Mathematical Modelling and Simulation with Comsol Multiphysics II
Winter term 2015/2016
Exercise 4
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Bearbeitung: Montag, 02.11.2015, 12:30-14:00 Uhr (während der Übung).

Exercise 4 (FitzHugh-Nagumo system: Traveling Front)**.**
Consider the **FitzHugh-Nagumo system**

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix}, \quad x \in \mathbb{R}, \, t \geqslant 0$$

for some $D \geqslant 0$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$, $\zeta \neq 0$ and $u_i = u_i(x,t) \in \mathbb{R}$ for $i = 1, 2$. Using the notation

$$u = (u_1, u_2) \in \mathbb{R}^2 \quad \text{and} \quad f : \mathbb{R}^2 \to \mathbb{R}^2, \, f(u) = \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix},$$

the FitzHugh-Nagumo system can also be written as

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} u_{xx} + f(u), \quad x \in \mathbb{R}, \, t \geqslant 0.$$

a) Solve the **nonfrozen FitzHugh-Nagumo system**

$$(1) \quad \begin{aligned} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix} & , x \in \Omega, \, t \in (0, T_1], \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} & , x \in \bar{\Omega}, \, t = 0, \\ \begin{pmatrix} u_{1,x} \\ u_{2,x} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & , x \in \partial\Omega, \, t \in [0, T_1], \end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$ for end time $T_1 = 80$, asymptotic states

$$u_- = \begin{pmatrix} 1.18769696080266 \\ 0.629232320266825 \end{pmatrix}, \quad u_+ = \begin{pmatrix} -1.56443178284120 \\ -0.288143927613547 \end{pmatrix},$$

inital data

$$u_0^{(1)}(x) = \begin{cases} u_-^{(1)} & , x < 0 \\ u_+^{(1)} & , x \geqslant 0 \end{cases}, \quad u_0^{(2)}(x) = \begin{cases} u_-^{(2)} & , x < 0 \\ u_+^{(2)} & , x \geqslant 0 \end{cases},$$

and parameters

$$D = \frac{1}{10}, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 3, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization use the BDF method of maximum order 5 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-2}$ and absolute tolerance atol $= 10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

b) Solve the **nonfrozen FitzHugh-Nagumo system**

$$(2) \quad \begin{aligned} \begin{pmatrix} \hat{v}_{1,t} \\ \hat{v}_{2,t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \hat{v}_{1,xx} \\ \hat{v}_{2,xx} \end{pmatrix} + \begin{pmatrix} \hat{v}_1 - \zeta \hat{v}_1^3 - \hat{v}_2 + \alpha \\ \beta(\gamma \hat{v}_1 - \delta \hat{v}_2 + \varepsilon) \end{pmatrix} &, x \in \Omega, \ t \in (0, T_2], \\[2mm] \begin{pmatrix} \hat{v}_1(0) \\ \hat{v}_2(0) \end{pmatrix} &= \begin{pmatrix} \hat{v}_0^{(1)} \\ \hat{v}_0^{(2)} \end{pmatrix} &, x \in \bar{\Omega}, \ t = 0, \\[2mm] \begin{pmatrix} \hat{v}_{1,x} \\ \hat{v}_{2,x} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} &, x \in \partial\Omega, \ t \in [0, T_2], \end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$ for end time $T_2 = 2$, $\hat{v}_0^{(1)}(x) = u_0^{(1)}(x)$, $\hat{v}_0^{(2)}(x) = u_0^{(2)}(x)$ and parameters

$$D = \frac{1}{10}, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 3, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization use the BDF method of maximum order 5 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-2}$ and absolute tolerance atol $= 10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

c) Solve the **frozen FitzHugh-Nagumo system**

$$(3) \quad \begin{aligned} \begin{pmatrix} v_{1,t} \\ v_{2,t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} v_{1,\xi\xi} \\ v_{2,\xi\xi} \end{pmatrix} + \mu \begin{pmatrix} v_{1,\xi} \\ v_{2,\xi} \end{pmatrix} + \begin{pmatrix} v_1 - \zeta v_1^3 - v_2 + \alpha \\ \beta(\gamma v_1 - \delta v_2 + \varepsilon) \end{pmatrix} &, \xi \in \Omega, \ t \in (0, T_3], \\[2mm] \begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix} &= \begin{pmatrix} v_0^{(1)} \\ v_0^{(2)} \end{pmatrix} &, \xi \in \bar{\Omega}, \ t = 0, \\[2mm] \begin{pmatrix} v_{1,\xi} \\ v_{2,\xi} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} &, \xi \in \partial\Omega, \ t \in [0, T_3], \\[2mm] 0 &= (v_1 - \hat{v}_1, \hat{v}_{1,\xi})_{L^2(\Omega,\mathbb{R})} + (v_2 - \hat{v}_2, \hat{v}_{2,\xi})_{L^2(\Omega,\mathbb{R})} &, t \in [0, T_3], \\[2mm] \gamma_t &= \mu &, t \in (0, T_3], \\[2mm] \gamma(0) &= 0 &, t = 0 \end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$ for end time $T_3 = 100$, initial data $v_0^{(1)}(\xi) = u_0^{(1)}(\xi)$, $v_0^{(2)}(\xi) = u_0^{(2)}(\xi)$, reference function $\hat{v}(\xi)$ as the solution of (2) at end time T_2 and parameters

$$D = \frac{1}{10}, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 3, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization use the BDF method of maximum order 2 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-2}$ and absolute tolerance atol $= 10^{-5}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method (automatic (Newton)).

d) Solve the **eigenvalue problem** for the linearization of the FitzHugh-Nagumo system

$$(4) \quad \begin{aligned} \lambda \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \mu_\star \begin{pmatrix} w_{1,\xi} \\ w_{2,\xi} \end{pmatrix} + Df(v_\star(\xi)) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &, \xi \in \Omega, \\[2mm] \begin{pmatrix} w_{1,\xi} \\ w_{2,\xi} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} &, \xi \in \partial\Omega \end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$, where $Df(u)$ denotes the derivative of f given by

$$Df(u) = \begin{pmatrix} 1 - 3\zeta u_1^2 & -1 \\ \beta\gamma & -\beta\delta \end{pmatrix}.$$

For $v_\star$ and $\mu_\star$ use the solutions v and μ of (3) at the end time $T_3 = 100$. Determine neigs $= 400$ eigenvalues λ and correspondig eigenfunctions $w = (w_1, w_2)^T$. The eigenvalues should be closest in absolute value around the shift -1.

e) **Postprocessing** and **Visualization** of results: Create the following plots to visualize the results of the computations:

   – Traveling Front, View 1: Plot the solution u_1 of (1) at time $t = 0, 20, 40$ and 60.

   – Traveling Front, View 1: Plot the solution u_2 of (1) at time $t = 0, 20, 40$ and 60.

   – Traveling Front, View 2: Create a time-space plot for the solution u_1 of (1).

   – Traveling Front, View 2: Create a time-space plot for the solution u_2 of (1).

   – Reference function: Plot the template solutions $\hat{v}_1$ and $\hat{v}_2$ of (2) at time $T2$.

   – Profile, View 1: Plot the solution v_1 and v_2 of (3) at the end time T_3.

   – Profile, View 2: Create a time-space plot for the solution v_1 of (3).

   – Profile, View 2: Create a time-space plot for the solution v_2 of (3).

   – Velocity: Plot the velocity μ of (3) for time t from 0 to T_3.

   – Position: Plot the position γ of (3) for time t from 0 to T_3.

   – Convergence indicator: Plot $\|v_t(t)\|_{L^2(\Omega, \mathbb{R}^2)}$ and $|\mu_t(t)|$ for time t from 0 to T_3.

   – Eigenvalues and Spectrum: Plot the eigenvalues λ of (4).

   – Eigenfunctions: Plot the eigenfunction w of (4) belonging to the zero eigenvalue.