configurations→Solution 3 and select Other>Solution Store from the list.
• Click on Study 2: Frozen Equation→Solver Configurations→Solution 3→Solution Store 1.
• Locate the Settings window for Solution Store.
• In the Label text field, type Eigenvalues and Eigenfunctions.

• Right-click on Study 1: Nonfrozen Equation and select compute from the list.
• Right-click on Study 2: Frozen Equation and select compute from the list.

11. Postprocessing and graphical output
In this section we generate 10 Plot groups for visualizing our results.

11.1. Results for nonfrozen equation.

Plot Group 1: Traveling Front, View 1
• Click on Results→1D Plot Group 1. Hint: If 1D Plot Group 1 does not exists, right-click on Results and select 1D Plot Group from the list.
• Locate the Settings window for 1D Plot Group.
• In the Label text field, type Traveling Front, View 1.
• In the Data section select Data set Study 1: Nonfrozen Equation/Nonfrozen Solution, Time selection Interpolated and Times (s) 0 40 80 120.
• In the Title section select Title type None.
• In the Plot Settings section select x-axis label x and y-axis label u(x,t).
• Click on Results→Traveling Front, View 1→Line Graph 1. Hint: If Line Graph 1 does not exists, right-click on Results→Traveling Front, View 1 and select Line Graph from the list.

• Locate the Settings window for Line Graph.

• In the Data section select Data set From parent.

• In the Selection section select Selection All domains.

• In the y-Axis Data section select Expression u.

• In the x-Axis Data section select Parameters Expression and Expression x.

• In the Coloring and Style section select Line Solid, Color Cycle and Width 2 in the Line style subsection.

• In the Legends section enable the Show legends checkbox, select Legends Manual and enter the legends t=0, t=40, t=80 and t=120.

Plot Group 2: Traveling Front, View 2

• Click on Results→1D Plot Group 2. Hint: If 1D Plot Group 2 does not exists, right-click on Results and select 1D Plot Group from the list.

• Locate the Settings window for 1D Plot Group.

• In the Label text field, type Traveling Front, View 2.

• In the Data section select Data set Study 1: Nonfrozen Equation/Nonfrozen Solution and Time selection All.

• In the Title section select Title type None.

• In the Plot Settings section select x-axis label x and y-axis label t.

• Click on Results→Traveling Front, View 2→Line Graph 1. Hint: If Line Graph 1 does not exists, right-click on Results→Traveling Front, View 2 and select Line Graph from the list.

• Locate the Settings window for Line Graph.

• In the Data section select Data set From parent.

• In the Selection section select Selection All domains.

• In the y-Axis Data section select Expression t.

• In the x-Axis Data section select Parameters Expression and Expression x.

• Right-click on Results→Traveling Front, View 2→Line Graph 1 and select Color Expression.

• Click on Results→Traveling Front, View 2→Line Graph 1→Color Expression 1.

• Locate the Settings window for Color Expression.

• In the Expression section select Expression u.

11.2. Results for frozen equation.

Plot Group 3: Profile, View 1

• Right-click on Results and select 1D Plot Group from the list.

• Locate the Settings window for 1D Plot Group.

• In the Label text field, type Profile, View 1.

• In the Data section select Data set Study 2: Frozen Equation/Frozen Solution and Time selection Last.
• In the Title section select Title type None.
• In the Plot Settings section select x-axis label x and y-axis label v(x,150).
• Right-click on Results→Profile, View 1 and select Line Graph from the list.
• Locate the Settings window for Line Graph.
• In the Data section select Data set From parent.
• In the Selection section select Selection All domains.
• In the y-Axis Data section select Expression v.
• In the x-Axis Data section select Parameters Expression and Expression x.
• In the Coloring and Style section select Line Solid, Color Blue and Width 2 in the Line style subsection.

Plot Group 4: Profile, View 2
• Right-click on Results and select 1D Plot Group from the list.
• Locate the Settings window for 1D Plot Group.
• In the Label text field, type Profile, View 2.
• In the Data section select Data set Study 1: Frozen Equation/Frozen Solution and Time selection All.
• In the Title section select Title type None.
• In the Plot Settings section select x-axis label x and y-axis label t.
• Right-click on Results→Profile, View 2 and select Line Graph from the list.
• Locate the Settings window for Line Graph.
• In the Data section select Data set From parent.
• In the Selection section select Selection All domains.
• In the y-Axis Data section select Expression t.
• In the x-Axis Data section select Parameters Expression and Expression x.
• Right-click on Results→Profile, View 2→Line Graph 1 and select Color Expression.
• Click on Results→Profile, View 2→Line Graph 1→Color Expression 1.
• Locate the Settings window for Color Expression.
• In the Expression section select Expression v.

Plot Group 5: Velocities
• Right-click on Results and select 1D Plot Group from the list.
• Locate the Settings window for 1D Plot Group.
• In the Label text field, type Velocities.
• In the Data section select Data set Study 2: Frozen Equation/Frozen Solution and Time selection All.
• In the Title section select Title type None.
• In the Plot Settings section select x-axis label t and y-axis label μ(t).
• Right-click on Results→Velocities and select Point Graph from the list.
• Locate the Settings window for Point Graph.
• In the Data section select Data set From parent.
• In the **Selection** section select **Selection All boundaries**. Now, select boundary point 2 and click on the minus sign - to remove the point from selection.
• In the **y-Axis Data** section select **Expression mu1**.
• In the **x-Axis Data** section select **Parameters Expression** and **Expression t**.
• In the **Coloring and Style** section select **Line Solid, Color Blue** and **Width 2** in the **Line style** subsection.

**Plot Group 6: Positions**
• Right-click on **Results** and select **1D Plot Group** from the list.
• Locate the **Settings** window for 1D Plot Group.
• In the **Label** text field, type **Positions**.
• In the **Data** section select **Data set Study 2: Frozen Equation/Frozen Solution and Time selection All**.
• In the **Title** section select **Title type None**.
• In the **Plot Settings** section select **x-axis label t** and **y-axis label gamma(t)**.
• Right-click on **Results→Positions** and select **Point Graph** from the list.
• Locate the **Settings** window for Point Graph.
• In the **Data** section select **Data set From parent**.
• In the **Selection** section select **Selection All boundaries**. Now, select boundary point 2 and click on the minus sign - to remove the point from selection.
• In the **y-Axis Data** section select **Expression g1**.
• In the **x-Axis Data** section select **Parameters Expression** and **Expression t**.
• In the **Coloring and Style** section select **Line Solid, Color Blue** and **Width 2** in the **Line style** subsection.

**Plot Group 7: Reference function**
• Right-click on **Results** and select **1D Plot Group** from the list.
• Locate the **Settings** window for 1D Plot Group.
• In the **Label** text field, type **Reference Function**.
• In the **Data** section select **Data set Study 2: Frozen Equation/Frozen Solution and Time selection First**.
• In the **Title** section select **Title type None**.
• In the **Plot Settings** section select **x-axis label x** and **y-axis label vh(x)**.
• Right-click on **Results→Reference Function** and select **Line Graph** from the list.
• Locate the **Settings** window for Line Graph.
• In the **Data** section select **Data set From parent**.
• In the **Selection** section select **Selection All domains**.
• In the **y-Axis Data** section select **Expression vh**.
• In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
• In the **Coloring and Style** section select **Line Solid, Color Blue** and **Width 2** in the **Line style** subsection.

**Plot Group 8: \( \|v_1\|_{L^2} \) and \( \mu_{1,t} \)**
• Right-click on **Results** and select **1D Plot Group** from the list.
• Locate the **Settings** window for 1D Plot Group.
• In the **Label** text field, type **L2-Norm of vt and abs of mu1t**.
• In the **Data** section select **Data set** Study 2: Frozen Equation/Frozen Solution and **Time selection** All.
• In the **Title** section select **Title type** None.
• In the **Plot Settings** section select **x-axis label** t. The **y-axis label** remains disabled.
• In the **Axis** section enable the checkbox for **y-axis log scale**.
• Right-click on **Results**→**L2-Norm of vt and abs of mu1t** and select **Point Graph** from the list.
• Locate the **Settings** window for **Point Graph**.
• In the **Data** section select **Data set** From parent.
• In the **Selection** section select **Selection** All boundaries. Now, select boundary point 2 and click on the minus sign - to remove the point from selection.
• In the **y-Axis Data** section select **Expression** \( \sqrt{\text{sqr}_v \text{t}} \).
• In the **x-Axis Data** section select **Parameters** Expression and **Expression** t.
• In the **Coloring and Style** section select **Line** Solid, **Color** Blue and **Width** 2.
• In the **Legends** section enable the **Show legends** checkbox, select **Legends** Manual and enter the legend \( ||v_\text{t}||_L^2 \).
• Right-click on **Results**→**L2-Norm of vt and abs of mu1t** and select **Point Graph** from the list.
• Locate the **Settings** window for **Point Graph**.
• In the **Data** section select **Data set** From parent.
• In the **Selection** section select **Selection** All boundaries. Now, select boundary point 2 and click on the minus sign - to remove the point from selection.
• In the **y-Axis Data** section select **Expression** abs(mu1t).
• In the **x-Axis Data** section select **Parameters** Expression and **Expression** t.
• In the **Coloring and Style** section select **Line** Dashed, **Color** Red and **Width** 2.
• In the **Legends** section enable the **Show legends** checkbox, select **Legends** Manual and enter the legend \( |\text{mu1}_\text{t}| \) in the **Line style** subsection.

11.3. Results for eigenvalue study.

**Plot Group 9: Eigenvalues and Spectrum**
• Right-click on **Results** and select **1D Plot Group** from the list.
• Locate the **Settings** window for 1D Plot Group.
• In the **Label** text field, type **Eigenvalues and Spectrum**.
• In the **Data** section select **Data set** Study 2: Frozen Equation/Eigenvalues and Eigenfunctions and **Eigenvalue selection** All.
• In the **Title** section select **Title type** None.
• In the **Plot Settings** section select **x-axis label** Re lambda and **y-axis label** Im lambda.
• In the **Axis** section enable the **Manual axis limits** checkbox and select **x minimum** -6, **x maximum** 2, **y minimum** -1 and **y maximum** 1.
• Right-click on **Results→Eigenvalues and Spectrum** and select **Line Graph** from the list.
• Locate the **Settings** window for Line Graph.
• In the **Data** section select **Data set** Study 2: Frozen Equation/Eigenvalues and Eigenfunctions, **Eigenvalue selection** From list and select all eigenvalues except the zero eigenvalue.
• In the **Selection** section select **Selection** All domains.
• In the **y-Axis Data** section select **Expression** imag(lambda).
• In the **x-Axis Data** section select **Parameters** Expression and **Expression** -real(lambda).
• In the **Coloring and Style** section select **Line** Solid, **Color** Red and **Width** 1 in the **Line style** subsection, and **Marker** Circle as well as **Positioning in data points** in the **Line markers** subsection.

Plot Group 10: Eigenfunctions

• Right-click on **Results→Eigenvalues and Spectrum** and select **Line Graph** from the list.
• Locate the **Settings** window for Line Graph.
• In the **Data** section select **Data set** Study 2: Frozen Equation/Eigenvalues and Eigenfunctions, **Eigenvalue selection** From list and select the zero eigenvalue.
• In the **Selection** section select **Selection** All domains.
• In the **y-Axis Data** section select **Expression** imag(lambda).
• In the **x-Axis Data** section select **Parameters** Expression and **Expression** -real(lambda).
• In the **Coloring and Style** section select **Line** None, **Color** Blue and **Width** 1 in the **Line style** subsection, and **Marker** Circle as well as **Positioning in data points** in the **Line markers** subsection.