Mathematical Modelling and Simulation with Comsol Multiphysics II
Winter term 2015/2016
Dr. Denny Otten
04.12.2015

7. Freezing Traveling Waves in Damped Wave Equations

7.1 Traveling waves in systems of damped waves equations

Consider a system of damped wave equations in one space dimension

$$(1)\qquad \begin{aligned} &Mu_{tt}(x,t)+Bu_t(x,t)=Au_{xx}(x,t)+f(u(x,t)) &&,\ x\in\mathbb{R},\ t>0,\\ &u(x,0)=u_0(x),\quad u_t(x,0)=v_0(x) &&,\ x\in\mathbb{R},\ t=0,\end{aligned}$$

with mass matrix $M\in\mathbb{R}^{m,m}$, damping matrix $B\in\mathbb{R}^{m,m}$, diffusion matrix $A\in\mathbb{R}^{m,m}$, nonlinearity $f:\mathbb{R}^m\to\mathbb{R}^m$, initial data $u_0:\mathbb{R}\to\mathbb{R}^m$ and $v_0:\mathbb{R}\to\mathbb{R}^m$ and solution $u:\mathbb{R}\times[0,\infty)\to\mathbb{R}^m$.
We are interested in traveling wave solutions of (1): A traveling wave of (1) is a solution $u_\star:\mathbb{R}\times[0,\infty)\to\mathbb{R}^m$ of the form

$$(2)\qquad u_\star(x,t)=v_\star(x-\mu_\star t)\quad,\ x\in\mathbb{R},\ t\geqslant 0,$$

with

$$\lim_{\xi\to+\infty}v_\star(\xi)=v_+\in\mathbb{R}^m,\quad \lim_{\xi\to-\infty}v_\star(\xi)=v_-\in\mathbb{R}^m,\quad f(v_+)=f(v_-)=0.$$

The function $v_\star:\mathbb{R}\to\mathbb{R}^m$ is called the profile and $\mu_\star\in\mathbb{R}$ the (translational) velocity of the traveling wave. The traveling wave $u_\star$ is called a traveling pulse, if $v_+=v_-$, and a traveling front, if $v_+\neq v_-$. The wave travels to the left, if $\mu_\star<0$ and to the right, if $\mu_\star>0$. In case $\mu_\star=0$, $u_\star$ is called a standing wave.
Our aim is to approximate traveling wave solutions of (1). The idea for approximating the traveling wave $u_\star$ is to determine the profile $v_\star$ and the velocity $\mu_\star$. This requires to transform (1) into a co-moving coordinate system.
Transforming (1) via $u(x,t)=v(\xi,t)$ with $\xi:=x-\mu_\star t$ in a co-moving frame yields

$$(3)\qquad \begin{aligned} &Mv_{tt}+Bv_t=(A-\mu_\star^2M)v_{\xi\xi}+2\mu_\star Mv_{\xi t}+\mu_\star Bv_\xi+f(v) &&,\ \xi\in\mathbb{R},\ t>0,\\ &v(\cdot,0)=u_0(\cdot),\quad v_t(\cdot,0)=v_0(\cdot)+\mu_\star u_{0,\xi}(\cdot) &&,\ \xi\in\mathbb{R},\ t=0,\end{aligned}$$

where we write $v=v(\xi,t)$ for reasons of readability. Inserting (2) into (1) shows, that $v_\star$ is a stationary solution of (3), i.e.

$$(4)\qquad 0=(A-\mu_\star^2M)v_{\star,\xi\xi}(\xi)+\mu_\star Bv_{\star,\xi}(\xi)+f(v_\star(\xi))\quad,\ \xi\in\mathbb{R}.$$

We are also interested in nonlinear stability of traveling waves. It is well known from the literature for a certain class of first-order evolution equations, that spectral stability implies nonlinear stability. For investigating spectral stability of a traveling wave, we must analyze the spectrum of the linearization of the right hand side in (3) at the wave profile $v_\star$, i.e.

$$\left[\mathcal{L}w\right](\xi)=(A-\mu_\star^2M)w_{\xi\xi}(\xi)+\mu_\star Bw_\xi(\xi)+Df(v_\star(\xi))w(\xi)\quad,\ \xi\in\mathbb{R}.$$

This requires to find solutions (λ,w) of the eigenvalue problem

$$(5)\qquad \lambda w(\xi)=(A-\mu_\star^2M)w_{\xi\xi}(\xi)+\mu_\star Bw_\xi(\xi)+Df(v_\star(\xi))w(\xi)\quad,\ \xi\in\mathbb{R},$$

with eigenfunction $v:\mathbb{R}\to\mathbb{C}^m$ and eigenvalue $\lambda\in\mathbb{C}$.
Approximating $v_\star$ via (3) requires the knowledge about the velocity $\mu_\star$ which is in general unknown. This motivates to introduce the freezing method, whose idea is it to approximate the profile $v_\star$ and the velocity $\mu_\star$ simultaneously.

7.2 Freezing method for traveling waves

Consider again a system of damped wave equations in one space dimension, cf. (1),

$$(6)\qquad \begin{aligned} &Mu_{tt}(x,t)+Bu_t(x,t)=Au_{xx}(x,t)+f(u(x,t)) &&, x\in\mathbb{R},\ t>0,\\ &u(x,0)=u_0(x),\quad u_t(x,0)=v_0(x) &&, x\in\mathbb{R},\ t=0,\end{aligned}$$

Introducing new unknowns $\gamma(t)\in\mathbb{R}$ (position) and $v(\xi,t)\in\mathbb{R}^m$ (profile) via the traveling wave ansatz

$$(7)\qquad u(x,t)=v(\xi,t),\ \xi:=x-\gamma(t)\quad , x\in\mathbb{R},\ t\geqslant 0$$

and inserting (7) into (6) yields

$$(8)\qquad Mv_{tt}+v_t=(A-\gamma_t^2M)v_{\xi\xi}+2\gamma_tMv_{\xi,t}+(\gamma_{tt}M+\gamma_tB)v_\xi+f(v)\quad , \xi\in\mathbb{R},\ t>0,$$

where we write $v=v(\xi,t)$ and $\gamma_t=\gamma_t(t)$ for reasons of readability. It is convenient to introduce two further unknowns $\mu_1(t)\in\mathbb{R}$ (velocity) and $\mu_2(t)\in\mathbb{R}$ (acceleration) via

$$\mu_1(t):=\gamma_t(t),\quad \mu_2(t):=\mu_{1,t}(t)=\gamma_{tt}(t).$$

Then, (8) reads as

$$(9)\qquad \begin{aligned} &Mv_{tt}+Bv_t=(A-\mu_1^2M)v_{\xi\xi}+2\mu_1Mv_{\xi,t}+(\mu_2M+\mu_1B)v_\xi+f(v) &&, \xi\in\mathbb{R},\ t>0,\\ &\mu_{1,t}=\mu_2 &&, t>0,\\ &\gamma_t=\mu_1 &&, t>0.\end{aligned}$$

Equ. (9) has to be equipped with suitable initial data. Requiring $\gamma(0)=0$ and $\mu_1(0)=\mu_1^0$, (7) and (6) imply

$$(10)\qquad v(\xi,0)=u_0(\xi),\quad v_t(\xi,0)=v_0(\xi)+\mu_1^0u_{0,\xi}(\xi),\quad \xi\in\mathbb{R},\ t=0.$$

Collecting the equations (9), $\gamma(0)=0$, $\mu_1(0)=\mu_1^0$ and (10) we obtain

$$(11)\qquad \begin{aligned} &Mv_{tt}+Bv_t=(A-\mu_1^2M)v_{\xi\xi}+2\mu_1Mv_{\xi,t}+(\mu_2M+\mu_1B)v_\xi+f(v) &&, \xi\in\mathbb{R},\ t>0,\\ &v(\cdot,0)=u_0(\cdot),\quad v_t(\cdot,0)=v_0(\cdot)+\mu_1^0u_{0,\xi}(\cdot) &&, \xi\in\mathbb{R},\ t=0,\\ &\mu_{1,t}=\mu_2 &&, t>0,\\ &\mu_1(0)=\mu_1^0 &&, t=0,\\ &\gamma_t=\mu_1 &&, t>0,\\ &\gamma(0)=0 &&, t=0.\end{aligned}$$

(11) contains the equations for v, μ_1 and γ. But so far, the system (11) is not well-posed, since there is still no equation for μ_2. To determine μ_2 we require an additional algebraic constraint, a so called phase condition: For this purpose let $\hat{v}:\mathbb{R}\to\mathbb{R}^m$ be a template (or reference) function, e.g. $\hat{v}=u_0$. The idea of the phase condition is to choose $v(\cdot,t)$ such that

$$\min_{g\in\mathbb{R}}\|v(\cdot,t)-\hat{v}(\cdot-g)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)}=\|v(\cdot,t)-\hat{v}(\cdot)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)}\quad , t\geqslant 0.$$

A necessary condition to guarantee that the left hand side attains its minimum at $g=0$ is that the first derivative of $\|v(\cdot,t)-\hat{v}(\cdot-g)\|^2_{L^2(\mathbb{R},\mathbb{R}^m)}$ evaluated at $g=0$ vanishes, i.e. for all $t\geqslant 0$

$$(12)\qquad 0\overset{!}{=}\left[\frac{d}{dg}\left(v(\cdot,t)-\hat{v}(\cdot-g),v(\cdot,t)-\hat{v}(\cdot-g)\right)_{L^2(\mathbb{R},\mathbb{R}^m)}\right]_{g=0}=2\left(v(\cdot,t)-\hat{v},\hat{v}_\xi\right)_{L^2(\mathbb{R},\mathbb{R}^m)}.$$

Combining (11) and (12) yields a partial differential algebraic evolution equation (PDAE)

$$
(13)\quad
\begin{aligned}
&Mv_{tt}+Bv_t=(A-\mu_1^2M)v_{\xi\xi}+2\mu_1Mv_{\xi,t}+(\mu_2M+\mu_1B)v_\xi+f(v) &&,\ \xi\in\mathbb{R},\ t>0,\\
&v(\cdot,0)=u_0,\quad v_t(\cdot,0)=v_0+\mu_1^0u_{0,\xi} &&,\ \xi\in\mathbb{R},\ t=0,\\
&0=(v(\cdot,t)-\hat{v},\hat{v}_\xi)_{L^2(\mathbb{R},\mathbb{R}^m)} &&,\ t\geqslant 0,\\
&\mu_{1,t}=\mu_2 &&,\ t>0,\\
&\mu_1(0)=\mu_1^0 &&,\ t=0,\\
&\gamma_t=\mu_1 &&,\ t>0,\\
&\gamma(0)=0 &&,\ t=0.
\end{aligned}
$$

The system (13) must be solved for (v,μ_1,μ_2,γ) for some given (u_0,v_0,μ_1^0). It consists of a PDE for v, two ODEs for μ_1 and γ and an algebraic constraint for μ_2. The ODE for γ is called the reconstruction equation, is decoupled from the other equations in (13) and can be solved in a postprocessing step. Since $(v,\mu_1,\mu_2)=(v_\star,\mu_\star,0)$ satisfy

$$
\begin{aligned}
&0=(A-\mu_\star^2M)v_{\star,\xi\xi}(\xi)+\mu_\star Bv_{\star,\xi}(\xi)+f(v_\star(\xi))\ ,\ \xi\in\mathbb{R},\\
&0=\mu_2,\\
&0=(v_\star-\hat{v},\hat{v}_\xi)_{L^2(\mathbb{R},\mathbb{R}^m)}\ ,
\end{aligned}
$$

we expect for stability reasons, that the solution (v,μ_1,μ_2,γ) of (13) satisfies

$$
(14)\qquad v(t)\to v_\star,\quad \mu_1(t)\to\mu_\star,\quad \mu_2(t)\to 0,\quad \text{as}\quad t\to\infty.
$$

As an indicator for the convergence in (14) we check the quantities

$$
(15)\qquad \|v_t(\cdot,t)\|_{L^2(\mathbb{R},\mathbb{R}^m)}\quad\text{and}\quad |\mu_{1,t}(t)|
$$

at each time instance t during the computation. In fact, both of these quantities should be small ($\approx 10^{-16}$), since $v_\star$ and $\mu_\star$ do not vary in time.

7.3 Numerical approximation of traveling waves via freezing method

Solving (1), (13) and (5) numerically, requires to truncate these equations to bounded domains. Let $\Omega\subset\mathbb{R}$ be a bounded open domain, then (1) must be satisfied for $x\in\Omega$, and equations (13) and (5) for $\xi\in\Omega$. To guarantee the well-posedness of these problems, we must equip the equations with appropriate boundary conditions. Normally, we choose homogeneous Neumann boundary conditions (also known as no-flux boundary conditions), i.e.

$$
u_x(x)=0,\ x\in\partial\Omega,\quad v_\xi(\xi)=0,\ \xi\in\partial\Omega.
$$

In this context, $\partial\Omega$ denotes the boundary of Ω and $\overline{\Omega}$ the closure of Ω, e.g. $\Omega=(a,b)$ with $-\infty<a<b<\infty$ then $\partial\Omega=\{a,b\}$ and $\overline{\Omega}=[a,b]$. Numerically, we solve the following equations:

Step 1: (Nonfrozen Equation)

$$
(16)\quad
\begin{aligned}
&Mu_{tt}(x,t)+Bu_t(x,t)=Au_{xx}(x,t)+f(u(x,t)) &&,\ x\in\Omega,\ t\in(0,T_1],\\
&u(x,0)=u_0(x),\quad u_t(x,0)=v_0(x) &&,\ x\in\overline{\Omega},\ t=0,\\
&u_x(x,t)=0 &&,\ x\in\partial\Omega,\ t\in[0,T_1].
\end{aligned}
$$

First, we determine the solution u of (16). The quantities A, M, B, f, u_0, v_0, Ω and T_1 are given.

Step 2: (**Frozen Equation**)

(17)

$$\begin{aligned}
&Mv_{tt}+Bv_t=(A-\mu_1^2M)v_{\xi\xi}+2\mu_1Mv_{\xi,t}+(\mu_2M+\mu_1B)v_\xi+f(v) &&,\ \xi\in\Omega,\ t\in(0,T_2],\\
&v(\cdot,0)=u_0,\quad v_t(\cdot,0)=v_0+\mu_1^0u_{0,\xi} &&,\ \xi\in\overline{\Omega},\ t=0,\\
&v_\xi(\cdot,t)=0 &&,\ \xi\in\partial\Omega,\ t\in[0,T_2],\\
&0=(v(\cdot,t)-\hat{v},\hat{v}_\xi)_{L^2(\Omega,\mathbb{R}^m)} &&,\ t\in[0,T_2],\\
&\mu_{1,t}=\mu_2 &&,\ t\in(0,T_2],\\
&\mu_1(0)=\mu_1^0 &&,\ t=0,\\
&\gamma_t=\mu_1 &&,\ t\in(0,T_2],\\
&\gamma(0)=0 &&,\ t=0.
\end{aligned}$$

Then, we determine the solution (v,μ_1,μ_2,γ) of (17). The quantities A, M, B, f, u_0, v_0, $\hat{v}$, μ_1^0, Ω and T_2 are given. The final time T_2 may be different to the end time T_1 from (16). The template function is often chosen as $\hat{v}(\xi)=u_0(\xi)$ or $\hat{v}(\xi)=u(\xi,T_1)$, where $u(\cdot,T_1)$ denotes the solution of (16) at the end time T_1. Also the initial data v_0 is often chosen as $v_0(\xi)=u_0(\xi)$ or $v_0(\xi)=u(\xi,T_1)$. The end time T_2 in (17) is often chosen such that the values of the quantities $\|v(\cdot,t)\|_{L^2(\Omega,\mathbb{R}^m)}$ and $|\mu_t(t)|$, cf. (15), are near 10^{-16}.

Step 3: (**Eigenvalue Problem**)

$$\begin{aligned}
&\lambda w(\xi)=(A-\mu_\star^2M)w_{\xi\xi}(\xi)+\mu_\star Bw_\xi(\xi)+Df(v_\star(\xi))w(\xi) &&,\ \xi\in\Omega,\\
&w_\xi(\xi)=0 &&,\ \xi\in\partial\Omega.
\end{aligned}\tag{18}$$

Finally, we determine (a predescribed number neig of) eigenvalues λ and associated eigenfunctions w of (18). The quantities A, M, B, $\mu_\star$, $v_\star$, f, Ω and neig are given. The profile $v_\star$ and the velocity $\mu_\star$ come actually from a simulation, more precisely we set $\mu_\star:=\mu_1(T_2)$ and $v_\star(\xi):=v(\xi,T_2)$, where $\mu_1(T_2)$ and $v(\cdot,T_2)$ denote two components of the solution of (17) at the end time T_2.

7.4 Spectra and eigenfunctions of traveling waves

We now look for solutions (λ,w) with $\lambda\in\mathbb{C}$ and $w:\mathbb{R}\to\mathbb{C}^m$ of the eigenvalue problem

$$\lambda w(x)=[\mathcal{L}w](x):=(A-\mu_\star^2M)w_{xx}(x)+\mu_\star Bw_x(x)+Df(v_\star(x))w(x),\ x\in\mathbb{R}.$$

7.4.1 Point spectrum of traveling waves on the imaginary axis

Consider the traveling wave equation

$$0=(A-\mu_\star^2M)v_{\star,xx}(x)+\mu_\star Bv_{\star,x}(x)+f(v_\star(x)),\ x\in\mathbb{R},\tag{19}$$

with mass matrix $M\in\mathbb{R}^{m,m}$, damping matrix $B\in\mathbb{R}^{m,m}$, diffusion matrix $A\in\mathbb{R}^{m,m}$, nonlinearity $f:\mathbb{R}^m\to\mathbb{R}^m$, translational velocity $\mu_\star\in\mathbb{R}$ and profile $v_\star:\mathbb{R}\to\mathbb{R}^m$.

For $g\in\mathbb{R}$ we define the group action $[a(g)v](x):=v(x-g)$. Applying $a(g)$ on both hand sides in (19) yields

$$\begin{aligned}
0&=a(g)\left[(A-\mu_\star^2M)v_{\star,xx}(x)+\mu_\star Bv_{\star,x}(x)+f(v_\star(x))\right]\\
&=(A-\mu_\star^2M)\frac{d^2}{dx^2}[a(g)v_\star(x)]+\mu_\star B\frac{d}{dx}[a(g)v_\star(x)]+a(g)f(v_\star(x))\\
&=(A-\mu_\star^2M)v_{\star,xx}(x-g)+\mu_\star Bv_{\star,x}(x-g)+f(v_\star(x-g)),\ x\in\mathbb{R}.
\end{aligned}\tag{20}$$

Taking the derivative $\frac{d}{dg}$ in (20) evaluated at $g=0$, we obtain (provided that $v_\star \in C^3(\mathbb{R}^d,\mathbb{R}^m)$ and $f \in C^1(\mathbb{R}^m,\mathbb{R}^m)$)

$$\begin{aligned}
0 &= \left[\frac{d}{dg}\Big((A-\mu_\star^2 M)v_{\star,xx}(x-g) + \mu_\star B v_{\star,x}(x-g) + f(v_\star(x-g))\Big)\right]_{g=0} \\
&= -\Big[(A-\mu_\star^2 M)v_{\star,xxx}(x-g) + \mu_\star B v_{\star,xx}(x-g) + Df(v_\star(x-g))v_{\star,x}(x-g)\Big]_{g=0} \\
&= -\Big((A-\mu_\star^2 M)v_{\star,xxx}(x) + \mu_\star B v_{\star,xx}(x) + Df(v_\star(x))v_{\star,x}(x)\Big),\ x\in\mathbb{R}.
\end{aligned}$$

This leads to the equation

$$\begin{aligned}
0 &= (A-\mu_\star^2 M)\frac{d^2}{dx^2}v_{\star,x}(x) + \mu_\star B\frac{d}{dx}v_{\star,x}(x) + Df(v_\star(x))v_{\star,x}(x) \\
&= \frac{d}{dx}\left((A-\mu_\star^2 M)v_{\star,xx}(x) + \mu_\star B v_{\star,x}(x) + f(v_\star(x))\right),\ x\in\mathbb{R}.
\end{aligned}$$

Therefore, $(\lambda, w(x)) := (0, v_{\star,x}(x))$, solves the eigenvalue problem

(21) $$\lambda w(x) = [\mathcal{L}w](x) := (A-\mu_\star^2 M)w_{xx}(x) + \mu_\star B w_x(x) + Df(v_\star(x))w(x),\ x\in\mathbb{R},$$

i.e. the function $w(x) = v_{\star,x}(x)$ is an eigenfunction associated with the eigenvalue $\lambda = 0$, provided the $v_\star$ is nontrivial (i.e. not constant), since otherwise we have $w(x)=0$.
Procedure: Applying $\frac{d}{dx}$ to (19) yields the solution $(\lambda, w(x)) := (0, v_{\star,x}(x))$ of (21).

Theorem 7.1 (Point spectrum of traveling waves). *Let $v_\star \in C^3(\mathbb{R},\mathbb{R}^m)$ be a nontrivial classical solution of (19) for some $A,B,M\in\mathbb{R}^{m,m}$, $\mu_\star\in\mathbb{R}$ and $f\in C^1(\mathbb{R}^m,\mathbb{R}^m)$. Then*

$$\lambda = 0,\quad w(x) = v_{\star,x}(x),\quad x\in\mathbb{R}$$

solves the eigenvalue problem (21). In particular,

$$\sigma_{\text{point}}^{\text{part}}(\mathcal{L}) := \{0\} \subseteq \sigma_{\text{point}}(\mathcal{L})$$

and the algebraic multiplicity of the eigenvalue $\lambda=0$ is greater or equal 1.

Example 7.2 (Damped Nagumo equation). The damped Nagumo equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u)(u-b),\ x\in\mathbb{R},\ t\geqslant 0,\ 0<b<1,\ \varepsilon>0$$

has two explicit traveling wave solutions $u_\star(x,t) = v_\star(x-\mu_\star t)$ with

$$v_\star(x) = \frac{1}{1+e^{\frac{kx}{\sqrt{2}}}},\quad \mu_\star = \frac{\sqrt{2}\left(\frac{1}{2}-b\right)}{k}\quad k = \pm\sqrt{1+2\varepsilon\left(\frac{1}{2}-b\right)^2},$$

i.e. $v_\star$ and $\mu_\star$ solve the associated traveling wave equation

$$0 = (1-\mu_\star^2\varepsilon)v_{\star,xx}(x) + \mu_\star v_{\star,x}(x) + v_\star(x)\left(1-v_\star(x)\right)\left(v_\star(x)-b\right),\ x\in\mathbb{R}.$$

The eigenvalue problem for the linearization

$$\lambda w(x) = (1-\mu_\star^2\varepsilon)w_{xx}(x) + \mu_\star w_x(x) - 3w^2(x) + 2(b+1)w(x) - b,\ x\in\mathbb{R},$$

(with $f(u) = u(1-u)(u-b)$ and $Df(u) = -3u^2+2(b+1)u-b$) has the solution

$$\lambda = 0,\quad w(x) = v_{\star,x}(x) = -\frac{k}{\sqrt{2}}\frac{e^{\frac{kx}{\sqrt{2}}}}{\left(1+e^{\frac{kx}{\sqrt{2}}}\right)^2},\quad x\in\mathbb{R}.$$

7.4.2 Essential spectrum of traveling waves

Consider again the traveling wave equation

(22) $$0 = (A - \mu_\star^2 M)v_{\star,xx}(x) + \mu_\star B v_{\star,x}(x) + f(v_\star(x)),\ x \in \mathbb{R},$$

with mass matrix $M \in \mathbb{R}^{m,m}$, damping matrix $B \in \mathbb{R}^{m,m}$, diffusion matrix $A \in \mathbb{R}^{m,m}$, nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$, constant asymptotic states $v_\pm \in \mathbb{R}^m$ (i.e. $f(v_\pm) = 0$), translational velocity $\mu_\star \in \mathbb{R}$ and profile $v_\star : \mathbb{R} \to \mathbb{R}^m$ satisfying $v_\star(x) \to v_\pm \in \mathbb{R}^m$ as $x \to \pm\infty$.
Initial value problem: The main idea to detecting the essential spectrum of $\mathcal{L}$ is to look for solutions of the initial value problem of the linearization, i.e.
(23)
$$\begin{aligned} Mv_{tt}(x,t) + Bv_t(x,t) &= (A - \mu_\star^2 M)v_{xx}(x,t) + \mu_\star B v_x(x,t) + Df(v_\star(x))v(x,t) &&, x \in \mathbb{R},\ t > 0,\\ v(x,0) &= u_0(x) &&, x \in \mathbb{R}, t = 0,\\ v_t(x,0) &= v_0(x) + \mu_\star u_{0,x}(x) &&, x \in \mathbb{R}, t = 0. \end{aligned}$$

Decomposition of $Df(v_\star(x))$: Introducing the matrices $Q_\pm(x) \in \mathbb{R}^{m,m}$ via

$$Q_\pm(x) := Df(v_\star(x)) - Df(v_\pm),\ x \in \mathbb{R},$$

we obtain from (23)
(24)
$$Mv_{tt}(x,t)+Bv_t(x,t) = (A-\mu_\star^2 M)v_{xx}(x,t)+\mu_\star B v_x(x,t)+Df(v_\pm)v(x,t)+Q_\pm(x)v(x,t),\ x \in \mathbb{R},\ t > 0.$$

Limiting operator (simplified operator, far-field operator): Since the essential spectrum depends only on the limiting equation for $x \to \pm\infty$, we let formally $x \to \pm\infty$ (but only in the coefficient matrices). Since $Q_+(x) \to 0$ as $x \to +\infty$ and $Q_-(x) \to 0$ as $x \to -\infty$, we can drop the term $Q_\pm(x)$ in (24) and obtain

(25) $$Mv_{tt}(x,t) + Bv_t(x,t) = (A - \mu_\star^2 M)v_{xx}(x,t) + \mu_\star B v_x(x,t) + Df(v_\pm)v(x,t),\ x \in \mathbb{R},\ t > 0.$$

Fourier transform: Since we seek for bounded solutions of (25), we perform a Fourier transformation in space and time. Inserting the Fourier transform

(26) $$v(x,t) = e^{\lambda t}e^{i\omega x}\hat{v},\ \lambda \in \mathbb{C},\ \omega \in \mathbb{R},\ \hat{v} \in \mathbb{C}^m,\ |\hat{v}| = 1$$

into (25) and dividing by $e^{\lambda t}e^{i\omega x}$ yields a finite dimensional quadratic eigenvalue problem

$$\left[\lambda^2 M + \lambda B\right]\hat{v} = \left[-\omega^2(A - \mu_\star^2 M) + i\omega\mu_\star B + Df(v_\pm)\right]\hat{v}$$

Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

(27) $$\det\left(-\omega^2(A - \mu_\star^2 M) + i\omega\mu_\star B + Df(v_\pm) - \lambda B - \lambda^2 M\right) = 0$$

for some $\omega \in \mathbb{R}$ belongs to the essential spectrum of $\mathcal{L}$. Note that the limiting case $M = 0$ and $B = I_m$ leads to the dispersion relation for traveling waves of first-order evolution equations.

Theorem 7.3 (Essential spectrum of traveling waves). *Let $v_\star \in C^2(\mathbb{R},\mathbb{R}^m)$ be a nontrivial classical solution of (22) satisfying $v_\star(x) \to v_\pm$ as $x \to \pm\infty$ for some $v_\pm \in \mathbb{R}$ and let $A, B, M \in \mathbb{R}^{m,m}$, $\mu_\star \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^m,\mathbb{R}^m)$ with $f(v_\pm) = 0$. Then both sets*

$$S_\pm := \{\lambda \in \mathbb{C} \mid \lambda \textit{ satisfies } (27) \textit{ for some } \omega \in \mathbb{R}\}$$

belong to the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ of $\mathcal{L}$, i.e.

$$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) := S_+ \cup S_- \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

Example 7.4 (Damped Fisher's equation). The damped Fisher's equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u),\ x \in \mathbb{R},\ t \geqslant 0,\ \varepsilon > 0$$

with coefficients $M = \varepsilon$, $B = 1$ and $A = 1$ has a traveling wave solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$\mu_\star = \frac{c_\star}{k},\quad k = \pm\sqrt{1 + c_\star^2\varepsilon},\quad c_\star = -2.$$

The profile $v_\star$ connects the asymptotic states $v_- = 0$ and $v_+ = 1$. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$, $f(u) = u(1-u)$ satisfies $f(v_\pm) = 0$, $f'(v_-) = 1$ and $f'(v_+) = -1$. The dispersion relation states that

$$S_\pm := \{\lambda \in \mathbb{C} \mid \lambda^2 + \frac{1}{\varepsilon}\lambda + \frac{1}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm)\right) = 0,\ \omega \in \mathbb{R}\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The quadratic problem has the solutions

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm)\right)}\right),\ \omega \in \mathbb{R}.$$

Defining the four sets

$$S_+^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+)\right)}\right) \omega \in \mathbb{R}\right\},$$

$$S_+^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+)\right)}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-)\right)}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-)\right)}\right) \mid \omega \in \mathbb{R}\right\},$$

the dispersion relation states that

$$S_\pm = S_+^1 \cup S_+^2 \cup S_-^1 \cup S_-^2 \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The essential spectrum of the traveling front in the damped Fisher's equation is illustrated in Figure 7.1.

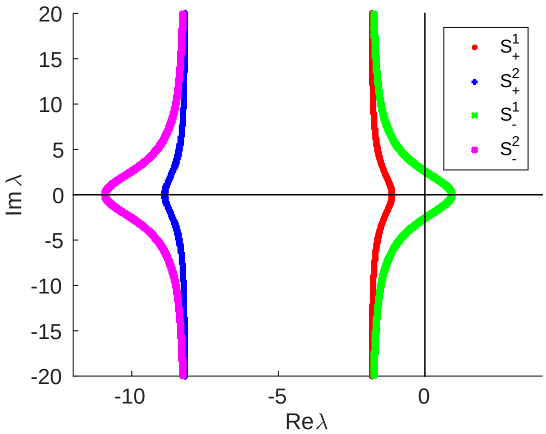

FIGURE 7.1. Essential spectrum of the damped Fisher's equation for parameter the $\varepsilon = \frac{1}{10}$

Example 7.5 (Damped Nagumo equation)**.** The damped Nagumo equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u)(u-b),\ x \in \mathbb{R},\ t \geqslant 0,\ 0 < b < 1,\ \varepsilon > 0$$

with coefficients $M = \varepsilon$, $B = 1$ and $A = 1$ has an explicit traveling front solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$v_\star(x) = \frac{1}{1 + e^{\frac{kx}{\sqrt{2}}}}, \quad \mu_\star = \frac{\sqrt{2}\left(\frac{1}{2} - b\right)}{k} \quad k = -\sqrt{1 + 2\varepsilon\left(\frac{1}{2} - b\right)^2},$$

$v_+ = 1$ and $v_- = 0$. The nonlinearity $f(u) = u(1-u)(u-b)$ satisfies $f(v_\pm) = 0$, $f'(v_+) = b-1$ and $f'(v_-) = -b$. Note that the parameters are $m = 1$, $M = \varepsilon$, $A = 1$ and $B = 1$. The dispersion relation states that

$$S_\pm := \{\lambda \in \mathbb{C} \mid \lambda^2 + \frac{1}{\varepsilon}\lambda + \frac{1}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm)\right) = 0,\, \omega \in \mathbb{R}\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The quadratic problem has the solutions

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm))}\right),\, \omega \in \mathbb{R}.$$

Defining the four sets

$$S_+^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+))}\right) \omega \in \mathbb{R}\right\},$$

$$S_+^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+))}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-))}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-))}\right) \mid \omega \in \mathbb{R}\right\},$$

the dispersion relation states that

$$S_\pm = S_+^1 \cup S_+^2 \cup S_-^1 \cup S_-^2 \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The essential spectrum of the traveling front in the damped Nagumo equation is illustrated in Figure 7.2.

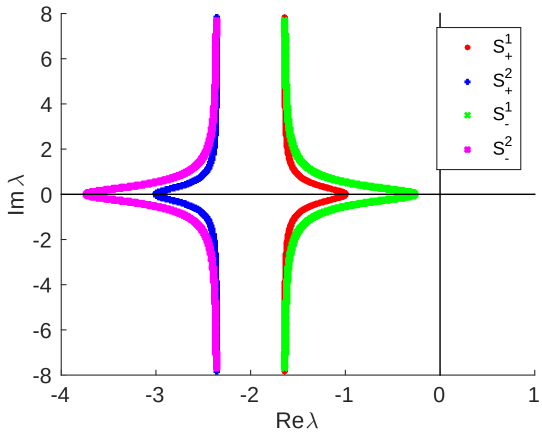

FIGURE 7.2. Essential spectrum of the damped Nagumo equation for parameters $b = \frac{1}{4}$ and $\varepsilon = \frac{1}{4}$

Example 7.6 (Damped quintic Nagumo equation)**.** The damped quintic Nagumo equation

$$\varepsilon u_{tt} + u_t = u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3),\ x \in \mathbb{R},\ t \geqslant 0,\ 0 < \alpha_1 < \alpha_2 < \alpha_3 < 1,\ \varepsilon > 0$$

with coefficients $M = \varepsilon$, $B = 1$, $A = 1$ and parameters

$$\alpha_1 = \frac{2}{5}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{17}{20}$$

has a traveling wave solution $u_\star(x,t) = v_\star(x-\mu_\star t)$ with

$$\mu_\star = \frac{c_\star}{k}, \quad k = \pm\sqrt{1+c_\star^2\varepsilon}, \quad c_\star = 0.07.$$

The profile $v_\star$ connects the asymptotic states $v_- = 0$ and $v_+ = 1$. The nonlinearity

$$f : \mathbb{R} \to \mathbb{R}, \quad f(u) = u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3)$$

satisfies

$$f(v_\pm) = 0, \quad f'(v_-) = -\alpha_1\alpha_2\alpha_3, \quad f'(v_+) = -(1-\alpha_1)(1-\alpha_2)(1-\alpha_3).$$

The dispersion relation states that

$$S_\pm := \{\lambda \in \mathbb{C} \mid \lambda^2 + \frac{1}{\varepsilon}\lambda + \frac{1}{\varepsilon}\left(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm)\right) = 0, \, \omega \in \mathbb{R}\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The quadratic problem has the solutions

$$\lambda_{1,2} = \frac{1}{2}\left(-\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_\pm))}\right), \, \omega \in \mathbb{R}.$$

Defining the four sets

$$S_+^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+))}\right) \omega \in \mathbb{R}\right\},$$

$$S_+^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_+))}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^1 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-))}\right) \mid \omega \in \mathbb{R}\right\},$$

$$S_-^2 := \left\{\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon} - \sqrt{\frac{1}{\varepsilon^2} - \frac{4}{\varepsilon}(\omega^2(1-\mu_\star^2\varepsilon) - i\omega\mu_\star - f'(v_-))}\right) \mid \omega \in \mathbb{R}\right\},$$

the dispersion relation states that

$$S_\pm = S_+^1 \cup S_+^2 \cup S_-^1 \cup S_-^2 \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}).$$

The essential spectrum of the traveling front in the damped quintic Nagumo equation is illustrated in Figure 7.3.

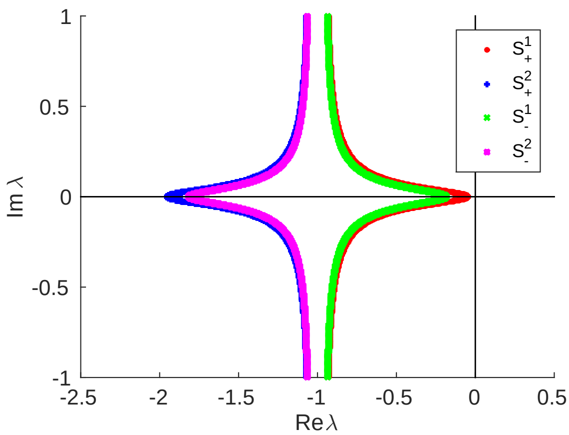

FIGURE 7.3. Essential spectrum of the damped quintic Nagumo equation for parameters $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{17}{20}$ and $\varepsilon = \frac{1}{2}$

Example 7.7 (Damped FitzHugh-Nagumo system)**.** The damped FitzHugh-Nagumo system

$$Mu_{tt} + u_t = Au_{xx} + \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi(u_1 + a - bu_2) \end{pmatrix},\ x \in \mathbb{R},\ t \geqslant 0$$

with coefficient matrices

$$M = \varepsilon I_2, \quad B = I_2, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho + c_\star^2 \varepsilon}{1 + c_\star^2 \varepsilon} \end{pmatrix}$$

and parameters

$$\rho = 0.1, \quad \phi = 0.08, \quad a = 0.7, \quad b > 0, \quad , \varepsilon > 0$$

has a traveling wave solution $u_\star(x,t) = v_\star(x - \mu_\star t)$, namely a traveling front solution for $b = 3$ and a traveling pulse solution for $b = 0.8$ with

$$\mu_\star = \frac{c_\star}{k}, \quad k = \pm\sqrt{1 + c_\star^2 \varepsilon}, \quad c_\star = \begin{cases} -0.8557 & \text{, if } b = 3 \text{ (front)}, \\ -0.7892 & \text{, if } b = 0.8 \text{ (pulse)}. \end{cases}$$

The profile $v_\star$ connects the asymptotic states

$$v_- = \begin{pmatrix} 1.18769696080266 \\ 0.629232320266825 \end{pmatrix} \quad \text{and} \quad v_+ = \begin{pmatrix} -1.56443178284120 \\ -0.288143927613547 \end{pmatrix}$$

if $b = 3$, and the asymptotic state

$$v_\pm = \begin{pmatrix} -1.19940803524404 \\ -0.624260044055044 \end{pmatrix}$$

if $b = 0.8$, i.e. $v_\star \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile $v_\star$ nor the velocity $\mu_\star$ are given explicitly. The nonlinearity

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(u) = \begin{pmatrix} u_1 - \frac{1}{3}u_1^3 - u_2 \\ \phi(u_1 + a - bu_2) \end{pmatrix}$$

satisfies

$$f(v_\pm) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Df(v_\pm) = \begin{pmatrix} 1 - \left(v_\pm^{(1)}\right)^2 & -1 \\ \phi & -b\phi \end{pmatrix}.$$

The dispersion relation states that

$$S_\pm := \{\lambda \in \mathbb{C} \mid \det \begin{pmatrix} a_1 - \lambda - \varepsilon\lambda^2 & -1 \\ \phi & a_2 - \lambda - \varepsilon\lambda^2 \end{pmatrix} = 0 \text{ for some } \omega \in \mathbb{R}\} \subseteq \sigma_{\text{ess}}(\mathcal{L})$$

where we used the abbreviations

$$\begin{aligned} a_1 &= -\omega^2(1 - \mu_\star^2 \varepsilon) + i\omega\mu_\star + 1 - \left(v_\pm^{(1)}\right)^2, \\ a_2 &= -\omega^2 \left(\frac{\rho + c_\star^2 \varepsilon}{1 + c_\star^2 \varepsilon} - \mu_\star^2 \varepsilon \right) + i\omega\mu_\star - b\phi. \end{aligned}$$

This leads to a quartic problem

$$\begin{aligned} 0 = \det \begin{pmatrix} a_1 - \lambda - \varepsilon\lambda^2 & -1 \\ \phi & a_2 - \lambda - \varepsilon\lambda^2 \end{pmatrix} &= (a_1 - \lambda - \varepsilon\lambda^2)(a_2 - \lambda - \varepsilon\lambda^2) + \phi \\ &= \varepsilon^2\lambda^4 + 2\varepsilon\lambda^3 + (1 - \varepsilon(a_1 + a_2))\lambda^2 - (a_1 + a_2)\lambda + \phi + a_1 a_2 \\ &= a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e \end{aligned}$$

with

$$a = \varepsilon^2, \quad b = 2\varepsilon, \quad c = (1 - \varepsilon(a_1 + a_2)), \quad d = -(a_1 + a_2), \quad e = \phi + a_1 a_2$$

Example 7.8 (Damped Barkley model)**.** The damped Barkley model

$$Mu_{tt} + u_t = Au_{xx} + \begin{pmatrix} \frac{1}{c}u_1\left(1-u_1\right)\left(u_1 - \frac{u_2+b}{a}\right) \\ u_1 - u_2 \end{pmatrix},\, x\in\mathbb{R},\, t\geqslant 0$$

with coefficient matrices

$$M = \varepsilon I_2, \quad B = I_2, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\rho + c_\star^2\varepsilon}{1+c_\star^2\varepsilon} \end{pmatrix},$$

parameters (parabolic case)

$$\rho = \frac{1}{10}, \quad a = \frac{3}{4}, \quad b = \frac{1}{100}, \quad c = \frac{1}{50}, \quad ,\varepsilon > 0$$

and parameters (parabolic-hyperbolic case)

$$\rho = 0, \quad a = \frac{3}{4}, \quad b = \frac{1}{100}, \quad c = \frac{1}{50}, \quad ,\varepsilon > 0$$

has a traveling pulse solution $u_\star(x,t) = v_\star(x - \mu_\star t)$ with

$$\mu_\star = \frac{c_\star}{k}, \quad k = \pm\sqrt{1 + c_\star^2\varepsilon}, \quad c_\star = \begin{cases} 4.6616 & \text{(parabolic case)}, \\ 4.6785 & \text{(parabolic-hyperbolic case)}. \end{cases}$$

The profile $v_\star$ connects the asymptotic state

$$v_\pm = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $v_\star \to v_\pm$ as $x \to \pm\infty$. Note that neither the profile $v_\star$ nor the velocity $\mu_\star$ are given explicitly. The nonlinearity

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(u) = \begin{pmatrix} \frac{1}{c}u_1\left(1-u_1\right)\left(u_1 - \frac{u_2+b}{a}\right) \\ u_1 - u_2 \end{pmatrix}$$

satisfies

$$f(v_\pm) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad Df(v_\pm) = \begin{pmatrix} -\frac{b}{ac} & 0 \\ 1 & -1 \end{pmatrix}.$$

The dispersion relation states that

$$S_\pm := \{\lambda \in \mathbb{C} \mid \det \begin{pmatrix} a_1 - \lambda - \varepsilon\lambda^2 & 0 \\ 1 & a_2 - \lambda - \varepsilon\lambda^2 \end{pmatrix} = 0 \text{ for some } \omega \in \mathbb{R}\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$

where we used the abbreviations

$$a_1 = -\omega^2(1 - \mu_\star^2\varepsilon) + i\omega\mu_\star - \frac{b}{ac},$$

$$a_2 = -\omega^2\left(\frac{\rho + c_\star^2\varepsilon}{1 + c_\star^2\varepsilon} - \mu_\star^2\varepsilon\right) + i\omega\mu_\star - 1.$$