Mathematical Modelling and Simulation with Comsol Multiphysics II

Winter term 2015/2016

Exercise 5

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Bearbeitung: Montag, 02.11.2015, 12:30-14:00 Uhr (während der Übung).

Exercise 5 (FitzHugh-Nagumo system: Traveling Pulse)**.**
Consider the **FitzHugh-Nagumo system**

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix}, \quad x \in \mathbb{R},\ t \geqslant 0$$

for some $D \geqslant 0$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$, $\zeta \neq 0$ and $u_i = u_i(x,t) \in \mathbb{R}$ for $i = 1, 2$. Using the notation

$$u = \big(u_1, u_2\big) \in \mathbb{R}^2 \quad \text{and} \quad f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(u) = \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix},$$

the FitzHugh-Nagumo system can also be written as

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} u_{xx} + f(u), \quad x \in \mathbb{R},\ t \geqslant 0.$$

a) Solve the **nonfrozen FitzHugh-Nagumo system**

$$(1) \qquad \begin{aligned} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix} &&,\ x \in \Omega,\ t \in (0, T_1], \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} &= \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \end{pmatrix} &&,\ x \in \bar{\Omega},\ t = 0, \\ \begin{pmatrix} u_{1,x} \\ u_{2,x} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} &&,\ x \in \partial\Omega,\ t \in [0, T_1], \end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$ for end time $T_1 = 120$, asymptotic states

$$u_{\pm} = \begin{pmatrix} -1.19940803524404 \\ -0.624260044055044 \end{pmatrix},$$

inital data

$$u_0^{(1)}(x) = \begin{cases} u_{\pm}^{(1)} & ,\ x \in [-60, -10) \\ u_{\pm}^{(1)} + \frac{x+10}{20} & ,\ x \in [-10, 10] \\ u_{\pm}^{(1)} + 1 & ,\ x \in (10, 60] \end{cases} \quad , \quad u_0^{(2)}(x) = u_{\pm}^{(2)},$$

and parameters

$$D = \frac{1}{10}, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 0.8, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization use the BDF method of maximum order 5 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance $\text{rtol} = 10^{-3}$ and absolute tolerance $\text{atol} = 10^{-5}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

b) Solve the **nonfrozen FitzHugh-Nagumo system**

(2)
$$\begin{aligned}
\begin{pmatrix}\hat v_{1,t}\\ \hat v_{2,t}\end{pmatrix} &= \begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}\hat v_{1,xx}\\ \hat v_{2,xx}\end{pmatrix} + \begin{pmatrix}\hat v_1-\zeta\hat v_1^3-\hat v_2+\alpha\\ \beta(\gamma\hat v_1-\delta\hat v_2+\varepsilon)\end{pmatrix} &&, x\in\Omega,\ t\in(0,T_2],\\
\begin{pmatrix}\hat v_1(0)\\ \hat v_2(0)\end{pmatrix} &= \begin{pmatrix}\hat v_0^{(1)}\\ \hat v_0^{(2)}\end{pmatrix} &&, x\in\bar\Omega,\ t=0,\\
\begin{pmatrix}\hat v_{1,x}\\ \hat v_{2,x}\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} &&, x\in\partial\Omega,\ t\in[0,T_2],
\end{aligned}$$

on the spatial domain $\Omega=(-60,60)$ for end time $T_2=1.2$, $\hat v_0^{(1)}(x)=u_0^{(1)}(x)$, $\hat v_0^{(2)}(x)=u_0^{(2)}(x)$ and parameters

$$D=\frac{1}{10},\quad \alpha=0,\quad \beta=\frac{2}{25},\quad \gamma=1,\quad \delta=0.8,\quad \varepsilon=\frac{7}{10},\quad \zeta=\frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x=0.1$. For the time discretization use the BDF method of maximum order 5 with intermediate time steps, time stepsize $\triangle t=0.1$, relative tolerance rtol $=10^{-2}$ and absolute tolerance atol $=10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

c) Solve the **frozen FitzHugh-Nagumo system**

(3)
$$\begin{aligned}
\begin{pmatrix}v_{1,t}\\ v_{2,t}\end{pmatrix} &= \begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}v_{1,\xi\xi}\\ v_{2,\xi\xi}\end{pmatrix} + \mu\begin{pmatrix}v_{1,\xi}\\ v_{2,\xi}\end{pmatrix} + \begin{pmatrix}v_1-\zeta v_1^3-v_2+\alpha\\ \beta(\gamma v_1-\delta v_2+\varepsilon)\end{pmatrix} &&, \xi\in\Omega,\ t\in(0,T_3],\\
\begin{pmatrix}v_1(0)\\ v_2(0)\end{pmatrix} &= \begin{pmatrix}v_0^{(1)}\\ v_0^{(2)}\end{pmatrix} &&, \xi\in\bar\Omega,\ t=0,\\
\begin{pmatrix}v_{1,\xi}\\ v_{2,\xi}\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} &&, \xi\in\partial\Omega,\ t\in[0,T_3],\\
0 &= (v_1-\hat v_1,\hat v_{1,\xi})_{L^2(\Omega,\mathbb{R})} + (v_2-\hat v_2,\hat v_{2,\xi})_{L^2(\Omega,\mathbb{R})} &&, t\in[0,T_3],\\
\gamma_t &= \mu &&, t\in(0,T_3],\\
\gamma(0) &= 0 &&, t=0
\end{aligned}$$

on the spatial domain $\Omega=(-60,60)$ for end time $T_3=150$, initial data $v_0^{(1)}(\xi)=u_0^{(1)}(\xi)$, $v_0^{(2)}(\xi)=u_0^{(2)}(\xi)$, reference function $\hat v(\xi)$ as the solution of (2) at end time T_2 and parameters

$$D=\frac{1}{10},\quad \alpha=0,\quad \beta=\frac{2}{25},\quad \gamma=1,\quad \delta=0.8,\quad \varepsilon=\frac{7}{10},\quad \zeta=\frac{1}{3}.$$

For the space discretization use linear Lagrange elements with maximal element size $\triangle x=0.1$. For the time discretization use the BDF method of maximum order 2 with intermediate time steps, time stepsize $\triangle t=0.1$, relative tolerance rtol $=10^{-2}$ and absolute tolerance atol $=10^{-5}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method (automatic (Newton)).

d) Solve the **eigenvalue problem** for the linearization of the FitzHugh-Nagumo system

(4)
$$\begin{aligned}
\lambda\begin{pmatrix}w_1\\ w_2\end{pmatrix} &= \begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} + \mu_\star\begin{pmatrix}w_{1,\xi}\\ w_{2,\xi}\end{pmatrix} + Df(v_\star(\xi))\begin{pmatrix}w_1\\ w_2\end{pmatrix} &&, \xi\in\Omega,\\
\begin{pmatrix}w_{1,\xi}\\ w_{2,\xi}\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix} &&, \xi\in\partial\Omega
\end{aligned}$$

on the spatial domain $\Omega = (-60, 60)$, where $Df(u)$ denotes the derivative of f given by

$$Df(u) = \begin{pmatrix} 1 - 3\zeta u_1^2 & -1 \\ \beta\gamma & -\beta\delta \end{pmatrix}.$$

For $v_\star$ and $\mu_\star$ use the solutions v and μ of (3) at the end time $T_3 = 150$. Determine neigs $= 400$ eigenvalues λ and correspondig eigenfunctions $w = (w_1, w_2)^T$. The eigenvalues should be closest in absolute value around the shift -1.

e) **Postprocessing** and **Visualization** of results: Create the following plots to visualize the results of the computations:

- Traveling Pulse, View 1: Plot the solution u_1 of (1) at time $t = 0, 20, 40$ and 60.
- Traveling Pulse, View 1: Plot the solution u_2 of (1) at time $t = 0, 20, 40$ and 60.
- Traveling Pulse, View 2: Create a time-space plot for the solution u_1 of (1).
- Traveling Pulse, View 2: Create a time-space plot for the solution u_2 of (1).
- Reference function: Plot the template solutions $\hat{v}_1$ and $\hat{v}_2$ of (2) at time $T2$.
- Profile, View 1: Plot the solution v_1 and v_2 of (3) at the end time T_3.
- Profile, View 2: Create a time-space plot for the solution v_1 of (3).
- Profile, View 2: Create a time-space plot for the solution v_2 of (3).
- Velocity: Plot the velocity μ of (3) for time t from 0 to T_3.
- Position: Plot the position γ of (3) for time t from 0 to T_3.
- Convergence indicator: Plot $\|v_t(t)\|_{L^2(\Omega,\mathbb{R}^2)}$ and $|\mu_t(t)|$ for time t from 0 to T_3.
- Eigenvalues and Spectrum: Plot the eigenvalues λ of (4).
- Eigenfunctions: Plot the eigenfunction w of (4) belonging to the zero eigenvalue.