Mathematical Modelling and Simulation with Comsol Multiphysics II

Winter term 2015/2016

Dr. Denny Otten

26.10.2015

1. Freezing Traveling Waves in Reaction Diffusion Systems

1.1 Traveling waves in reaction diffusion systems

Consider a system of reaction diffusion equations in one space dimension

(1)
$$\begin{aligned} u_t(x,t) &= Au_{xx}(x,t) + f(u(x,t)) \quad &, x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x) \quad &, x \in \mathbb{R},\ t = 0, \end{aligned}$$

with diffusion matrix $A \in \mathbb{R}^{m,m}$, nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$, initial data $u_0 : \mathbb{R} \to \mathbb{R}^m$ and solution $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$.

We are interested in traveling wave solutions of (1): A traveling wave of (1) is a solution $u_\star : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ of the form

(2)
$$u_\star(x,t) = v_\star(x - \mu_\star t) \quad , x \in \mathbb{R},\ t \geqslant 0,$$

with

$$\lim_{\xi\to+\infty} v_\star(\xi) = v_+ \in \mathbb{R}^m, \quad \lim_{\xi\to-\infty} v_\star(\xi) = v_- \in \mathbb{R}^m, \quad f(v_+) = f(v_-) = 0.$$

The function $v_\star : \mathbb{R} \to \mathbb{R}^m$ is called the profile and $\mu_\star \in \mathbb{R}$ the (translational) velocity of the traveling wave. The traveling wave $u_\star$ is called a traveling pulse, if $v_+ = v_-$, and a traveling front, if $v_+ \neq v_-$. The wave travels to the left, if $\mu_\star < 0$ and to the right, if $\mu_\star > 0$. In case $\mu_\star = 0$, in which we are not interested, $u_\star$ is called a standing wave.

Our aim is to approximate traveling wave solutions of (1). The idea for approximating the traveling wave $u_\star$ is to determine the profile $v_\star$ and the velocity $\mu_\star$. This requires to transform (1) into a co-moving coordinate system.

Transforming (1) via $u(x,t) = v(\xi,t)$ with $\xi := x - \mu_\star t$ in a co-moving frame yields

(3)
$$\begin{aligned} v_t(\xi,t) &= Av_{\xi\xi}(\xi,t) + \mu_\star v_\xi(\xi,t) + f(v(\xi,t)) \quad &, \xi \in \mathbb{R},\ t > 0, \\ v(\xi,0) &= u_0(\xi) \quad &, \xi \in \mathbb{R},\ t = 0. \end{aligned}$$

Inserting (2) into (1) shows, that $v_\star$ is a stationary solution of (3), i.e.

(4)
$$0 = Av_{\star,\xi\xi}(\xi) + \mu_\star v_{\star,\xi}(\xi) + f(v_\star(\xi)) \quad , \xi \in \mathbb{R}.$$

We are also interested in nonlinear stability of traveling waves. It is well known from the literature, that in many cases spectral stability implies nonlinear stability. For investigating spectral stability of a traveling wave, we must analyze the spectrum of the linearization of the right hand side in (3) at the wave profile $v_\star$, i.e.

$$[\mathcal{L}w](\xi) = Aw_{\xi\xi}(\xi) + \mu_\star w_\xi(\xi) + Df(v_\star(\xi))w(\xi) \quad , \xi \in \mathbb{R}.$$

This requires to find solutions (λ, w) of the eigenvalue problem

(5)
$$\lambda w(\xi) = Aw_{\xi\xi}(\xi) + \mu_\star w_\xi(\xi) + Df(v_\star(\xi))w(\xi) \quad , \xi \in \mathbb{R},$$

with eigenfunction $w : \mathbb{R} \to \mathbb{C}^m$ and eigenvalue $\lambda \in \mathbb{C}$.

Approximating $v_\star$ via (3) requires the knowledge about the velocity $\mu_\star$ which is in general unknown. This motivates to introduce the freezing method, whose idea is it to approximate the profile $v_\star$ and the velocity $\mu_\star$ simultaneously.

1.2 Freezing method for traveling waves

Consider again a system of reaction diffusion equations in one space dimension, cf. (1),

$$
(6)\qquad \begin{aligned} u_t(x,t) &= Au_{xx}(x,t) + f(u(x,t)) &&, x\in\mathbb{R},\, t>0,\\ u(x,0) &= u_0(x) &&, x\in\mathbb{R},\, t=0. \end{aligned}
$$

Introducing new unknowns $\gamma(t)\in\mathbb{R}$ (position) and $v(\xi,t)\in\mathbb{R}^m$ (profile) via the traveling wave ansatz

$$
(7)\qquad u(x,t) = v(\xi,t),\, \xi := x-\gamma(t) \quad, x\in\mathbb{R},\, t\geqslant 0
$$

and inserting (7) into (6) yields

$$
(8)\qquad v_t(\xi,t) = Av_{\xi\xi}(\xi,t) + \gamma_t(t)v_\xi(\xi,t) + f(v(\xi,t)) \quad, \xi\in\mathbb{R},\, t>0.
$$

It is convenient to introduce a further unknown $\mu(t)\in\mathbb{R}$ (velocity) via $\gamma_t(t)=\mu(t)$. Then, (8) reads as

$$
(9)\qquad \begin{aligned} v_t(\xi,t) &= Av_{\xi\xi}(\xi,t) + \mu(t)v_\xi(\xi,t) + f(v(\xi,t)) &&, \xi\in\mathbb{R},\, t>0,\\ \gamma_t(t) &= \mu(t) &&, t>0. \end{aligned}
$$

Equ. (9) has to be equipped with suitable initial data. Requiring $\gamma(0)=0$, (7) and (6) imply

$$
(10)\qquad v(\xi,0) = u_0(\xi) \quad, \xi\in\mathbb{R},\, t=0.
$$

Collecting the equations (9), $\gamma(0)=0$ and (10) we obtain

$$
(11)\qquad \begin{aligned} v_t(\xi,t) &= Av_{\xi\xi}(\xi,t) + \mu(t)v_\xi(\xi,t) + f(v(\xi,t)) &&, \xi\in\mathbb{R},\, t>0,\\ v(\xi,0) &= u_0(\xi) &&, \xi\in\mathbb{R},\, t=0,\\ \gamma_t(t) &= \mu(t) &&, t>0,\\ \gamma(0) &= 0 &&, t=0. \end{aligned}
$$

(11) contains the equations for v and γ. But so far, the system (11) is not well-posed, since there is still no equation for μ. To determine μ we require an additional algebraic constraint, a so called phase condition: For this purpose let $\hat{v}:\mathbb{R}\to\mathbb{R}^m$ be a template function, e.g. $\hat{v}=u_0$. The idea of the phase condition is to choose $v(\cdot,t)$ such that

$$
\min_{g\in\mathbb{R}} \|v(\cdot,t)-\hat{v}(\cdot-g)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)} = \|v(\cdot,t)-\hat{v}(\cdot)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)} \quad, t\geqslant 0.
$$

A necessary condition to guarantee that the left hand side attains its minimum at $g=0$ is that the first derivative of $\|v(\cdot,t)-\hat{v}(\cdot-g)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)}$ evaluated at $g=0$ vanishes, i.e. for all $t\geqslant 0$

$$
(12)\qquad 0 \overset{!}{=} \left[\frac{d}{dg}\left(v(\cdot,t)-\hat{v}(\cdot-g), v(\cdot,t)-\hat{v}(\cdot-g)\right)_{L^2(\mathbb{R},\mathbb{R}^m)}\right]_{g=0} = 2\left(v(\cdot,t)-\hat{v},\hat{v}_\xi\right)_{L^2(\mathbb{R},\mathbb{R}^m)}.
$$

Combining (11) and (12) yields a partial differential algebraic evolution equation (PDAE)

$$
(13)\qquad \begin{aligned} v_t(\xi,t) &= Av_{\xi\xi}(\xi,t) + \mu(t)v_\xi(\xi,t) + f(v(\xi,t)) &&, \xi\in\mathbb{R},\, t>0,\\ v(\xi,0) &= u_0(\xi) &&, \xi\in\mathbb{R},\, t=0,\\ 0 &= \left(v(\cdot,t)-\hat{v},\hat{v}_\xi\right)_{L^2(\mathbb{R},\mathbb{R}^m)} &&, t\geqslant 0,\\ \gamma_t(t) &= \mu(t) &&, t>0,\\ \gamma(0) &= 0 &&, t=0. \end{aligned}
$$

The last two equations in (13) for the position γ are decoupled from the other equations in (13). Therefore, the γ-equation can be solved in a postprocessing step. The γ-equation is called the reconstruction equation for the traveling wave. Since $(v_\star,\mu_\star)$ satisfy

$$
\begin{aligned} 0 &= Av_{\star,\xi\xi}(\xi) + \mu_\star v_{\star,\xi}(\xi) + f(v_\star(\xi))\,,\ \xi\in\mathbb{R},\\ 0 &= \left(v_\star-\hat{v},\hat{v}_\xi\right)_{L^2(\mathbb{R},\mathbb{R}^m)}, \end{aligned}
$$

we expect for stability reasons, that the solution (v, μ, γ) of (13) satisfies

$$v(t) \to v_\star, \quad \mu(t) \to \mu_\star \quad \text{as} \quad t \to \infty. \tag{14}$$

As an indicator for the convergence in (14) we check the quantities

$$\|v_t(\cdot, t)\|_{L^2(\mathbb{R},\mathbb{R}^m)} \quad \text{and} \quad |\mu_t(t)| \tag{15}$$

at each time instance t during the computation. In fact, both of these quantities should be small ($\approx 10^{-16}$), since $v_\star$ and $\mu_\star$ do not vary in time.

1.3 Numerical approximation of traveling waves via freezing method

Solving (1), (13) and (5) numerically, requires to truncate these equations to bounded domains. Let $\Omega \subset \mathbb{R}$ be a bounded open domain, then (1) must be satisfied for $x \in \Omega$, and equations (13) and (5) for $\xi \in \Omega$. To guarantee the well-posedness of these problems, we must equip the equations with appropriate boundary conditions. Normally, we choose homogeneous Neumann boundary conditions (also known as no-flux boundary conditions), i.e.

$$u_x(x) = 0,\ x \in \partial\Omega, \quad v_\xi(\xi) = 0,\ \xi \in \partial\Omega.$$

In this context, $\partial\Omega$ denotes the boundary of Ω and $\overline{\Omega}$ the closure of Ω, e.g. $\Omega = (a, b)$ with $-\infty < a < b < \infty$ then $\partial\Omega = \{a, b\}$ and $\overline{\Omega} = [a, b]$. Numerically, we solve the following equations:

Step 1: (Nonfrozen Equation)

$$\begin{aligned} &u_t(x,t) = Au_{xx}(x,t) + f(u(x,t)) && ,\ x \in \Omega,\ t \in (0, T_1], \\ &u(x,0) = u_0(x) && ,\ x \in \overline{\Omega},\ t = 0, \\ &u_x(x,t) = 0 && ,\ x \in \partial\Omega,\ t \in [0, T_1]. \end{aligned} \tag{16}$$

First, we determine the solution u of (16). The quantities A, f, u_0, Ω and T_1 are given.

Step 2: (Frozen Equation)

$$\begin{aligned} &v_t(\xi,t) = Av_{\xi\xi}(\xi,t) + \mu(t)v_\xi(\xi,t) + f(v(\xi,t)) && ,\ \xi \in \Omega,\ t \in (0, T_2], \\ &v(\xi,0) = v_0(\xi),\ \xi \in \overline{\Omega} && ,\ t = 0, \\ &v_\xi(\xi,t) = 0 && ,\ \xi \in \partial\Omega,\ t \in [0, T_2], \\ &0 = (v(\cdot,t) - \hat{v}, \hat{v}_\xi)_{L^2(\Omega,\mathbb{R}^m)} && ,\ t \in [0, T_2], \\ &\gamma_t(t) = \mu(t) && ,\ t \in (0, T_2], \\ &\gamma(0) = 0 && ,\ t = 0. \end{aligned} \tag{17}$$

Then, we determine the solution (v, μ, γ) of (17). The quantities A, f, v_0, $\hat{v}$, Ω and T_2 are given. The final time T_2 may be different to the end time T_1 from (16). The template function is often chosen as $\hat{v}(\xi) = u_0(\xi)$ or $\hat{v}(\xi) = u(\xi, T_1)$, where $u(\cdot, T_1)$ denotes the solution of (16) at the end time T_1. Also the initial data v_0 is often chosen as $v_0(\xi) = u_0(\xi)$ or $v_0(\xi) = u(\xi, T_1)$. The end time T_2 in (17) is often chosen such that the values of the quantities $\|v(\cdot, t)\|_{L^2(\Omega,\mathbb{R}^m)}$ and $|\mu_t(t)|$, cf. (15), are near 10^{-16}.

Step 3: (Eigenvalue Problem)

$$\begin{aligned} &\lambda w(\xi) = Aw_{\xi\xi}(\xi) + \mu_\star w_\xi(\xi) + Df(v_\star(\xi))w(\xi) && ,\ \xi \in \Omega, \\ &w_\xi(\xi) = 0 && ,\ \xi \in \partial\Omega. \end{aligned} \tag{18}$$

Finally, we determine (a predescribed number neig of) eigenvalues λ and associated eigenfunctions w of (18). The quantities A, $\mu_\star$, $v_\star$, f, Ω and neig are given. The profile $v_\star$ and the velocity

$\mu_\star$ come actually from a simulation, more precisely we set $\mu_\star := \mu(T_2)$ and $v_\star(\xi) := v(\xi, T_2)$, where $\mu(T_2)$ and $v(\cdot, T_2)$ denote two components of the solution of (17) at the end time T_2.

1.4 Spectra and eigenfunctions of traveling waves

We now look for solutions (λ, w) of the eigenvalue problem

$$\lambda w(x) = [\mathcal{L}w](x) := A\triangle w(x) + \mu_\star^T \nabla w(x) + Df(v_\star(x))w(x),\ x \in \mathbb{R}^d.$$

1.4.1 Point spectrum of traveling waves on the imaginary axis

Consider the traveling wave equation

$$0 = A\triangle v_\star(x) + \mu_\star^T \nabla v_\star(x) + f(v_\star(x)),\ x \in \mathbb{R}^d,\ d \geqslant 1, \tag{19}$$

with diffusion matrix $A \in \mathbb{R}^{m,m}$, nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$, translational velocity $\mu_\star \in \mathbb{R}^d$ and profile $v_\star : \mathbb{R}^d \to \mathbb{R}^m$.
For $g \in \mathbb{R}^d$ we define the group action $[a(g)v](x) := v(x-g)$. Applying $a(g)$ on both hand sides in (19) yields

$$\begin{aligned} 0 &= a(g)\left[A\triangle v_\star(x) + \mu_\star^T \nabla v_\star(x) + f(v_\star(x))\right] \\ &= A\triangle[a(g)v_\star(x)] + \mu_\star^T \nabla[a(g)v_\star(x)] + a(g)f(v_\star(x)) \\ &= A\triangle v_\star(x-g) + \mu_\star^T \nabla v_\star(x-g) + f(v_\star(x-g)),\ x \in \mathbb{R}^d. \end{aligned} \tag{20}$$

Taking the derivative $\frac{d}{dg}$ in (20) evaluated at $g = 0$, we obtain (provided that $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$)

$$\begin{aligned} 0 &= \left[\frac{d}{dg}\left(A\triangle v_\star(x-g) + \mu_\star^T \nabla v_\star(x-g) + f(v_\star(x-g))\right)\right]_{g=0} \\ &= \left[-\left(A\triangle D_j v_\star(x-g) + \mu_\star^T \nabla D_j v_\star(x-g) + Df(v_\star(x-g))D_j v_\star(x-g)\right)_{j=1,\ldots,d}\right]_{g=0} \\ &= -\left(A\triangle D_j v_\star(x) + \mu_\star^T \nabla D_j v_\star(x) + Df(v_\star(x))D_j v_\star(x)\right)_{j=1,\ldots,d},\ x \in \mathbb{R}^d. \end{aligned}$$

This leads to a total of d equations

$$\begin{aligned} 0 &= A\triangle D_j v_\star(x) + \mu_\star^T \nabla D_j v_\star(x) + Df(v_\star(x))D_j v_\star(x) \\ &= D_j\left(A\triangle v_\star(x) + \mu_\star^T \nabla v_\star(x) + f(v_\star(x))\right),\ x \in \mathbb{R}^d,\ j = 1, \ldots, d. \end{aligned}$$

Therefore, $(\lambda, w(x)) := (0, D_j v_\star(x))$, $j = 1, \ldots, d$, solves the eigenvalue problem

$$\lambda w(x) = [\mathcal{L}w](x) := A\triangle w(x) + \mu_\star^T \nabla w(x) + Df(v_\star(x))w(x),\ x \in \mathbb{R}^d, \tag{21}$$

i.e. the function $w(x) = D_j v_\star(x)$ is an eigenfunction associated with the eigenvalue $\lambda = 0$, provided the $v_\star$ is nontrivial (i.e. not constant), since otherwise we have $w(x) = 0$.
Procedure: Applying D_j to (19) yields the solution $(\lambda, w(x)) := (0, D_j v_\star(x))$ of (21).

Theorem 1.1 (Point spectrum of traveling waves)**.** *Let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ be a nontrivial classical solution of (19) for some $A \in \mathbb{R}^{m,m}$, $\mu_\star \in \mathbb{R}^d$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. Then*

$$\lambda = 0, \quad w(x) = D_j v_\star(x), \quad x \in \mathbb{R}^d, \quad j = 1, \ldots, d$$

solves the eigenvalue problem (21). In particular, the algebraic multiplicity of the eigenvalue $\lambda = 0$ is greater or equal d.

Example 1.2 (Nagumo equation)**.** The Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-b),\ x \in \mathbb{R},\ t \geqslant 0,\ 0 < b < 1,$$

has an explicit traveling wave solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$v_\star(x) = \frac{1}{1 + e^{-\frac{x}{\sqrt{2}}}}, \quad \mu_\star = -\sqrt{2}\left(\frac{1}{2} - b\right) \quad \text{(Huxley wave)},$$

i.e. $v_\star$ and $\mu_\star$ solve the associated traveling wave equation

$$0 = v_{\star,xx}(x) + \mu_\star v_{\star,x}(x) + v_\star(x)\left(1 - v_\star(x)\right)\left(v_\star(x) - b\right), \; x \in \mathbb{R}.$$

The eigenvalue problem for the linearization

$$\lambda w(x) = w_{xx}(x) + \mu_\star w_x(x) - 3w^2(x) + 2(b+1)w(x) - b, \; x \in \mathbb{R},$$

(with $f(u) = u(1-u)(u-b)$ and $Df(u) = -3u^2 + 2(b+1)u - b$) has the solution

$$\lambda = 0, \quad w(x) = v_{\star,x}(x) = \frac{1}{\sqrt{2}} \frac{e^{-\frac{x}{\sqrt{2}}}}{\left(1 + e^{-\frac{x}{\sqrt{2}}}\right)^2}, \quad x \in \mathbb{R}.$$

Consider the traveling wave equation in divergence form

(22) $$0 = A\nabla^T(Q\nabla v_\star(x)) + \mu_\star^T \nabla v_\star(x) + f(v_\star(x)), \; x \in \mathbb{R}^d, \; d \geqslant 1,$$

with $A \in \mathbb{R}^{m,m}$, $Q \in \mathbb{R}^{d,d}$ and

$$A\nabla^T(Q\nabla v_\star(x)) = A \sum_{i=1}^{d} \sum_{j=1}^{d} D_i\left(Q_{ij} D_j v_\star(x)\right).$$

Applying D_j to (22), $j = 1, \ldots, d$, yields (provided that $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$)

$$0 = A\nabla^T(Q\nabla D_j v_\star(x)) + \mu_\star^T \nabla D_j v_\star(x) + Df(v_\star(x)) D_j v_\star(x), \; x \in \mathbb{R}^d,$$

since A, Q and $\mu_\star$ do not depend on x. Therefore, $(\lambda, w(x)) = (0, D_j v_\star(x))$, $j = 1, \ldots, d$, solves the eigenvalue problem

(23) $$\lambda w(x) = [\mathcal{L}w](x) := A\nabla^T(Q\nabla w(x)) + \mu_\star^T \nabla w(x) + Df(v_\star(x))w(x), \; x \in \mathbb{R}^d.$$

Corollary 1.3 (Point spectrum of traveling waves, divergence form)**.** *Let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ be a nontrivial classical solution of (22) for some $A \in \mathbb{R}^{m,m}$, $Q \in \mathbb{R}^{d,d}$, $\mu_\star \in \mathbb{R}^d$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$. Then*

$$\lambda = 0, \quad w(x) = D_j v_\star(x), \quad x \in \mathbb{R}^d, \quad j = 1, \ldots, d$$

solve the eigenvalue problem (23). In particular, the algebraic multiplicity of the eigenvalue $\lambda = 0$ is greater or equal d.

1.4.2 Essential spectrum of traveling waves

For simplicity consider the traveling wave equation in one space dimension ($d = 1$)

(24) $$0 = Av_{\star,xx}(x) + \mu_\star v_{\star,x}(x) + f(v_\star(x)), \; x \in \mathbb{R}.$$

with diffusion matrix $A \in \mathbb{R}^{m,m}$, nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$, constant asymptotic states $v_\pm \in \mathbb{R}^m$ (i.e. $f(v_\pm) = 0$), translational velocity $\mu_\star \in \mathbb{R}$ and profile $v_\star : \mathbb{R} \to \mathbb{R}^m$ satisfying $v_\star(x) \to v_\pm \in \mathbb{R}^m$ as $x \to \pm\infty$.

Initial value problem: The main idea to detecting the essential spectrum of $\mathcal{L}$ is to look for solutions of

(25) $$\begin{aligned} v_t(x,t) &= [\mathcal{L}v](x,t) := Av_{xx}(x,t) + \mu_\star v_x(x,t) + Df(v_\star(x))v(x,t) &&, \; x \in \mathbb{R}, \; t > 0, \\ v(x,0) &= v_0(x) &&, \; x \in \mathbb{R}, t = 0. \end{aligned}$$

Decomposition of $Df(v_\star(x))$: Introducing the matrices $Q_\pm(x) \in \mathbb{R}^{m,m}$ via

$$Q_\pm(x) := Df(v_\star(x)) - Df(v_\pm), \; x \in \mathbb{R},$$

we obtain from (25)

(26)
$$v_t(x,t) = [\mathcal{L}_\pm v](x,t) := Av_{xx}(x,t) + \mu_\star v_x(x,t) + Df(v_\pm)v(x,t) + Q_\pm(x)v(x,t), \; x \in \mathbb{R}, \; t > 0.$$

Limiting operator (simplified operator, far-field operator): Since the essential spectrum depends only on the limiting equation for $x \to \pm\infty$, we let formally $x \to \pm\infty$ (but only in the coefficient matrices). Since $Q_+(x) \to 0$ as $x \to +\infty$ and $Q_-(x) \to 0$ as $x \to -\infty$, we can drop the term $Q_\pm(x)$ in (26) and obtain

$$v_t(x,t) = Av_{xx}(x,t) + \mu_\star v_x(x,t) + Df(v_\pm)v(x,t),\ x \in \mathbb{R},\ t > 0. \tag{27}$$

Fourier transform: Since we seek for bounded solutions of (27), we perform a Fourier transformation in space and time. Inserting the Fourier transform

$$v(x,t) = e^{\lambda t}e^{i\omega x}\hat{v},\ \lambda \in \mathbb{C},\ \omega \in \mathbb{R},\ \hat{v} \in \mathbb{C}^m,\ |\hat{v}| = 1 \tag{28}$$

into (27) and dividing by $e^{\lambda t}e^{i\omega x}$ yields a finite dimensional eigenvalue problem

$$\lambda\hat{v} = \left(-\omega^2 A + i\omega\mu_\star I_m + Df(v_\pm)\right)\hat{v}$$

Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

$$\det\left(-A\omega^2 + i\omega\mu_\star I_m + Df(v_\pm) - \lambda I_m\right) = 0$$

for some $\omega \in \mathbb{R}$ belongs to the essential spectrum of $\mathcal{L}$.

Theorem 1.4 (Essential spectrum of traveling waves, $d = 1$)**.** *Let $v_\star \in C^2(\mathbb{R}, \mathbb{R}^m)$ be a nontrivial classical solution of (24) satisfying $v_\star(x) \to v_\pm$ as $x \to \pm\infty$ for some $v_\pm \in \mathbb{R}$ and let $A \in \mathbb{R}^{m,m}$, $\mu_\star \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ with $f(v_\pm) = 0$. Then algebraic curves (asymptotic parabolas)*

$$\begin{aligned} S_\pm &:= \left\{\lambda \in \mathbb{C} \mid \det\left(-A\omega^2 + i\omega\mu_\star I_m + Df(v_\pm) - \lambda I_m\right) = 0 \text{ for some } \omega \in \mathbb{R}\right\} \\ &= \left\{\lambda \in \sigma\left(-A\omega^2 + i\omega\mu_\star I_m + Df(v_\pm)\right) \mid \omega \in \mathbb{R}\right\} \end{aligned}$$

belongs to the essential spectrum $\sigma_{\text{ess}}(\mathcal{L})$ of $\mathcal{L}$, i.e. $S_\pm \subseteq \sigma_{\text{ess}}(\mathcal{L})$.

Example 1.5 (Fisher's equation)**.** The Fisher's equation

$$u_t = u_{xx} + u(1-u),\ x \in \mathbb{R},\ t \geqslant 0,$$

has a traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with velocity $\mu_\star \approx -2$ and profile $v_\star$ connecting the asymptotic states $v_+ = 1$ and $v_- = 0$, i.e. $v_\star(x) \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile nor the velocity are given explicitly. The nonlinearity $f(u) = u(1-u)$ satisfies $f(v_\pm) = 0$, $f'(v_+) = -1$ and $f'(v_-) = 1$. The dispersion relation states that

$$S_\pm := \{\lambda = -\omega^2 + i\omega\mu_\star + f'(v_\pm) \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L})$$

i.e.

$$\begin{aligned} S_+ &:= \{\lambda = -\omega^2 + i\omega\mu_\star - 1 \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}), \\ S_- &:= \{\lambda = -\omega^2 + i\omega\mu_\star + 1 \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}). \end{aligned}$$

The essential spectrum of the traveling front in Fisher's equation is illustrated in Figure 1.1.

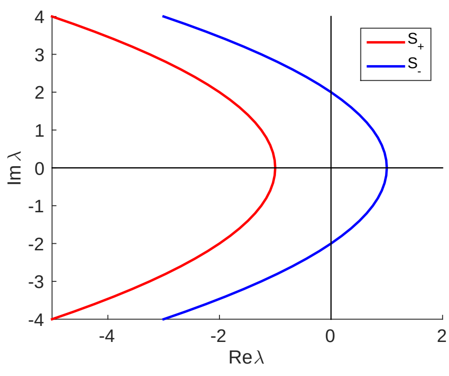

FIGURE 1.1. Essential spectrum of Fisher's equation

Example 1.6 (Nagumo equation)**.** The Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-b),\ x \in \mathbb{R},\ t \geqslant 0,\ 0 < b < 1.$$

has an explicit traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$v_\star(x) = \frac{1}{1+e^{-\frac{x}{\sqrt{2}}}}, \quad \mu_\star = -\sqrt{2}\left(\frac{1}{2} - b\right) \quad \text{(Huxley wave)},$$

$v_+ = 1$ and $v_- = 0$. The nonlinearity $f(u) = u(1-u)(u-b)$ satisfies $f(v_\pm) = 0$, $f'(v_+) = b-1$ and $f'(v_-) = -b$. The dispersion relation states that

$$S_\pm := \{\lambda = -\omega^2 + i\omega\mu_\star + f'(v_\pm) \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L})$$

i.e.

$$S_+ := \{\lambda = -\omega^2 + i\omega\mu_\star - b \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}),$$
$$S_- := \{\lambda = -\omega^2 + i\omega\mu_\star + (b-1) \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}).$$

The essential spectrum of the traveling front in the Nagumo equation is illustrated in Figure 1.2.

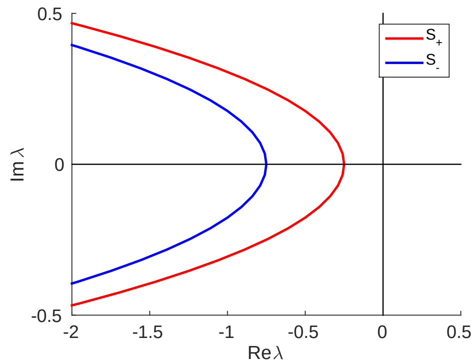

FIGURE 1.2. Essential spectrum of the Nagumo equation for parameter $b = \frac{1}{4}$

Example 1.7 (Quintic Nagumo equation)**.** Consider the quintic Nagumo equation

$$u_t = u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_3)(u-\alpha_3),\ x \in \mathbb{R},\ t \geqslant 0,\ 0 < \alpha_1 < \alpha_2 < \alpha_3 < 1.$$

For the parameters $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{17}{20}$ the quintic Nagumo equation has a traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with velocity $\mu_\star \approx 0.07$ and profile $v_\star$ conntecting the asymptotic states $v_+ = 1$ and $v_- = 0$, i.e. $v_\star(x) \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile nor the velocity are given explicitly. The nonlinearity $f(u) = u(1-u)(u-\alpha_1)(u-\alpha_3)(u-\alpha_3)$ satisfies $f(v_\pm) = 0$, $f'(v_+) = -(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)$ and $f'(v_-) = -\alpha_1\alpha_2\alpha_3$. The dispersion relation states that

$$S_\pm := \{\lambda = -\omega^2 + i\omega\mu_\star + f'(v_\pm) \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L})$$

i.e.

$$S_+ := \{\lambda = -\omega^2 + i\omega\mu_\star - (1-\alpha_1)(1-\alpha_2)(1-\alpha_3) \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}),$$
$$S_- := \{\lambda = -\omega^2 + i\omega\mu_\star - \alpha_1\alpha_2\alpha_3 \mid \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L}).$$

The essential spectrum of the traveling front in the quintic Nagumo equation is illustrated in Figure 1.3.

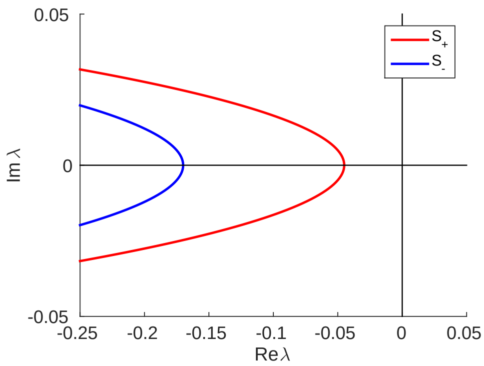

FIGURE 1.3. Essential spectrum of the quintic Nagumo equation for parameters $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{17}{20}$

Example 1.8 (Fitz-Hugh Nagumo system)**.** Consider the FitzHugh-Nagumo system

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix}, \quad x \in \mathbb{R},\, t \geqslant 0$$

for some $D \geqslant 0$, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$, $\zeta \neq 0$ and $u_i = u_i(x,t) \in \mathbb{R}$ for $i = 1, 2$. Using the notation

$$u = \left(u_1, u_2\right)^T \in \mathbb{R}^2 \quad \text{and} \quad f : \mathbb{R}^2 \to \mathbb{R}^2,\, f(u) = \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix},$$

the FitzHugh-Nagumo system can also be written as

$$u_t = A u_{xx} + f(u), \quad x \in \mathbb{R},\, t \geqslant 0, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}.$$

(a) For the parameters (parabolic case)

$$D = \frac{1}{10}, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 3, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}$$

and for the parameters (parabolic-hyperbolic case)

$$D = 0, \quad \alpha = 0, \quad \beta = \frac{2}{25}, \quad \gamma = 1, \quad \delta = 3, \quad \varepsilon = \frac{7}{10}, \quad \zeta = \frac{1}{3}$$

the Fitz-Hugh Nagumo system has a traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with velocity $\mu_\star \approx -0.8560$ (parabolic case) and $\mu_\star \approx -0.8664$ (parabolic-hyperbolic case) and profile $v_\star$ conntecting the asymptotic states

$$v_- = \begin{pmatrix} 1.18769696080266 \\ 0.629232320266825 \end{pmatrix} \quad \text{and} \quad v_+ = \begin{pmatrix} -1.56443178284120 \\ -0.288143927613547 \end{pmatrix},$$

i.e. $v_\star(x) \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile nor the velocity are given explicitly. The nonlinearity

$$f(u) = \begin{pmatrix} u_1 - \zeta u_1^3 - u_2 + \alpha \\ \beta(\gamma u_1 - \delta u_2 + \varepsilon) \end{pmatrix}$$

satisfies

$$f(v_\pm) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Df(v_\pm) = \begin{pmatrix} 1 - 3\zeta \left(v_\pm^{(1)}\right)^2 & -1 \\ \beta\gamma & -\beta\delta \end{pmatrix}.$$

The dispersion relation states that

$$S_\pm := \left\{ \lambda \in \mathbb{C} \mid \det\left(-\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} + i\omega\mu_\star I_2 + \begin{pmatrix} 1 - 3\zeta \left(v_\pm^{(1)}\right)^2 & -1 \\ \beta\gamma & -\beta\delta \end{pmatrix} - \lambda I_2 \right) = 0,\, \omega \in \mathbb{R} \right\}$$

belongs to $\sigma_{\mathrm{ess}}(\mathcal{L})$, i.e. both sets

$$S_+ := \left\{\lambda \in \mathbb{C} \mid \det\left(-\omega^2\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix} + i\omega\mu_\star I_2 + \begin{pmatrix}1-3\zeta\left(v_+^{(1)}\right)^2 & -1\\ \beta\gamma & -\beta\delta\end{pmatrix} - \lambda I_2\right) = 0,\ \omega\in\mathbb{R}\right\},$$

$$S_- := \left\{\lambda \in \mathbb{C} \mid \det\left(-\omega^2\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix} + i\omega\mu_\star I_2 + \begin{pmatrix}1-3\zeta\left(v_-^{(1)}\right)^2 & -1\\ \beta\gamma & -\beta\delta\end{pmatrix} - \lambda I_2\right) = 0,\ \omega\in\mathbb{R}\right\}$$

are contained in $\sigma_{\mathrm{ess}}(\mathcal{L})$. Due to

$$\begin{aligned}0 =& \det\begin{pmatrix}-\omega^2 + i\omega\mu_\star + 1 - 3\zeta\left(v_\pm^{(1)}\right)^2 - \lambda & -1\\ \beta\gamma & -\omega^2 D + i\omega\mu_\star - \beta\delta - \lambda\end{pmatrix}\\ =& \lambda^2 - (a+b)\lambda + (ab+c)\end{aligned}$$

with abbreviations

$$a := -\omega^2 + i\omega\mu_\star + 1 - 3\zeta\left(v_\pm^{(1)}\right)^2,\quad b := -\omega^2 D + i\omega\mu_\star - \beta\delta,\quad c := \beta\gamma,$$

every $\lambda_{1,2}^{\pm} = \lambda_{1,2}^{\pm}(\omega) \in \mathbb{C}$ satisfying

$$\lambda_{1,2}^{\pm} = \frac{1}{2}\left((a+b) \pm \sqrt{(a+b)^2 - 4(ab+c)}\right),$$

for some $\omega \in \mathbb{R}$ belongs to the essential spectrum of $\mathcal{L}$. The essential spectrum of the traveling front in the FitzHugh-Nagumo system is illustrated in Figure 1.4(a).

(b) For the parameters (parabolic case)

$$D = \frac{1}{10},\quad \alpha = 0,\quad \beta = \frac{2}{25},\quad \gamma = 1,\quad \delta = 0.8,\quad \varepsilon = \frac{7}{10},\quad \zeta = \frac{1}{3}$$

and for the parameters (parabolic-hyperbolic case)

$$D = 0,\quad \alpha = 0,\quad \beta = \frac{2}{25},\quad \gamma = 1,\quad \delta = 0.8,\quad \varepsilon = \frac{7}{10},\quad \zeta = \frac{1}{3}$$

the Fitz-Hugh Nagumo system has a traveling pulse solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with velocity $\mu_\star \approx -0.7892$ (parabolic case) and $\mu_\star \approx -0.8121$ (parabolic-hyperbolic case) and profile $v_\star$ conntecting the asymptotic state

$$v_\pm = \begin{pmatrix}-1.19940803524404\\ -0.624260044055044\end{pmatrix},$$

i.e. $v_\star(x) \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile nor the velocity are given explicitly. The nonlinearity

$$f(u) = \begin{pmatrix}u_1 - \zeta u_1^3 - u_2 + \alpha\\ \beta(\gamma u_1 - \delta u_2 + \varepsilon)\end{pmatrix}$$

satisfies

$$f(v_\pm) = \begin{pmatrix}0\\0\end{pmatrix}\quad\text{and}\quad Df(v_\pm) = \begin{pmatrix}1-3\zeta\left(v_\pm^{(1)}\right)^2 & -1\\ \beta\gamma & -\beta\delta\end{pmatrix}.$$

The dispersion relation states that

$$S_\pm := \left\{\lambda \in \mathbb{C} \mid \det\left(-\omega^2\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix} + i\omega\mu_\star I_2 + \begin{pmatrix}1-3\zeta\left(v_\pm^{(1)}\right)^2 & -1\\ \beta\gamma & -\beta\delta\end{pmatrix} - \lambda I_2\right) = 0,\ \omega\in\mathbb{R}\right\}$$

belongs to $\sigma_{\mathrm{ess}}(\mathcal{L})$. Similarly to (a), we obtain that every $\lambda_{1,2}^{\pm} = \lambda_{1,2}^{\pm}(\omega) \in \mathbb{C}$ satisfying

$$\lambda_{1,2}^{\pm} = \frac{1}{2}\left((a+b) \pm \sqrt{(a+b)^2 - 4(ab+c)}\right),$$

for some $\omega \in \mathbb{R}$, where

$$a := -\omega^2 + i\omega\mu_\star + 1 - 3\zeta\left(v_\pm^{(1)}\right)^2, \quad b := -\omega^2 D + i\omega\mu_\star - \beta\delta, \quad c := \beta\gamma,$$

belongs to the essential spectrum of $\mathcal{L}$. The essential spectrum of the traveling pulse in the FitzHugh-Nagumo system is illustrated in Figure 1.4(b).

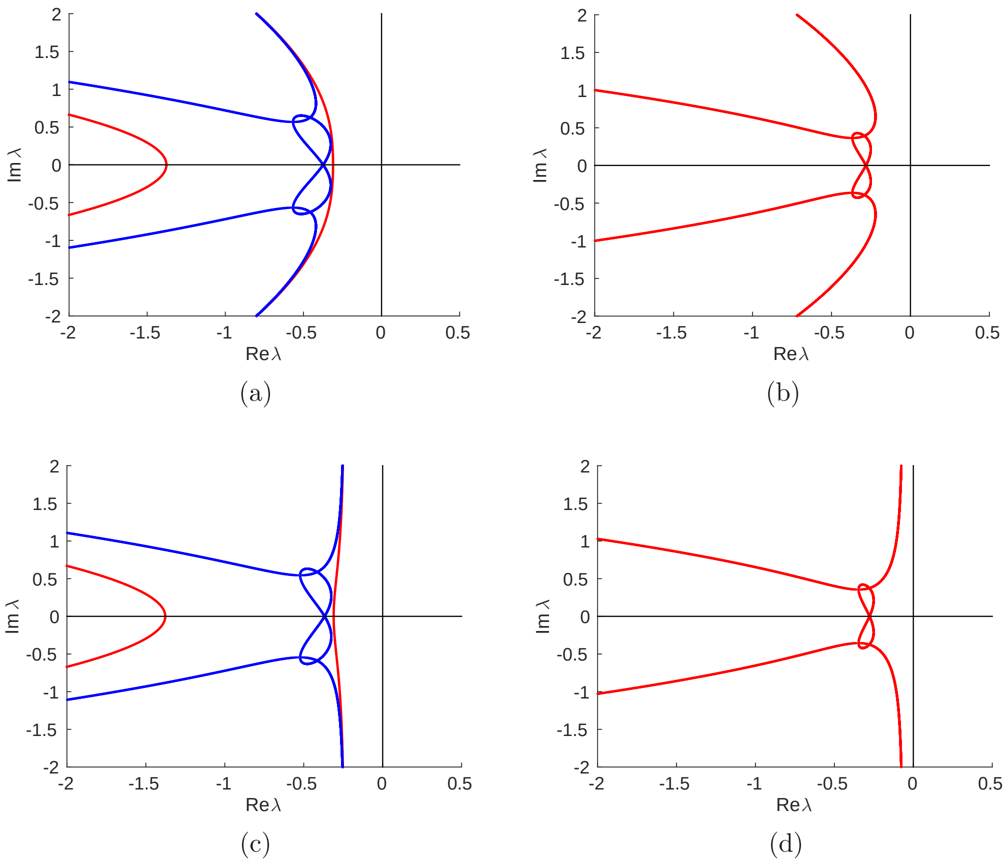

FIGURE 1.4. Essential spectrum of the Nagumo equation for a traveling front with $D = \frac{1}{10}$ in (a) and with $D = 0$ in (c), and for a traveling pulse with $D = \frac{1}{10}$ in (b) and with $D = 0$ in (d)

Example 1.9 (Barkley model)**.** Consider the Barkley model

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u_{1,xx} \\ u_{2,xx} \end{pmatrix} + \begin{pmatrix} \frac{1}{\varepsilon} u_1(1-u_1)\left(u_1 - \frac{u_2+b}{a}\right) \\ u_1 - u_2 \end{pmatrix}, \quad x \in \mathbb{R},\, t \geqslant 0$$

for some $D \geqslant 0$, $a, b, \varepsilon > 0$ and $u_i = u_i(x,t) \in \mathbb{R}$ for $i = 1, 2$. Using the notation

$$u = \left(u_1, u_2\right)^T \in \mathbb{R}^2 \quad \text{and} \quad f : \mathbb{R}^2 \to \mathbb{R}^2,\, f(u) = \begin{pmatrix} \frac{1}{\varepsilon} u_1(1-u_1)\left(u_1 - \frac{u_2+b}{a}\right) \\ u_1 - u_2 \end{pmatrix},$$

the Barkley model can also be written as

$$u_t = Au_{xx} + f(u), \quad x \in \mathbb{R},\, t \geqslant 0, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}.$$

For the parameters (parabolic case)

$$D = \frac{1}{10}, \quad a = \frac{3}{4}, \quad b = \frac{1}{100}, \quad \varepsilon = \frac{1}{50}$$

and for the parameters (parabolic-hyperbolic case)

$$D = 0, \quad a = \frac{3}{4}, \quad b = \frac{1}{100}, \quad \varepsilon = \frac{1}{50}$$

the Barkley model has a traveling pulse solution $u_\star(x,t) = v_\star(x-\mu_\star t)$ with velocity $\mu_\star \approx 4.6616$ (parabolic case) and $\mu_\star \approx 4.6785$ (parabolic-hyperbolic case) and profile $v_\star$ conntecting the asymptotic state

$$v_\pm = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $v_\star(x) \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile nor the velocity are given explicitly. The nonlinearity

$$f(u) = \begin{pmatrix} \frac{1}{\varepsilon} u_1(1-u_1)\left(u_1 - \frac{u_2+b}{a}\right) \\ u_1 - u_2 \end{pmatrix}$$

satisfies

$$f(v_\pm) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Df(v_\pm) = \begin{pmatrix} -\frac{b}{\varepsilon a} & 0 \\ 1 & -1 \end{pmatrix}.$$

The dispersion relation states that

$$S_\pm := \left\{ \lambda \in \mathbb{C} \mid \det\left(-\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} + i\omega\mu_\star I_2 + \begin{pmatrix} -\frac{b}{\varepsilon a} & 0 \\ 1 & -1 \end{pmatrix} - \lambda I_2 \right) = 0,\ \omega \in \mathbb{R} \right\}$$

belongs to $\sigma_{\text{ess}}(\mathcal{L})$. Due to

$$\begin{aligned} 0 &= \det \begin{pmatrix} -\omega^2 + i\omega\mu_\star - \frac{b}{\varepsilon a} - \lambda & 0 \\ 1 & -\omega^2 D + i\omega\mu_\star - 1 - \lambda \end{pmatrix} \\ &= (-\omega^2 + i\omega\mu_\star - \frac{b}{\varepsilon a} - \lambda)(-\omega^2 D + i\omega\mu_\star - 1 - \lambda) \end{aligned}$$

we obtain that every $\lambda_{1,2}^\pm = \lambda_{1,2}^\pm(\omega) \in \mathbb{C}$ satisfying

$$\lambda_1^\pm = -\omega^2 + i\omega\mu_\star - \frac{b}{\varepsilon a}, \quad \lambda_2^\pm = -\omega^2 D + i\omega\mu_\star - 1$$

for some $\omega \in \mathbb{R}$, belongs to the essential spectrum of $\mathcal{L}$. The essential spectrum of the traveling pulse in the Barkley model is illustrated in Figure 1.5(b).

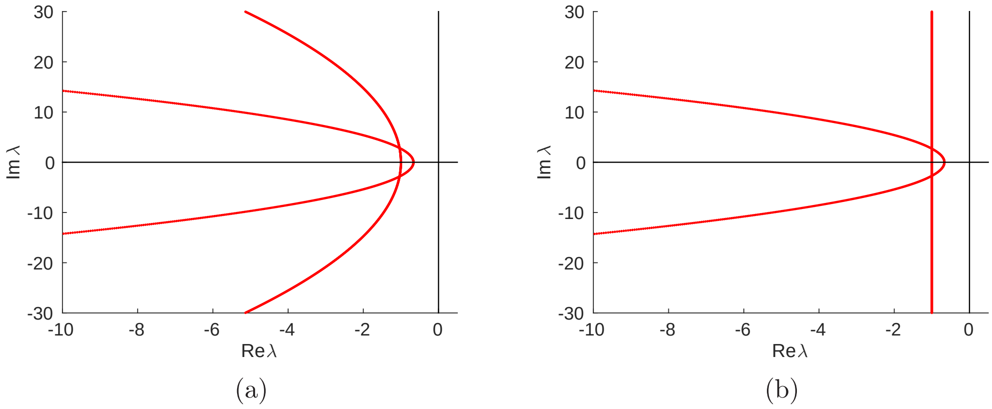

FIGURE 1.5. Essential spectrum of the Barkley model for a traveling pulse with $D = \frac{1}{10}$ in (a) and with $D = 0$ in (b)