Mathematical Modelling and Simulation with Comsol Multiphysics II

Winter term 2015/2016

Dr. Denny Otten

09.11.2015

2. Freezing Oscillating Waves in Reaction Diffusion Systems

2.1 Oscillating waves in reaction diffusion systems

Consider a system of reaction diffusion equations on $\mathbb{R}^d$

$$\begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)) \quad &, x\in\mathbb{R}^d,\ t>0,\\ u(x,0) &= u_0(x) &, x\in\mathbb{R}^d,\ t=0, \end{aligned} \tag{1}$$

with diffusion matrix $A \in \mathbb{C}^{m,m}$, nonlinearity $f : \mathbb{C}^m \to \mathbb{C}^m$, initial data $u_0 : \mathbb{R}^d \to \mathbb{C}^m$ and solution $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{C}^m$. We assume that f satisfies $f(e^{i\varphi}z) = e^{i\varphi}f(z)$ for any $\varphi \in \mathbb{R}$ and $z \in \mathbb{C}^m$. The operator $\triangle$ denotes the Laplacian given by

$$\triangle u(x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x),\ x\in\mathbb{R}^d.$$

We are interested in oscillating wave solutions of (1): An oscillating (or phaserotating) wave of (1) is a solution $u_\star : \mathbb{R}^d \times [0,\infty) \to \mathbb{C}^m$ of the form

$$u_\star(x,t) = e^{-i\mu_\star t}v_\star(x),\ x\in\mathbb{R}^d,\ t\geqslant 0. \tag{2}$$

The function $v_\star : \mathbb{R}^d \to \mathbb{C}^m$ is called the profile and $\mu_\star \in \mathbb{R}$ the (oscillation or phase) velocity of the oscillating wave.

Our aim is to approximate oscillating wave solutions of (1). The idea for approximating the oscillating wave $u_\star$ is to determine the profile $v_\star$ and the velocity $\mu_\star$ simultaneously. This requires to transform (1) into a co-oscillating coordinate system.

Transforming (1) via $u(x,t) = e^{-i\mu_\star t}v(x,t)$ in a co-oscillating frame yields

$$\begin{aligned} v_t(x,t) &= A\triangle v(x,t) + i\mu_\star v(x) + f(v(x,t)) \quad &, x\in\mathbb{R}^d,\ t>0,\\ v(x,0) &= u_0(x) &, x\in\mathbb{R}^d,\ t=0. \end{aligned} \tag{3}$$

Inserting (2) into (1) shows, that $v_\star$ is a stationary solution of (3), i.e.

$$0 = A\triangle v_\star(x) + i\mu_\star v_\star(x) + f(v_\star(x)) \quad , x\in\mathbb{R}^d. \tag{4}$$

We are also interested in nonlinear stability of oscillating waves. It is well known from the literature, that in many cases spectral stability implies nonlinear stability. For investigating spectral stability of a oscillating wave, we must analyze the spectrum of the linearization of the right hand side in (3) at the wave profile $v_\star$, i.e.

$$\left[\mathcal{L}w\right](x) = A\triangle w(x) + i\mu_\star w(x) + Df(v_\star(x))w(x) \quad , x\in\mathbb{R}^d.$$

This requires to find solutions (λ, w) of the eigenvalue problem

$$\lambda w(x) = Aw_x(x) + i\mu_\star w(x) + Df(v_\star(x))w(x) \quad , x\in\mathbb{R}^d, \tag{5}$$

with eigenfunction $w : \mathbb{R} \to \mathbb{C}^m$ and eigenvalue $\lambda \in \mathbb{C}$.

Approximating $v_\star$ via (3) requires the knowledge about the velocity $\mu_\star$ which is in general unknown. This motivates to introduce the freezing method, whose idea is it to approximate the profile $v_\star$ and the velocity $\mu_\star$ simultaneously.

2.2 Freezing method for oscillating waves

Consider again a system of reaction diffusion equations on $\mathbb{R}^d$, cf. (1),

$$(6)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)) &&, x\in\mathbb{R}^d,\, t>0,\\ u(x,0) &= u_0(x) &&, x\in\mathbb{R}^d,\, t=0. \end{aligned}$$

Introducing new unknowns $\gamma(t)\in\mathbb{R}$ (position) and $v(x,t)\in\mathbb{C}^m$ (profile) via the oscillating wave ansatz

$$(7)\qquad u(x,t) = e^{-i\gamma(t)}v(x,t) \quad, x\in\mathbb{R}^d,\, t\geqslant 0$$

and inserting (7) into (6) yields

$$(8)\qquad v_t(x,t) = A\triangle v(x,t) + i\gamma_t(t)v(x,t) + f(v(x,t)) \quad, x\in\mathbb{R}^d,\, t>0.$$

It is convenient to introduce a further unknown $\mu(t)\in\mathbb{R}$ (velocity) via $\gamma_t(t)=\mu(t)$. Then, (8) reads as

$$(9)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + i\mu(t)v(x,t) + f(v(x,t)) &&, x\in\mathbb{R}^d,\, t>0,\\ \gamma_t(t) &= \mu(t) &&, t>0. \end{aligned}$$

Equ. (9) has to be equipped with suitable initial data. Requiring $\gamma(0)=0$, (7) and (6) imply

$$(10)\qquad v(x,0) = u_0(x) \quad, x\in\mathbb{R}^d,\, t=0.$$

Collecting the equations (9), $\gamma(0)=0$ and (10) we obtain

$$(11)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + i\mu(t)v(x,t) + f(v(x,t)) &&, x\in\mathbb{R}^d,\, t>0,\\ v(x,0) &= u_0(x) &&, x\in\mathbb{R}^d,\, t=0,\\ \gamma_t(t) &= \mu(t) &&, t>0,\\ \gamma(0) &= 0 &&, t=0. \end{aligned}$$

(11) contains the equations for v and γ. But so far, the system (11) is not well-posed, since there is still no equation for μ. To determine μ we require an additional algebraic constraint, a so called phase condition: For this purpose let $\hat{v}:\mathbb{R}^d\to\mathbb{C}^m$ be a template function, e.g. $\hat{v}=u_0$. The idea of the phase condition is to choose $v(\cdot,t)$ such that

$$\min_{g\in\mathbb{R}} \left\| v(\cdot,t) - e^{-ig}\hat{v}(\cdot)\right\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^m)} = \|v(\cdot,t)-\hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^m)} \quad, t\geqslant 0.$$

A necessary condition to guarantee that the left hand side attains its minimum at $g=0$ is that the first derivative of $\|v(\cdot,t)-e^{-ig}\hat{v}(\cdot)\|^2_{L^2(\mathbb{R}^d,\mathbb{C}^m)}$ evaluated at $g=0$ vanishes, i.e. for all $t\geqslant 0$

$$(12)\quad 0 \overset{!}{=} \left[\frac{d}{dg}\left(v(\cdot,t)-e^{-ig}\hat{v}(\cdot), v(\cdot,t)-e^{-ig}\hat{v}(\cdot)\right)_{L^2(\mathbb{R}^d,\mathbb{C}^m)}\right]_{g=0} = 2\mathrm{Re}\,\left(v(\cdot,t)-\hat{v}, i\hat{v}\right)_{L^2(\mathbb{R}^d,\mathbb{C}^m)}.$$

Combining (11) and (12) yields a partial differential algebraic evolution equation (PDAE)

$$(13)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + i\mu(t)v(x,t) + f(v(x,t)) &&, x\in\mathbb{R}^d,\, t>0,\\ v(x,0) &= u_0(x) &&, x\in\mathbb{R}^d,\, t=0,\\ 0 &= \mathrm{Re}\,\left(v(\cdot,t)-\hat{v}, i\hat{v}\right)_{L^2(\mathbb{R}^d,\mathbb{C}^m)} &&, t\geqslant 0,\\ \gamma_t(t) &= \mu(t) &&, t>0,\\ \gamma(0) &= 0 &&, t=0. \end{aligned}$$

The last two equations in (13) for the position γ are decoupled from the other equations in (13). Therefore, the γ-equation can be solved in a postprocessing step. The γ-equation is called the reconstruction equation for the oscillating wave. Since $(v_\star,\mu_\star)$ satisfy

$$\begin{aligned} 0 &= A\triangle v_\star(x) + i\mu_\star v_\star(x) + f(v_\star(x))\,, x\in\mathbb{R},\\ 0 &= \mathrm{Re}\,\left(v_\star-\hat{v}, i\hat{v}\right)_{L^2(\mathbb{R}^d,\mathbb{C}^m)}, \end{aligned}$$

we expect for stability reasons, that the solution (v,μ,γ) of (13) satisfies

$$v(t)\to v_\star,\quad \mu(t)\to\mu_\star \quad\text{as}\quad t\to\infty. \tag{14}$$

As an indicator for the convergence in (14) we check the quantities

$$\|v_t(\cdot,t)\|_{L^2(\mathbb{R}^d,\mathbb{C}^m)} \quad\text{and}\quad |\mu_t(t)| \tag{15}$$

at each time instance t during the computation. In fact, both of these quantities should be small ($\approx 10^{-16}$), since $v_\star$ and $\mu_\star$ do not vary in time.

2.3 Numerical approximation of traveling waves via freezing method

Solving (1), (13) and (5) numerically, requires to truncate these equations to bounded domains. Let $\Omega\subset\mathbb{R}$ be a bounded open domain, then (1), (13) and (5) must be satisfied for $x\in\Omega$. To guarantee the well-posedness of these problems, we must equip the equations with appropriate boundary conditions. Normally, we choose homogeneous Neumann boundary conditions (also known as no-flux boundary conditions), i.e.

$$\frac{\partial u}{\partial n}(x)=0,\ x\in\partial\Omega,\quad \frac{\partial v}{\partial n}(x)=0,\ x\in\partial\Omega.$$

where $\frac{\partial u}{\partial n}(x):=\nabla u(x)\cdot n$ denotes the directional derivative of u along the (unit) normal vector $n=n(x)\in\mathbb{R}^d$ at the boundary point $x\in\partial\Omega$. In this context, $\partial\Omega$ denotes the boundary of Ω and $\overline{\Omega}$ the closure of Ω. E.g. if $\Omega=(a,b)$ with $-\infty<a<b<\infty$ then $\partial\Omega=\{a,b\}$ and $\overline{\Omega}=[a,b]$. If $\Omega=B_R(0):=\{x\in\mathbb{R}^d\mid |x|<R\}$ with $R>0$ then $\partial\Omega=\{x\in\mathbb{R}^d\mid |x|=R\}$ and $\overline{\Omega}=\{x\in\mathbb{R}^d\mid |x|\leqslant R\}$. Numerically, we solve the following equations:

Step 1: (**Nonfrozen Equation**)

$$\begin{aligned}
&u_t(x,t)=A\triangle u(x,t)+f(u(x,t)) &&,\ x\in\Omega,\ t\in(0,T_1],\\
&u(x,0)=u_0(x) &&,\ x\in\overline{\Omega},\ t=0,\\
&\frac{\partial u}{\partial n}(x,t)=0 &&,\ x\in\partial\Omega,\ t\in[0,T_1].
\end{aligned}\tag{16}$$

First, we determine the solution u of (16). The quantities A, f, u_0, Ω and T_1 are given.

Step 2: (**Frozen Equation**)

$$\begin{aligned}
&v_t(x,t)=A\triangle v(x,t)+i\mu(t)v(x,t)+f(v(x,t)) &&,\ x\in\Omega,\ t\in(0,T_2],\\
&v(x,0)=v_0(x),\ x\in\overline{\Omega} &&,\ t=0,\\
&\frac{\partial v}{\partial n}(x,t)=0 &&,\ x\in\partial\Omega,\ t\in[0,T_2],\\
&0=\mathrm{Re}\,(v(\cdot,t)-\hat{v},i\hat{v})_{L^2(\Omega,\mathbb{C}^m)} &&,\ t\in[0,T_2],\\
&\gamma_t(t)=\mu(t) &&,\ t\in(0,T_2],\\
&\gamma(0)=0 &&,\ t=0.
\end{aligned}\tag{17}$$

Then, we determine the solution (v,μ,γ) of (17). The quantities A, f, v_0, $\hat{v}$, Ω and T_2 are given. The final time T_2 may be different to the end time T_1 from (16). The template function is often chosen as $\hat{v}(x)=u_0(x)$ or $\hat{v}(x)=u(x,T_1)$, where $u(\cdot,T_1)$ denotes the solution of (16) at the end time T_1. Also the initial data v_0 is often chosen as $v_0(x)=u_0(x)$ or $v_0(x)=u(x,T_1)$. The end time T_2 in (17) is often chosen such that the values of the quantities $\|v(\cdot,t)\|_{L^2(\Omega,\mathbb{C}^m)}$ and $|\mu_t(t)|$, cf. (15), are near 10^{-16}.

Step 3: (**Eigenvalue Problem**)

$$\begin{aligned}
&\lambda w(x)=A\triangle w(x)+i\mu_\star w(x)+Df(v_\star(x))w(x) &&,\ x\in\Omega,\\
&\frac{\partial w}{\partial n}(x)=0 &&,\ x\in\partial\Omega.
\end{aligned}\tag{18}$$

Finally, we determine (a predescribed number neig of) eigenvalues λ and associated eigenfunctions w of (18). The quantities A, $\mu_\star$, $v_\star$, f, Ω and neig are given. The profile $v_\star$ and the velocity

$\mu_\star$ come actually from a simulation, more precisely we set $\mu_\star := \mu(T_2)$ and $v_\star(\xi) := v(\xi, T_2)$, where $\mu(T_2)$ and $v(\cdot, T_2)$ denote two components of the solution of (17) at the end time T_2.
Note that equation (18) requires f to be holomorphic. This is very restrictive and in many applications not satisfied. For instance, the nonlinearities of the Ginzburg-Landau equation, the Schrödinger equation and the Gross-Pitaevskii equation are not holomorphic at the origin 0, but they are real-differentiable. This motivates to formulate the real-valued versions of (16), (17) and (18).

2.4 Spectra and eigenfunctions of oscillating waves

We now look for solutions (λ, w) of the eigenvalue problem

$$\lambda w(x) = [\mathcal{L}w](x) := A\triangle w(x) + i\mu_\star w(x) + Df(v_\star(x))w(x),\; x \in \mathbb{R}^d.$$

2.4.1 Point spectrum of oscillating waves on the imaginary axis

Case 1: (f holomorphic). Consider the oscillating wave equation

$$0 = A\triangle v_\star(x) + i\mu_\star v_\star(x) + f(v_\star(x)),\; x \in \mathbb{R}^d,\; d \geqslant 1, \tag{19}$$

with diffusion matrix $A \in \mathbb{C}^{m,m}$, nonlinearity $f : \mathbb{C}^m \to \mathbb{C}^m$, phase velocity $\mu_\star \in \mathbb{R}$ and profile $v_\star : \mathbb{R}^d \to \mathbb{C}^m$.
For $g \in \mathbb{R}$ we define the group action $[a(g)v](x) := e^{-ig}v(x)$. Applying $a(g)$ on both hand sides in (19) yields (provided that f satisfies $f(e^{i\varphi}z) = e^{i\varphi}f(z)$ for any $\varphi \in \mathbb{R}$ and $z \in \mathbb{C}^m$)

$$\begin{aligned} 0 &= a(g)\left[A\triangle v_\star(x) + i\mu_\star v_\star(x) + f(v_\star(x))\right] \\ &= A\triangle[a(g)v_\star(x)] + i\mu_\star[a(g)v_\star(x)] + f(a(g)v_\star(x)) \\ &= A\triangle e^{-ig}v_\star(x) + i\mu_\star e^{-ig}v_\star(x) + f(e^{-ig}v_\star(x)),\; x \in \mathbb{R}^d. \end{aligned} \tag{20}$$

Taking the derivative $\frac{d}{dg}$ in (20) evaluated at $g = 0$, we obtain (provided that $v_\star \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ and $f \in C^1(\mathbb{C}^m, \mathbb{C}^m)$ holomorphic)

$$\begin{aligned} 0 &= \left[\frac{d}{dg}\Big(A\triangle e^{-ig}v_\star(x) + i\mu_\star e^{-ig}v_\star(x) + f(e^{-ig}v_\star(x))\Big)\right]_{g=0} \\ &= -i\Big[A\triangle e^{-ig}v_\star(x) + i\mu_\star e^{-ig}v_\star(x) + Df(e^{-ig}v_\star(x))e^{-ig}v_\star(x)\Big]_{g=0} \\ &= -i\Big(A\triangle v_\star(x) + i\mu_\star v_\star(x) + Df(v_\star(x))v_\star(x)\Big),\; x \in \mathbb{R}^d. \end{aligned}$$

This leads to the equation

$$0 = A\triangle v_\star(x) + i\mu_\star v_\star(x) + Df(v_\star(x))v_\star(x),\; x \in \mathbb{R}^d.$$

Therefore, $(\lambda, w(x)) := (0, v_\star(x))$ solves the eigenvalue problem

$$\lambda w(x) = [\mathcal{L}w](x) := A\triangle w(x) + i\mu_\star w(x) + Df(v_\star(x))w(x),\; x \in \mathbb{R}^d, \tag{21}$$

i.e. the function $w(x) = v_\star(x)$ is an eigenfunction associated with the eigenvalue $\lambda = 0$, provided the $v_\star$ is not identically 0.
Procedure: Considering (19) as an operator equation, differentiating (19) w.r.t. (the function!!!) $v_\star$ and evaluating at $v_\star$ yields the solution $(\lambda, w(x)) := (0, v_\star(x))$ of (21).

Theorem 2.1 (Point spectrum of oscillating waves, complex version)**.** *Let $v_\star \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ be a nontrivial classical solution of (19) for some $A \in \mathbb{C}^{m,m}$, $\mu_\star \in \mathbb{R}$ and a holomorphic nonlinearity $f \in C^1(\mathbb{C}^m, \mathbb{C}^m)$ satisfying*

$$f(e^{i\varphi}z) = e^{i\varphi}f(z) \quad \text{for any} \quad \varphi \in \mathbb{R} \quad \text{and} \quad z \in \mathbb{C}^m.$$

Then

$$\lambda = 0, \quad w(x) = v_\star(x), \quad x \in \mathbb{R}^d$$

solves the eigenvalue problem (21)*. In particular, the algebraic multiplicity of the eigenvalue* $\lambda=0$ *is at least* 1*.*

Case 2: (f real-differentiable). Consider the real-valued version of (19) which is

$$0=\mathbf{A}\triangle\mathbf{v}_\star(x)+\mu_\star S_2\mathbf{v}_\star(x)+\mathbf{f}(\mathbf{v}_\star(x)),\; x\in\mathbb{R}^d,\; d\geqslant 1, \tag{22}$$

with diffusion matrix $\mathbf{A}=\begin{pmatrix}A_1 & -A_2\\ A_2 & A_1\end{pmatrix}\in\mathbb{R}^{2m,2m}$, skew-symmetric matrix $S_2=\begin{pmatrix}0 & -I_m\\ I_m & 0\end{pmatrix}\in\mathbb{R}^{2m,2m}$, nonlinearity $\mathbf{f}=\begin{pmatrix}f_1\\ f_2\end{pmatrix}:\mathbb{R}^{2m}\to\mathbb{R}^{2m}$, phase velocity $\mu_\star\in\mathbb{R}$ and profile $\mathbf{v}_\star:\mathbb{R}^d\to\mathbb{R}^{2m}$, where $A_1=\operatorname{Re}A$, $A_2=\operatorname{Im}A$, $f_1=\operatorname{Re}f$ and $f_2=\operatorname{Im}f$.
For $g\in\mathbb{R}$ we define the group action $[a(g)v](x):=E(g)v(x)$, where

$$E(g)=\begin{pmatrix}\cos(-g)I_m & -\sin(-g)I_m\\ \sin(-g)I_m & \cos(-g)I_m\end{pmatrix}\in\mathbb{R}^{2m,2m}.$$

Applying $a(g)$ on both hand sides in (22) yields (note that $E(g)\mathbf{A}=\mathbf{A}E(g)$, $E(g)S_2=S_2E(g)$ and $E(g)\mathbf{f}(\mathbf{v}_\star(x))=\mathbf{f}(E(g)\mathbf{v}_\star(x))$ since $f(e^{i\varphi}z)=e^{i\varphi}f(z)$ for any $\varphi\in\mathbb{R}$ and $z\in\mathbb{C}^m$)

$$\begin{aligned}0&=a(g)\left[\mathbf{A}\triangle\mathbf{v}_\star(x)+\mu_\star S_2\mathbf{v}_\star(x)+\mathbf{f}(\mathbf{v}_\star(x))\right]\\ &=E(g)\mathbf{A}\triangle\mathbf{v}_\star(x)+\mu_\star E(g)S_2\mathbf{v}_\star(x)+E(g)\mathbf{f}(\mathbf{v}_\star(x))\\ &=\mathbf{A}\triangle E(g)\mathbf{v}_\star(x)+\mu_\star S_2E(g)\mathbf{v}_\star(x)+\mathbf{f}(E(g)\mathbf{v}_\star(x)),\; x\in\mathbb{R}^d.\end{aligned}\tag{23}$$

Taking the derivative $\frac{d}{dg}$ in (23) evaluated at $g=0$, we obtain (provided that $v_\star\in C^2(\mathbb{R}^d,\mathbb{R}^{2m})$ and $f\in C^1(\mathbb{R}^{2m},\mathbb{R}^{2m})$ real-differentiable)

$$\begin{aligned}0&=\left[\frac{d}{dg}\Big(\mathbf{A}\triangle E(g)\mathbf{v}_\star(x)+\mu_\star S_2E(g)\mathbf{v}_\star(x)+\mathbf{f}(E(g)\mathbf{v}_\star(x))\Big)\right]_{g=0}\\ &=\Big[\mathbf{A}\triangle E'(g)\mathbf{v}_\star(x)+\mu_\star S_2E'(g)\mathbf{v}_\star(x)+\mathbf{Df}(E(g)\mathbf{v}_\star(x))E'(g)\mathbf{v}_\star(x)\Big]_{g=0}\\ &=-\Big(\mathbf{A}\triangle S_2\mathbf{v}_\star(x)+\mu_\star S_2^2\mathbf{v}_\star(x)+\mathbf{Df}(\mathbf{v}_\star(x))S_2\mathbf{v}_\star(x)\Big),\; x\in\mathbb{R}^d.\end{aligned}$$

where we used

$$E'(g)=\begin{pmatrix}\sin(-g)I_m & \cos(-g)I_m\\ -\cos(-g)I_m & \sin(-g)I_m\end{pmatrix},\quad [E'(g)]_{g=0}=\begin{pmatrix}0 & I_m\\ -I_m & 0\end{pmatrix}=-S_2.$$

This leads to the equation

$$0=\mathbf{A}\triangle S_2\mathbf{v}_\star(x)+\mu_\star S_2^2\mathbf{v}_\star(x)+\mathbf{Df}(\mathbf{v}_\star(x))S_2\mathbf{v}_\star(x),\; x\in\mathbb{R}^d.$$

Therefore, $(\lambda,w(x)):=(0,S_2\mathbf{v}_\star(x))$ solves the eigenvalue problem

$$\lambda w(x)=[\mathcal{L}w](x):=\mathbf{A}\triangle w(x)+\mu_\star S_2w(x)+\mathbf{Df}(\mathbf{v}_\star(x))w(x),\; x\in\mathbb{R}^d, \tag{24}$$

i.e. the function $w(x)=\mathbf{v}_\star(x)$ is an eigenfunction associated with the eigenvalue $\lambda=0$, provided the $\mathbf{v}_\star$ is not identically 0.
Procedure: Considering (22) as an operator equation, differentiating (22) w.r.t. (the function!!!) $\mathbf{v}_\star$ and evaluating at $S_2\mathbf{v}_\star$ yields the solution $(\lambda,w(x)):=(0,S_2\mathbf{v}_\star(x))$ of (24).

Theorem 2.2 (Point spectrum of oscillating waves, real version)**.** *Let* $\mathbf{v}_\star\in C^2(\mathbb{R}^d,\mathbb{R}^{2m})$ *be a nontrivial classical solution of* (22) *for some* $\mathbf{A}=\begin{pmatrix}A_1 & -A_2\\ A_2 & A_1\end{pmatrix}\in\mathbb{R}^{2m,2m}$*,* $A_1,A_2\in\mathbb{R}^{m,m}$*,* $\mu_\star\in\mathbb{R}$ *and a real-differentiable nonlinearity* $\mathbf{f}\in C^1(\mathbb{R}^{2m},\mathbb{R}^{2m})$ *satisfying*

$$\mathbf{f}(E(g)z)=E(g)\mathbf{f}(z)\quad\textit{for any}\quad g\in\mathbb{R}\quad\textit{and}\quad z\in\mathbb{R}^{2m}.$$

Then

$$\lambda=0,\quad w(x)=S_2v_\star(x),\quad x\in\mathbb{R}^d$$

solves the eigenvalue problem (24). In particular, the algebraic multiplicity of the eigenvalue $\lambda = 0$ is at least 1.

2.4.2 Essential spectrum of localized oscillating waves

Case 1: (f holomorphic). Consider the oscillating wave equation in $\mathbb{R}^d$

$$0 = A\triangle v_\star(x) + i\mu_\star v_\star(x) + f(v_\star(x)),\ x \in \mathbb{R}^d. \tag{25}$$

with diffusion matrix $A \in \mathbb{C}^{m,m}$, nonlinearity $f : \mathbb{C}^m \to \mathbb{C}^m$, constant asymptotic states $v_\infty \in \mathbb{C}^m$ (i.e. $f(v_\infty) = 0$), phase velocity $\mu_\star \in \mathbb{R}$ and profile $v_\star : \mathbb{R}^d \to \mathbb{C}^m$ satisfying $v_\star(x) \to v_\infty \in \mathbb{C}^m$ as $|x| \to \infty$.

Initial value problem: The main idea to detecting the essential spectrum of $\mathcal{L}$ is to look for solutions of

$$\begin{aligned} &v_t(x,t) = [\mathcal{L}v](x,t) := A\triangle v(x,t) + i\mu_\star v(x,t) + Df(v_\star(x))v(x,t) &&,\ x \in \mathbb{R}^d,\ t > 0,\\ &v(x,0) = v_0(x) &&,\ x \in \mathbb{R}^d, t = 0. \end{aligned} \tag{26}$$

Decomposition of $Df(v_\star(x))$: Introducing the matrix $Q(x) \in \mathbb{C}^{m,m}$ via

$$Q(x) := Df(v_\star(x)) - Df(v_\infty),\ x \in \mathbb{R}^d,$$

we obtain from (26)

$$v_t(x,t) = [\mathcal{L}_Q v](x,t) := A\triangle v(x,t) + i\mu_\star v(x,t) + Df(v_\infty)v(x,t) + Q(x)v(x,t),\ x \in \mathbb{R}^d,\ t > 0. \tag{27}$$

Limiting operator (simplified operator, far-field operator): Since the essential spectrum depends only on the limiting equation for $|x| \to \infty$, we let formally $|x| \to \infty$ (but only in the coefficient matrices). Since $Q(x) \to 0$ as $|x| \to \infty$, we can drop the term $Q(x)$ in (27) and obtain

$$v_t(x,t) = [\mathcal{L}_\infty v](x,t) := A\triangle v(x,t) + i\mu_\star v(x,t) + Df(v_\infty)v(x,t),\ x \in \mathbb{R}^d,\ t > 0. \tag{28}$$

Fourier transform: Since we seek for bounded solutions of (28), we perform a Fourier transformation in space and time. Inserting the Fourier transform

$$v(x,t) = e^{\lambda t}e^{i\omega\cdot x}\hat{v},\ \lambda \in \mathbb{C},\ \omega \in \mathbb{R}^d,\ \hat{v} \in \mathbb{C}^m,\ |\hat{v}| = 1,\ \omega \cdot x := \sum_{j=1}^d \omega_j x_j \tag{29}$$

into (28) and dividing by $e^{\lambda t}e^{i\omega\cdot x}$ yields a finite dimensional eigenvalue problem

$$\lambda\hat{v} = \left(-|\omega|^2 A + i\mu_\star I_m + Df(v_\infty)\right)\hat{v}$$

Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

$$\det\left(-|\omega|^2 A + i\mu_\star I_m + Df(v_\infty) - \lambda I_m\right) = 0$$

for some $\omega \in \mathbb{R}^d$ belongs to the essential spectrum of $\mathcal{L}$, see Figure 2.1.

Theorem 2.3 (Essential spectrum of oscillating waves, complex version)**.** *Let $v_\star \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ be a nontrivial classical solution of (25) satisfying $v_\star(x) \to v_\infty$ as $|x| \to \infty$ for some $v_\infty \in \mathbb{C}^m$ and let $A \in \mathbb{C}^{m,m}$, $\mu_\star \in \mathbb{R}$ and a holomorphic function $f \in C^1(\mathbb{C}^m, \mathbb{C}^m)$ with $f(v_\infty) = 0$. Then, the line*

$$\begin{aligned} S_\infty &:= \left\{\lambda \in \mathbb{C} \mid \det\left(-|\omega|^2 A + i\mu_\star I_m + Df(v_\infty) - \lambda I_m\right) = 0 \text{ for some } \omega \in \mathbb{R}^d\right\}\\ &= \left\{\lambda \in \sigma\left(-|\omega|^2 A + i\mu_\star I_m + Df(v_\infty)\right) \mid \omega \in \mathbb{R}^d\right\} \end{aligned}$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$, i.e. $S_\infty \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$.

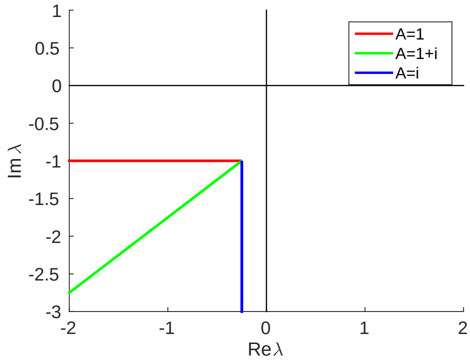

FIGURE 2.1. Subset S_∞ of the essential spectrum of an oscillating wave for a scalar-valued ($m = 1$), holomorphic nonlinearity and for parameters $\mu_\star = -1$, $Df(v_\infty) = -\frac{1}{4}$ and $A = 1$ (red), $A = 1 + i$ (green), $A = i$ (blue).

Case 2: (f real-differentiable). Consider the real-valued version of (25) which is

$$0 = \mathbf{A}\triangle\mathbf{v}_\star(x) + \mu_\star S_2\mathbf{v}_\star(x) + \mathbf{f}(\mathbf{v}_\star(x)),\ x \in \mathbb{R}^d,\ d \geqslant 1, \tag{30}$$

with diffusion matrix $\mathbf{A} = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{R}^{2m,2m}$, skew-symmetric matrix $S_2 = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \in \mathbb{R}^{2m,2m}$, nonlinearity $\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, constant asymptotic states $\mathbf{v}_\infty \in \mathbb{R}^{2m}$ (i.e. $\mathbf{f}(\mathbf{v}_\infty) = 0$), phase velocity $\mu_\star \in \mathbb{R}$ and profile $\mathbf{v}_\star : \mathbb{R}^d \to \mathbb{R}^{2m}$ satisfying $\mathbf{v}_\star(x) \to \mathbf{v}_\infty \in \mathbb{R}^{2m}$ as $|x| \to \infty$.

Initial value problem: The main idea to detecting the essential spectrum of $\mathcal{L}$ is to look for solutions of

$$\begin{aligned} &\mathbf{v}_t(x,t) = [\mathcal{L}\mathbf{v}](x,t) := \mathbf{A}\triangle\mathbf{v}(x,t) + \mu_\star S_2\mathbf{v}(x,t) + \mathbf{Df}(\mathbf{v}_\star(x))\mathbf{v}(x,t) &&,\ x \in \mathbb{R}^d,\ t > 0, \\ &\mathbf{v}(x,0) = \mathbf{v_0}(x) &&,\ x \in \mathbb{R}^d, t = 0. \end{aligned} \tag{31}$$

Decomposition of $\mathbf{Df}(\mathbf{v}_\star(x))$: Introducing the matrix $\mathbf{Q}(x) \in \mathbb{R}^{2m,2m}$ via

$$\mathbf{Q}(x) := \mathbf{Df}(\mathbf{v}_\star(x)) - \mathbf{Df}(\mathbf{v}_\infty),\ x \in \mathbb{R}^d,$$

we obtain from (31)

$$\mathbf{v}_t(x,t) = [\mathcal{L}_\mathbf{Q}\mathbf{v}](x,t) := \mathbf{A}\triangle\mathbf{v}(x,t) + \mu_\star S_2\mathbf{v}(x,t) + \mathbf{Df}(\mathbf{v}_\infty)\mathbf{v}(x,t) + \mathbf{Q}(x)\mathbf{v}(x,t),\ x \in \mathbb{R}^d,\ t > 0. \tag{32}$$

Limiting operator (simplified operator, far-field operator): Since the essential spectrum depends only on the limiting equation for $|x| \to \infty$, we let formally $|x| \to \infty$ (but only in the coefficient matrices). Since $\mathbf{Q}(x) \to 0$ as $|x| \to \infty$, we can drop the term in (32) and obtain

$$\mathbf{v}_t(x,t) = [\mathcal{L}_\infty\mathbf{v}](x,t) := \mathbf{A}\triangle\mathbf{v}(x,t) + \mu_\star S_2\mathbf{v}(x,t) + \mathbf{Df}(\mathbf{v}_\infty)\mathbf{v}(x,t),\ x \in \mathbb{R}^d,\ t > 0. \tag{33}$$

Fourier transform: Since we seek for bounded solutions of (33), we perform a Fourier transformation in space and time. Inserting the Fourier transform

$$\mathbf{v}(x,t) = e^{\lambda t}e^{i\omega\cdot x}\hat{v},\ \lambda \in \mathbb{C},\ \omega \in \mathbb{R}^d,\ \hat{\mathbf{v}} \in \mathbb{C}^{2m},\ |\hat{\mathbf{v}}| = 1,\ \omega\cdot x := \sum_{j=1}^{d}\omega_j x_j \tag{34}$$

into (33) and dividing by $e^{\lambda t}e^{i\omega\cdot x}$ yields a finite dimensional eigenvalue problem

$$\lambda\hat{v} = \left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty)\right)\hat{\mathbf{v}}$$

Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

$$\det\left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty) - \lambda I_{2m}\right) = 0$$

for some $\omega \in \mathbb{R}^d$ belongs to the essential spectrum of $\mathcal{L}$, see Figure 2.2.

Theorem 2.4 (Essential spectrum of oscillating waves, real version)**.** *Let* $\mathbf{v}_\star \in C^2(\mathbb{R}^d, \mathbb{R}^{2m})$ *be a nontrivial classical solution of* (30) *satisfying* $\mathbf{v}_\star(x) \to \mathbf{v}_\infty$ *as* $|x| \to \infty$ *for some* $\mathbf{v}_\infty \in \mathbb{R}^m$ *and let* $\mathbf{A} = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \in \mathbb{R}^{2m,2m}$, $A_1, A_2 \in \mathbb{R}^{m,m}$, $\mu_\star \in \mathbb{R}$ *and a real-differentiable function* $\mathbf{f} \in C^1(\mathbb{R}^{2m}, \mathbb{R}^{2m})$ *with* $\mathbf{f}(\mathbf{v}_\infty) = 0$*. Then, the line*

$$\begin{aligned} S_\infty &:= \left\{\lambda \in \mathbb{C} \mid \det\left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty) - \lambda I_{2m}\right) = 0 \text{ for some } \omega \in \mathbb{R}^d\right\} \\ &= \left\{\lambda \in \sigma\left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty)\right) \mid \omega \in \mathbb{R}^d\right\} \end{aligned}$$

belongs to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ *of* $\mathcal{L}$*, i.e.* $S_\infty \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$*.*

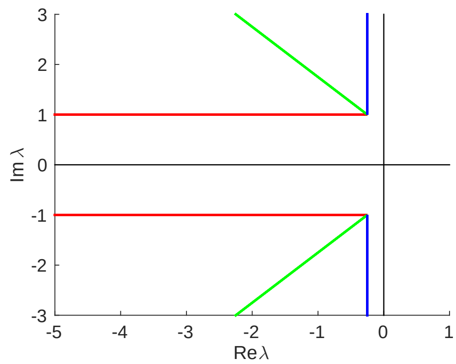

FIGURE 2.2. Subset S_∞ of the essential spectrum of an oscillating wave for a scalar-valued ($m = 1$), real-differentiable nonlinearity and for parameters $\mu_\star = -1$, $\mathbf{Df}(\mathbf{v}_\infty) = -\frac{1}{4}I_2$ and $\mathbf{A} = I_2$ (red), $\mathbf{A} = I_2 + S_2$ (green), $\mathbf{A} = S_2$ (blue).

Example 2.5 (Cubic-quintic Ginzburg-Landau equation)**.** Consider the cubic-quintic Ginzburg-Landau equation

$$u_t = \alpha \triangle u + \delta u + \beta|u|^2 u + \gamma|u|^4 u,\; x \in \mathbb{R}^d,\; t \geqslant 0,$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$, $\delta \in \mathbb{R}$ and $u = u(x,t) \in \mathbb{C}$. The nonlinearity

$$f : \mathbb{C} \to \mathbb{C}, \quad f(u) = (\delta + \beta|u|^2 + \gamma|u|^4)u,$$

which is of the form $f(u) = g(|u|^2)u$ with polynomial $g(w) := \delta + \beta w + \gamma w^2$, is not holomorphic at $u = 0$, but it is real-differentiable. Decomposing

$$u = u_1 + iu_2, \quad \alpha = a_1 + ia_2, \quad \beta = b_1 + ib_2, \quad \gamma = c_1 + ic_2$$

with $u_1, u_2, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and introducing

$$\mathbf{A} = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} c_1 & -c_2 \\ c_2 & c_1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{2,2}$ and $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^2$, the associated real-valued system reads as

$$\mathbf{u}_t = \mathbf{A}\triangle\mathbf{u} + \delta\mathbf{u} + \mathbf{B}|\mathbf{u}|^2\mathbf{u} + \mathbf{C}|\mathbf{u}|^4\mathbf{u}, \quad x \in \mathbb{R}^d,\; t \geqslant 0,$$

where the (real-valued) nonlinearity is given by

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{f}(\mathbf{u}) = (\delta I_2 + \mathbf{B}|\mathbf{u}|^2 + \mathbf{C}|\mathbf{u}|^4)\mathbf{u}$$

For the parameters

$$\alpha = 1, \quad \beta = 3, \quad \gamma = -\frac{11}{4} + i, \quad \delta = -\frac{1}{10}$$

the cubic-quintic Ginzburg-Landau equation has an oscillating pulse solution $u_\star(x,t) = e^{-\mu_\star t}v_\star(x)$ with velocity $\mu_\star \approx -1.3$ and profile $v_\star$ conntecting the asymptotic state $v_\infty = 0$, i.e. $v_\star(x) \to v_\infty$ as $|x| \to \infty$. Note that neither the profile nor the velocity are given explicitly. The (real-valued) nonlinearity $\mathbf{f}$ satisfies $\mathbf{f}(\mathbf{v}_\infty) = 0$ and $\mathbf{Df}(\mathbf{v}_\infty) = \delta I_2$. The dispersion relation states that

$$S_\infty := \left\{\lambda \in \mathbb{C} \mid \det\left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty) - \lambda I_2\right) = 0 \text{ for some } \omega \in \mathbb{R}^d\right\}$$

belongs to $\sigma_{\mathrm{ess}}(\mathcal{L})$. Due to

$$0 = \det\begin{pmatrix} -|\omega|^2 a_1 + \delta - \lambda & |\omega|^2 a_2 - \mu_\star \\ -|\omega|^2 a_2 + \mu_\star & -|\omega|^2 a_1 + \delta - \lambda \end{pmatrix} = (-|\omega|^2 a_1 + \delta - \lambda)^2 + (|\omega|^2 a_2 - \mu_\star)^2,$$

every $\lambda_{1,2} = \lambda_{1,2}(\omega) \in \mathbb{C}$ satisfying

$$\lambda_{1,2} = -|\omega|^2 a_1 + \delta \pm i(|\omega|^2 a_2 - \mu_\star)$$

for some $\omega \in \mathbb{R}^d$ belongs to the essential spectrum of $\mathcal{L}$. The essential spectrum of the oscillating pulse in the cubic-quintic Ginzburg-Landau equation is illustrated in Figure 2.3.

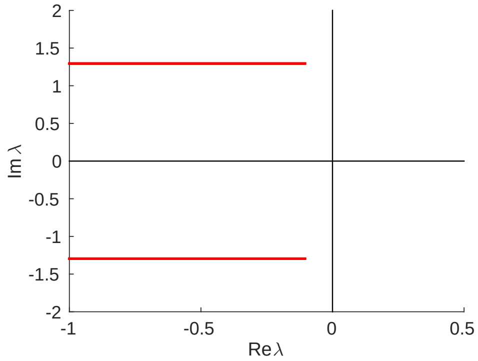

FIGURE 2.3. Essential spectrum of the cubic-quintic Ginzburg-Landau equation for an oscillating pulse with $\alpha = 1$, $\beta = 3$, $\gamma = -\frac{11}{4} + i$, $\delta = -\frac{1}{10}$ and $d = 1$

Example 2.6 (Nonlinear Schrödinger equation)**.** Consider the nonlinear Schrödinger equation

$$u_t = i\triangle u + \beta|u|^2 u,\ x \in \mathbb{R}^d,\ t \geqslant 0,$$

for some $\beta \in \mathbb{C}$ and $u = u(x,t) \in \mathbb{C}$. The nonlinearity

$$f : \mathbb{C} \to \mathbb{C}, \quad f(u) = \beta|u|^2 u,$$

which is of the form $f(u) = g(|u|^2)u$ with polynomial $g(w) := \beta w$, is not holomorphic at $u = 0$, but it is real-differentiable. Decomposing

$$u = u_1 + iu_2, \quad \beta = b_1 + ib_2,$$

with $u_1, u_2, b_1, b_2 \in \mathbb{R}$ and introducing

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2,2}$ and $\mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^2$, the associated real-valued system reads as

$$\mathbf{u}_t = \mathbf{A}\triangle\mathbf{u} + \mathbf{B}|\mathbf{u}|^2\mathbf{u}, \quad x \in \mathbb{R}^d,\ t \geqslant 0,$$

where the (real-valued) nonlinearity is given by

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{f}(\mathbf{u}) = \mathbf{B}|\mathbf{u}|^2\mathbf{u}$$

For the parameter $\beta = 2i$ the nonlinear Schrödinger equation has an oscillating pulse solution $u_\star(x,t) = e^{-\mu_\star t} v_\star(x)$ with velocity $\mu_\star \approx -1$ and profile $v_\star$ conntecting the asymptotic state $v_\infty = 0$, i.e. $v_\star(x) \to v_\infty$ as $|x| \to \infty$. The (real-valued) nonlinearity $\mathbf{f}$ satisfies $\mathbf{f}(\mathbf{v}_\infty) = 0$ and $\mathbf{Df}(\mathbf{v}_\infty) = 0$. The (real version of the) dispersion relation states that

$$S_\infty := \left\{\lambda \in \mathbb{C} \mid \det\left(-|\omega|^2\mathbf{A} + \mu_\star S_2 + \mathbf{Df}(\mathbf{v}_\infty) - \lambda I_2\right) = 0 \text{ for some } \omega \in \mathbb{R}^d\right\}$$

belongs to $\sigma_{\mathrm{ess}}(\mathcal{L})$. Due to

$$0 = \det\begin{pmatrix} -\lambda & |\omega|^2 - \mu_\star \\ -(|\omega|^2 - \mu_\star) & -\lambda \end{pmatrix} = \lambda^2 + \left||\omega|^2 - \mu_\star\right|^2,$$

every $\lambda_{1,2} = \lambda_{1,2}(\omega) \in \mathbb{C}$ satisfying

$$\lambda_{1,2} = \pm i \left| |\omega|^2 - \mu_\star \right|$$

for some $\omega \in \mathbb{R}^d$ belongs to the essential spectrum of $\mathcal{L}$. The essential spectrum of the oscillating pulse in the nonlinear Schrödinger equation is illustrated in Figure 2.4.

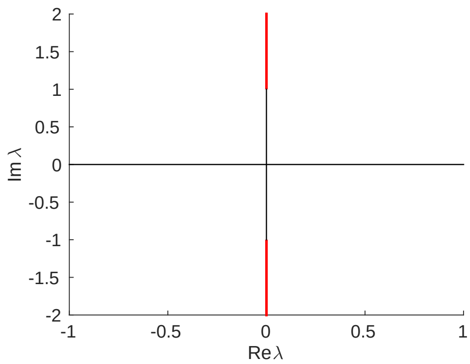

FIGURE 2.4. Essential spectrum of the nonlinear Schrödinger equation for an oscillating pulse with $\beta = 2i$ and $d = 1$