Mathematical Models of Reaction Diffusion Systems, their Numerical Solutions and the Freezing Method with Comsol Multiphysics

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1 Mathematical Models

In this chapter I want to give a large assortment of examples for reaction-diffusion equations, their phenomena and some background stuff. The focus of this investigations will be the freezing method ([12]) and their numerical computations with Comsol Multiphysics 3.5a. For this purpose we use the following notions:

\( d \)  
spatial dimension, \( d \in \{1, 2, 3\} \)

\( \mathbb{R}^d \)  
\( d \)-dimensional euclidean space

\( x \)  
spatial variable, \( x \in \mathbb{R}^d \)

\( t \)  
temporal variable, \( t \in [0, \infty[ \)

\( R \)  
ratio, \( R \in \mathbb{R} \) with \( R > 0 \)

\( B_R(0) \)  
ball in \( \mathbb{R}^d \) with ratio \( R \) and center \( 0 \)

\( \partial B_R(0) \)  
boundary of \( B_R(0) \)

\( \triangle x \)  
spatial discretization parameter, spatial stepsize

\( \triangle t \)  
temporal discretization parameter, temporal stepsize

\( \text{Re}(u) \)  
real part of \( u \)

\( \text{Im}(u) \)  
imaginary part of \( u \)

\( i \)  
imaginary unit

\( \partial \)  
normal derivative, derivative in outer direction
1.1 Fisher’s equation

**Name:** Fisher’s equation (sometimes called Kolmogorov-Petrovsky-Piskounov equation (KPP) or Fisher-Kolmogorov equation)

**Equations:**

\[ u_t = \Delta u + au(1 - u) \]

where \( u = u(x,t) \in \mathbb{R} \), \( x \in \mathbb{R}^d \), \( d \in \{1, 2, 3\} \), \( t \in [0, \infty) \) and \( a \in \mathbb{R} \).

**Notations:**

- \( u \): frequency of the mutant gene
- \( a \): system parameter

**Short description:** The Fisher’s equation ([31]), named after Ronald Aylmer Fisher (1890-1962) and Andrey Nicolaevich Kolmogorov (1903-1987), describes the spreading of biological populations, chemical reaction processes, heat and mass transfer ([28]) and is also used in genetics. This very simple model exhibits traveling front solutions.

**Phenomena:**
- Traveling 1-front (traveling front)

**Set of parameter values:**

<table>
<thead>
<tr>
<th>( d )</th>
<th>( a )</th>
<th>( R )</th>
<th>( \Delta x )</th>
<th>( \Delta t )</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>75</td>
<td>0.1</td>
<td>0.1</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
<td>1-front</td>
</tr>
</tbody>
</table>

**Numerical results:**
- Traveling 1-front (traveling front):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= u_{xx} + au(1 - u) , \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0 , \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0 , \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( a = 1 \), \( R = 75 \) and initial data \( u_0(x) = \frac{1}{2} \tanh(x) + \frac{1}{2} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-2} \), \( atol = 10^{-3} \) and intermediate timesteps.

Nonfrozen solution:
1.1 Fisher’s equation

Spatial-temporal pattern:

II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{align*}
  v_t &= v_{xx} + av(1 - v) + \lambda_1 v_x, & x \in B_R(0), & t \in [0, \infty[ \\
  \frac{\partial v}{\partial n} &= 0, & x \in \partial B_R(0), & t \in [0, \infty[ \\
  v(0) &= u_0, & x \in B_R(0), & t = 0 \\
  \gamma_t &= \lambda_1, & t \in [0, \infty[ \\
  \gamma(0) &= 0, & t = 0 \\
  0 &= \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{R})}, & t \in [0, \infty[ 
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use again parameters \( a = 1, R = 75 \), initial data \( u_0(x) = \frac{1}{2} \tanh(x) + \frac{1}{2} \) and reference function \( \hat{v}(x) = u_0(x) \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \triangle x = 0.1 \). For the temporal discretization we use BDF(2) with \( \triangle t = 0.01 \), \( rtol = 10^{-3} \), \( atol = 10^{-4} \) and intermediate timesteps.

Frozen solution, velocity and spatial-temporal pattern:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
The increasing convergence error indicates that Fisher’s-Front cannot be frozen by the freezing method. This can be recognized by the essential spectra. Because to ensure the stability of the front the essential spectra must exhibit a spectral gap, which doesn’t exist.

Explicit solutions:
- **1d:**
  \[
  u(x,t) = \frac{1}{(1 + C \exp \left( -\frac{5}{6} \lambda t \pm \frac{x}{6} \sqrt{6\lambda} \right))^{2}}, \text{ if } \lambda > 0
  \]
  \[
  u(x,t) = \frac{1 + 2C \exp \left( -\frac{5}{6} \lambda t \pm \frac{x}{6} \sqrt{-6\lambda} \right)}{(1 + C \exp \left( -\frac{5}{6} \lambda t \pm \frac{x}{6} \sqrt{-6\lambda} \right))^{2}}, \text{ if } \lambda < 0
  \]
  are two possible solutions of the nonfrozen system with parameter \( \lambda \in \mathbb{R} \) and an arbitrary constant \( C \in \mathbb{R} \) ([1],[51],[48]).

**Literature:** [28], [1], [51], [22], [31], [48]
1.2 Nagumo equation

**Name:** Nagumo equation

**Equations:**

\[ u_t = \Delta u + u (1 - u) (u - \alpha) \]

\[ u = u(x, t) \in \mathbb{R}, \ x \in \mathbb{R}^d, \ d \in \{1, 2, 3\}, \ t \in [0, \infty[, \ \alpha \in ]0, 1[. \]

**Notations:** not available

**Short description:** The Nagumo equation ([62]), named after Jin-Ichi Nagumo (1926-1999), describes propagation of nerve pulses in a nerve axon ([62]), spread of genetic traits, shape and speed of pulses in the nerve and is widely used in biology, circuit theory, heat and mass transfer and other fields. This model exhibits traveling front and traveling multifront solutions as well as sources and sinks ([4]).

**Phenomena:**

- Traveling 1-front (traveling front)
- Traveling 2-front (traveling multifront)
- Front interaction (3-front collision)

**Set of parameter values**:

<table>
<thead>
<tr>
<th>(d)</th>
<th>(\alpha)</th>
<th>(R)</th>
<th>(\Delta x)</th>
<th>(\Delta t)</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{3})</td>
<td>75</td>
<td>0.1</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n} = 0) on (\partial B_R(0))</td>
<td>1-front</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{3})</td>
<td>100</td>
<td>0.1</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n} = 0) on (\partial B_R(0))</td>
<td>2-front</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{3})</td>
<td>150</td>
<td>0.3</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n} = 0) on (\partial B_R(0))</td>
<td>3-front collision</td>
</tr>
</tbody>
</table>

**Numerical results:**

- Traveling 1-front (traveling front):
  
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u)(u - \alpha) , \ x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0 , \ x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0 , \ x \in B_R(0), \ t = 0
\end{align*}
\]

where \(d = 1\) (i.e. \(B_R(0) = [-R, R]\)). For the numerical computations we use parameters \(\alpha = \frac{1}{4}\), \(R = 50\) and initial data \(u_0(x) = \frac{1}{2} \tanh(x) + \frac{1}{2}\) (also \(u_0 = \begin{cases} 0 & x < 0 \\ \frac{1}{R} & x \geq 0 \end{cases}\) possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.1\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.1\), \(rtol = 10^{-2}\), \(atol = 10^{-3}\) and intermediate timesteps ([13],[12]).

Nonfrozen solution:
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[\begin{align*}
v_t & = v_{xx} + v(1 - v)(v - \alpha) + \lambda_1 v_x, & x \in B_R(0), & t \in [0, \infty] \\
\partial_tv & = 0, & x \in \partial B_R(0), & t \in [0, \infty] \\
v(0) & = u_0, & x \in B_R(0), & t = 0 \\
\gamma_t & = \lambda_1, & t \in [0, \infty] \\
\gamma(0) & = 0, & t = 0 \\
0 & = \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{R})}, & t \in [0, \infty]
\end{align*}\]

where \(d = 1\) (i.e. \(B_R(0) = [-R, R]\)). For the numerical computations we use again parameters \(\alpha = \frac{1}{4}\), \(R = 50\), initial data \(u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}\) and reference function \(\hat{v}(x) = u_0(x)\). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.1\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.1\), \(rtol = 10^{-3}\), \(atol = 10^{-4}\) and intermediate timesteps.

Frozen solution, velocity and spatial-temporal pattern:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
1.2 Nagumo equation

- Traveling 2-front (traveling multifront):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
    u_t &= u_{xx} + u(1 - u)(u - \alpha) \quad x \in B_R(0), \ t \in [0, \infty[ \\
    \frac{\partial u}{\partial n} &= 0 \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
    u(0) &= u_0 \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha = \frac{1}{4} \), \( R = 100 \) and initial data \( u_0(x) = -\frac{1}{2} \tanh (x - 50) + \frac{1}{2} + \begin{cases} 
\frac{1}{2} \tanh (x + 50) - \frac{1}{2} & x \leq 0 \\
0 & x > 0
\end{cases} \) (also \( u_0 = \exp \left( -\frac{x^2}{4R} \right) \) possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-2}, atol = 10^{-3} \) and intermediate timesteps ([11])).

Nonfrozen solution:

Spatial-temporal pattern:

II. Frozen system (with freezing method ([11],[73])): Consider the associated frozen system (\( j = 1, 2 \))
\[
\begin{align*}
  v_{j,t} &= v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{2} \varphi(-g_{k} + g_{j})} \cdot f \left( \sum_{k=1}^{2} v_{k}(\cdot - g_{k} + g_{j}) \right), \\
  \frac{\partial v_{j}}{\partial n} &= 0, \\
  v_{j}(0) &= u_{j_{0}}, \\
  g_{j,t} &= \mu_{j}, \\
  g_{j}(0) &= g_{j_{0}}, \\
  0 &= \left\langle (\hat{v}_{j})_{x}, v_{j} - \hat{v}_{j} \right\rangle_{L^{2}(B_{R}(0), \mathbb{R})},
\end{align*}
\]

where \( f(v) = v(1 - v)(v - \alpha) \) and \( d = 1 \) (i.e. \( B_{R}(0) = [-R, R] \)). For the numerical computations we use again parameters \( \alpha = \frac{1}{4}, R = 50 \), initial data \( u_{10}(x) = \frac{1}{2} \tanh(x) + \frac{1}{2}, u_{20}(x) = -\frac{1}{2} \tanh(x) - \frac{1}{2} \), initial positions \( \gamma_{10} = -50, \gamma_{20} = 50 \), reference functions \( \hat{v}_{1}(x) = u_{10}(x), \hat{v}_{2}(x) = u_{20}(x) \) and bump function \( \varphi(x) = \frac{\exp(\frac{1}{2} x) + \exp(-\frac{1}{2} x)}{2} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, rtol = 10^{-3}, atol = 10^{-4} \) and intermediate timesteps.

Frozen solutions:

Velocities and positions:

Spatial-temporal patterns:
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

\[ |v_t|_{L^2} \]

\[ t \]

\[ ||v_t|| \]

\[ \sqrt{||v_t||_{L^2}} \]

\[ \text{Convergence error} \]

\[ \text{Eigenvalues} \]

- Front interaction (3-front collision):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t & = u_{xx} + u(1 - u)(u - \alpha), \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} & = 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) & = u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha = \frac{1}{4}, R = 150 \) and initial data \( u_0 = \begin{cases} \
  \frac{1}{2} \tanh(x + 75) + \frac{1}{2} & x \leq -50 \\
  \frac{1}{2} \tanh(x) + \frac{1}{2} & x \in [-50, 50] \quad \text{possible}. \\
  \frac{1}{2} \tanh(x + 75) + \frac{1}{2} & x \geq 50
\end{cases} \)

spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \triangle x = 0.3 \). For the temporal discretization we use BDF(2) with \( \triangle t = 0.1, rtol = 10^{-2}, atol = 10^{-3} \) and intermediate timesteps ([13],[12]).

Nonfrozen solution:

Spatial-temporal pattern:
II. Frozen system (with freezing method ([11])): Consider the associated frozen system \((j = 1, 2, 3)\)

\[
v_{j,t} = v_{j,xx} + \frac{\varphi(v)}{\sum_{k=1}^{3} \varphi(-g_k + g_j)} \cdot f \left( \sum_{k=1}^{3} v_k(-g_k + g_j) \right) + \mu_j v_{j,x}, \quad x \in B_R(0), \quad t \in [0, \infty[ \\
\frac{\partial v_j}{\partial n} = 0, \quad x \in \partial B_R(0), \quad t \in [0, \infty[ \\
v_j(0) = u_{j0} \\
g_j,t = \mu_j \\
g_j(0) = g_{j0} \\
0 = \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0), \mathbb{R})}
\]

where \(f(v) = v(1-v)(v-\alpha)\) and \(d = 1\) (i.e. \(B_R(0) = [-R, R]\)). For the numerical computations we use again parameters \(\alpha = \frac{1}{4}, \ R = 50\), initial data \(u_{10}(x) = \begin{cases} 1 & x > 25 \\ \frac{1}{50} x + \frac{1}{2} & x \in [-25, 25] \\ 0 & x < 25 \end{cases}\), \(u_{20}(x) = \begin{cases} -1 & x > 25 \\ -\frac{1}{50} x - \frac{1}{2} & x \in [-25, 25] \\ 0 & x < 25 \end{cases}\), initial positions \(\gamma_{10} = -50, \ \gamma_{20} = 0, \ \gamma_{30} = 50\), reference functions \(\hat{v}_1(x) = u_{10}(x), \ \hat{v}_2(x) = u_{20}(x), \ \hat{v}_3(x) = u_{30}(x)\) and bump function \(\varphi(x) = \frac{\exp(\frac{1}{2} x) + \exp(-\frac{1}{2} x)}{2}\).

Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.1\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.1, \ rtol = 10^{-2}, \ atol = 10^{-2}\) and intermediate timesteps.

Frozen solutions:

Velocities, positions and Spatial-temporal patterns:

Spatial-temporal patterns:
1.2 Nagumo equation

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Explicit solutions:

• 1d: \( u(x) = \overline{u}(\xi) \) with \( \xi = x - \mu t \) where

\[
\overline{u}(\xi) = \left( 1 + \exp \left( -\frac{\xi}{\sqrt{2}} \right) \right)^{-1} \quad \text{with speed} \quad \mu = -\sqrt{2} \left( \frac{1}{2} - \alpha \right)
\]

\[
\overline{u}(\xi) = \left( 1 + \exp \left( \frac{\xi}{\sqrt{2}} \right) \right)^{-1} \quad \text{with speed} \quad \mu = \sqrt{2} \left( \frac{1}{2} - \alpha \right)
\]

are two possible profiles \( \overline{u} \) with their velocity \( \mu \) and parameter \( \alpha \in [0, \frac{1}{2}] \) ([23],[4]).

• 1d:

\[
u(x,t) = \frac{A \exp \left( \pm \frac{\sqrt{2}}{2} x + \left( \frac{1}{2} - \alpha \right) t \right) + \alpha B \exp \left( \pm \frac{\sqrt{2}}{2} x \alpha x + \alpha \left( \frac{a}{2} - 1 \right) t \right)}{A \exp \left( \pm \frac{\sqrt{2}}{2} x + \left( \frac{1}{2} - \alpha \right) t \right) + B \exp \left( \pm \frac{\sqrt{2}}{2} x \alpha x + \alpha \left( \frac{a}{2} - 1 \right) t \right) + C}
\]

where \( A, B, C \) are arbitrary constants ([49]).

Additional informations: The Nagumo equation (sometimes called Allen-Cahn model or Zeldovich equation arising in combustion theory ([86])) is a simplified form of the FitzHugh-Nagumo equations (to this later). Especially for \( \alpha = -1 \) we receive a special case of the Chafee-Infante equation (sometimes called Newell-Whitehead-Segel equation describing Rayleigh-Benard convections ([63],[72]))

\[
u_t = \Delta u + \lambda (u - u^3), \quad \lambda \in \mathbb{R}
\]

or more general

\[
u_t = \Delta u + au - bu^3, \quad a, b \in \mathbb{R}
\]

which in general possesses an attractor as well as a pitchfork bifurcation ([65],[43]).

Literature: [62], [4], [13], [11], [23], [49], [12]
1.3 Quintic Nagumo equation

Name: **Quintic Nagumo equation**

**Equations:**

\[ u_t = \Delta u + u (1 - u)(u - \alpha_1)(u - \alpha_2)(u - \alpha_3) \]

\[ u = u(x, t) \in \mathbb{R}, x \in \mathbb{R}^d, d \in \{1, 2, 3\}, t \in [0, \infty[, \alpha_1, \alpha_2, \alpha_3 \in ]0, 1[ \text{ with } \alpha_1 < \alpha_2 < \alpha_3. \]

**Notations:** not available

**Short description:** The Quintic Nagumo equation, named after Jin-Ichi Nagumo (1926-1999) describes an extension of the Nagumo equation and was developed only for mathematical investigations ([11],[73]). This model exhibits traveling front and traveling multiform solutions.

**Phenomena:**

- Traveling 1-front (traveling front)
- Traveling 2-front (traveling multiform)
- Front interaction (2-front collision)
- Traveling 3-front (traveling multiform)
- Traveling 4-front (traveling multiform)
- Front interaction (4-front double collision)

**Set of parameter values:**

<table>
<thead>
<tr>
<th>d</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_3 )</th>
<th>( R )</th>
<th>( \Delta x )</th>
<th>( \Delta t )</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{2}{7} )</td>
<td>( \frac{7}{17} )</td>
<td>( \frac{17}{20} )</td>
<td>100</td>
<td>0.3</td>
<td>0.3</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
<td>1-front</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{3}{17} )</td>
<td>( \frac{7}{10} )</td>
<td>( \frac{17}{20} )</td>
<td>100</td>
<td>0.3</td>
<td>0.3</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
<td>2-front</td>
</tr>
<tr>
<td>1</td>
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<td>( \frac{7}{10} )</td>
<td>( \frac{17}{20} )</td>
<td>250</td>
<td>0.4</td>
<td>1.0</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
<td>3-front</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{3}{10} )</td>
<td>( \frac{7}{10} )</td>
<td>( \frac{17}{20} )</td>
<td>150</td>
<td>0.3</td>
<td>0.5</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
<td>4-front</td>
</tr>
</tbody>
</table>

**Numerical results:**

- Traveling 1-front (traveling front):
  - Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= u_{xx} + u(1 - u)(u - \alpha_1)(u - \alpha_2)(u - \alpha_3), \quad x \in B_R(0), t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{2}{7}, \alpha_2 = \frac{3}{17}, \alpha_3 = \frac{17}{20}, R = 100 \) and initial data \( u_0(x) = \frac{1}{2} \tanh(x) + \frac{1}{2} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.3, rtol = 10^{-2}, atol = 10^{-3} \) and intermediate timesteps.

Nonfrozen solution:
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{align*}
    v_t &= v_{xx} + v(1 - v)(v - \alpha_1)(v - \alpha_2)(v - \alpha_3) + \lambda_1 v_x, \quad x \in B_R(0), \ t \in [0, \infty] \\
    \frac{\partial v}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty] \\
    v(0) &= u_0, \quad x \in B_R(0), \ t = 0 \\
    \gamma_t &= \lambda_1, \quad t \in [0, \infty] \\
    \gamma(0) &= 0, \quad t = 0 \\
    0 &= \langle \tilde{v}_x, v - \tilde{v} \rangle_{L^2(B_R(0), \mathbb{R})}
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use again parameters \( \alpha_1 = \frac{2}{5}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{17}{20}, R = 100 \), initial data \( u_0(x) = \frac{1}{2} \tanh(x) + \frac{1}{2} \) and reference function \( \tilde{v}(x) = u_0(x) \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.2, rtol = 10^{-3}, atol = 10^{-4} \) and intermediate timesteps.

Frozen solution, velocity and spatial-temporal pattern:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
• Traveling 2-front (traveling multifront):
  I. Nonfrozen solution: Consider the nonfrozen system

  \[
  \begin{align*}
  u_t &= u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3), \quad x \in B_R(0), \; t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \; t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \; t = 0
  \end{align*}
  \]

  where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{1}{16}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{7}{10}, R = 100 \) and initial data \( u_0(x) = \frac{1}{4} \tanh \left( \frac{x+25}{5} \right) + \frac{1}{4} \tanh \left( \frac{x-25}{5} \right) + \frac{1}{2} \) (also \( u_0 = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{2} \) possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.3, \; \text{rtol} = 10^{-2}, \; \text{atol} = 10^{-3} \) and intermediate timesteps.

  Nonfrozen solution:

  ![Nonfrozen solution](image1)

  Spatial-temporal pattern:

  ![Spatial-temporal pattern](image2)

  II. Frozen system (with freezing method ([11])): Consider the associated frozen system \( (j = 1, 2) \)
\[ v_{j,t} = v_{j,xx} + \varphi(-g_k + g_j) \cdot f \left( \sum_{k=1}^{2} v_k (\cdot - g_k + g_j, \cdot) \right) + \mu_j v_{j,x}, \quad x \in B_{R}(0), \ t \in [0, \infty[ \]

\[ \frac{\partial v_j}{\partial n} = 0, \quad x \in \partial B_{R}(0), \ t \in [0, \infty[ \]

\[ v_j(0) = u_{j0}, \quad x \in B_{R}(0), \ t = 0 \]

\[ g_{j,t} = \mu_j, \quad t \in [0, \infty[ \]

\[ g_j(0) = g_{j0}, \quad t = 0 \]

\[ 0 = \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_{R}(0), \mathbb{R})}, \quad t \in [0, \infty[ \]

where \( f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3) \) and \( d = 1 \) (i.e. \( B_{R}(0) = [-R,R] \)). For the numerical computations we use again parameters \( \alpha_1 = \frac{1}{16}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{7}{16}, R = 50 \), initial data \( u_{t0}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4}, \) \( u_{20}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4} \), initial positions \( \gamma_{10} = -50, \gamma_{20} = 50 \), reference functions \( \hat{v}_1(x) = u_{10}(x), \hat{v}_2(x) = u_{20}(x) \) and bump function \( \varphi(x) = \frac{1}{2} \exp \left( \frac{1}{2} x \right) + \exp \left( -\frac{1}{2} x \right) \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \triangle x = 0.3 \). For the temporal discretization we use BDF(2) with \( \triangle t = 0.3, rtol = 10^{-3}, atol = 10^{-4} \) and intermediate timesteps.

Frozen solutions:

Velocities and positions:

Spatial-temporal patterns:
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Front interaction (2-front collision):
  I. Nonfrozen solution: Consider the nonfrozen system

  \[ u_t = u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3), \quad x \in B_R(0), \ t \in [0, \infty[ \]

  \[ \frac{\partial u}{\partial n} = 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \]

  \[ u(0) = u_0, \quad x \in B_R(0), \ t = 0 \]

  where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{3}{4}, \ R = 100 \) and initial data \( u_0(x) = \frac{1}{2} \text{tanh} \left( \frac{x+25}{5} \right) + \frac{1}{2} \text{tanh} \left( \frac{x-25}{5} \right) + \frac{1}{2} \) (also \( u_0 = \frac{1}{2} \text{tanh} \left( \frac{x}{5} \right) + \frac{1}{2} \) possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.3, \ rtol = 10^{-2}, \ atol = 10^{-3} \) and intermediate timesteps.

  Nonfrozen solution:

  \[ \text{Spatial-temporal pattern:} \]

II. Frozen system (with freezing method ([11])): Consider the associated frozen system \((j = 1, 2)\)
1.3 Quintic Nagumo equation

\[ v_{j,t} = v_{j,xx} + \sum_{k=1}^{2} \frac{\varphi(-g_{k}+g_{j})}{\varphi(-g_{k}+g_{j})} \cdot f \left( \sum_{k=1}^{2} v_{k} \left( -g_{k} + g_{j} \right) \right) + \mu_{j} v_{j,x}, \quad x \in B_{R}(0), \ t \in [0, \infty[ \]

\[ \frac{\partial v_{j}}{\partial n} = 0, \quad x \in \partial B_{R}(0), \ t \in [0, \infty[ \]

\[ v_{j}(0) = u_{j0}, \quad x \in B_{R}(0), \ t = 0 \]

\[ g_{j,t} = \mu_{j}, \quad x \in [0, \infty[ \]

\[ g_{j}(0) = g_{j0}, \quad t = 0 \]

\[ 0 = \langle (\hat{v}_{j})_{x}, v_{j} - \hat{v}_{j} \rangle_{L^{2}(B_{R}(0), \mathbb{R})} \quad , \ t \in [0, \infty[ \]

where \( f(v) = v(1-v)(v-\alpha_{1})(v-\alpha_{2})(v-\alpha_{3}) \) and \( d = 1 \) (i.e. \( B_{R}(0) = [-R, R] \)). For the numerical computations we use again parameters \( \alpha_{1} = \frac{1}{4}, \ \alpha_{2} = \frac{1}{2}, \ \alpha_{3} = \frac{3}{4}, \ R = 50 \), initial data \( u_{10}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4}, \ u_{20}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4} \), initial positions \( \gamma_{10} = -50, \ \gamma_{20} = 50 \), reference functions \( \hat{v}_{1}(x) = u_{10}(x), \ \hat{v}_{2}(x) = u_{20}(x) \) and bump function \( \varphi(x) = \frac{1}{2} e^{\frac{1}{2} x} + e^{-\frac{1}{2} x} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 1.5, \ rtol = 10^{-3}, \ atol = 5 \cdot 10^{-4} \) and intermediate timesteps.

Frozen solutions:

Velocities and positions:

Spatial-temporal patterns:
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 3-front (traveling multiframe):
  I. Nonfrozen solution: Consider the nonfrozen system
  \[
  \begin{align*}
  u_t &= u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3), \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \ t = 0
  \end{align*}
  \]
  where \( d = 1 \) (i. e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{1}{16}, \ \alpha_2 = \frac{1}{2}, \ \alpha_3 = \frac{7}{10}, \ R = 250 \) and initial data \( u_0(x) = \frac{1}{4} \tanh(x+50) + \frac{1}{4} \tanh(x) - \frac{1}{4} \tanh(x-50) + \frac{1}{4} \).
  Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.4. \) For the temporal discretization we use BDF(2) with \( \Delta t = 1.0, \ rtol = 10^{-3}, \ atol = 10^{-4} \) and intermediate timesteps.
  Nonfrozen solution:

  ![Solution u, t=100](image1)
  ![Solution u, t=1000](image2)
  ![Solution u, t=2000](image3)

  Spatial-temporal pattern:

  ![u(x,t)](image4)

  II. Frozen system (with freezing method ([11])): Consider the associated frozen system (\( j = 1, 2, 3 \))
\[ v_{j,t} = v_{j,xx} + \frac{\varphi(\cdot)}{\mu_j v_{j,x}} \cdot f \left( \sum_{k=1}^{3} v_k (\cdot - g_k + g_j, \cdot) \right) \]

\[ \frac{\partial v_j}{\partial n} = 0, \quad (x \in \partial B_R(0), t \in [0, \infty[) \]

\[ v_j(0) = u_{j0}, \quad g_j(0) = g_{j0} \]

\[ 0 = \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0), \mathbb{R})}, \quad t \in [0, \infty] \]

where \( f(v) = v(1 - v)(v - \alpha_1)(v - \alpha_2)(v - \alpha_3) \) and \( d = 1 \) (i.e., \( B_R(0) = [-R, R] \)). For the numerical computations we use again parameters \( \alpha_1 = \frac{1}{16}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{7}{16}, R = 200 \), initial data \( u_{10}(x) = \frac{1}{2} \tanh \left( \frac{x}{5} \right) + \frac{1}{4}, u_{20}(x) = u_{10}(x), u_{30}(x) = -\frac{1}{4} \tanh \left( \frac{x}{5} \right) - \frac{1}{4} \), initial positions \( \gamma_{10} = -50, \gamma_{20} = 0, \gamma_{30} = 50 \), reference functions \( \hat{v}_1(x) = u_{10}(x), \hat{v}_2(x) = u_{20}(x), \hat{v}_3(x) = u_{30}(x) \) and bump function \( \varphi(x) = \frac{\exp(\frac{x}{2}) + \exp(-\frac{x}{2})}{4} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.4 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.3, \text{rtol} = 10^{-3}, \text{atol} = 10^{-4} \) and intermediate timesteps.

Frozen solutions:

Velocities, positions and Spatial-temporal patterns:

Spatial-temporal patterns:
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 4-front (traveling multiframe):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
    u_t &= u_{xx} + u(1-u)(u - \alpha_1)(u - \alpha_2)(u - \alpha_3), & x \in B_R(0), t \in [0, \infty[ \\
    \frac{\partial u}{\partial n} &= 0, & x \in \partial B_R(0), t \in [0, \infty[ \\
    u(0) &= u_0, & x \in B_R(0), t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{1}{16}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{7}{40}, R = 150 \) and initial data \( u_0(x) = \frac{1}{4} \tanh \left( \frac{x+100}{5} \right) + \frac{1}{4} \tanh \left( \frac{x+50}{5} \right) - \frac{1}{4} \tanh \left( \frac{x-50}{5} \right) - \frac{1}{4} \tanh \left( \frac{x-100}{5} \right) \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.5, rtol = 10^{-3}, atol = 10^{-4} \) and intermediate timesteps.

Nonfrozen solution:

- Frozen system (with freezing method ([11])): Consider the associated frozen system \( (j = 1, 2, 3, 4) \).
1.3 Quintic Nagumo equation

\begin{align*}
v_{j,t} &= v_{j,xx} + \frac{\varphi(\cdot)}{\varphi(-g_k + g_j)} \cdot f\left( \sum_{k=1}^{4} v_k (\cdot - g_k + g_j) \right) \\
\frac{\partial v_j}{\partial n} &= 0, ~ x \in BR(0), \quad t \in [0, \infty[ \\
v_j(0) &= u_{j0}, \quad x \in BR(0), \quad t = 0 \\
g_{j,t} &= \mu_j g_j, \quad t \in [0, \infty[ \\
g_j(0) &= g_{j0}, \quad t = 0
\end{align*}

where \( f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3) \) and \( d = 1 \) (i.e. \( BR(0) = [-R, R] \)). For the numerical computations we use again parameters \( \alpha_1 = \frac{1}{10}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{7}{10}, R = 50 \), initial data \( u_{10}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4}, u_{20}(x) = u_{10}(x), u_{30}(x) = -\frac{1}{4} \tanh \left( \frac{5}{5} \right) - \frac{1}{4}, u_{40}(x) = u_{30}(x) \), initial positions \( \gamma_{10} = -100, \gamma_{20} = -50, \gamma_{30} = 50, \gamma_{40} = 100 \), reference functions \( \hat{v}_1(x) = u_{10}(x), \hat{v}_2(x) = u_{20}(x), \hat{v}_3(x) = u_{30}(x), \hat{v}_4(x) = u_{40}(x) \) and bump function \( \varphi(x) = \frac{2}{\exp(\frac{1}{2}x) + \exp(-\frac{1}{2}x)} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.5 \). For the temporal discretization we use BDF(2) with \( \Delta t = 1.0, rtol = 10^{-2}, atol = 10^{-3} \) and intermediate timesteps.

Frozen solutions, velocities and positions:

Spatial-temporal patterns:
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Front interaction (4-front double collision):
  I. Nonfrozen solution: Consider the nonfrozen system

\[ u_t = u_{xx} + u(1 - u)(u - \alpha_1)(u - \alpha_2)(u - \alpha_3), \quad x \in B_R(0), \ t \in [0, \infty[ \]

\[ \frac{\partial u}{\partial n} = 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \]

\[ u(0) = u_0, \quad x \in B_R(0), \ t = 0 \]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha_1 = \frac{1}{4} \), \( \alpha_2 = \frac{1}{2} \), \( \alpha_3 = \frac{3}{4} \), \( R = 150 \) and initial data \( u_0(x) = \frac{1}{4} \tanh \left( \frac{x + 100}{5} \right) + \frac{1}{4} \tanh \left( \frac{x + 20}{5} \right) - \frac{1}{4} \tanh \left( \frac{x - 20}{5} \right) - \frac{1}{4} \tanh \left( \frac{x - 100}{5} \right) \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.3 \). For the temporal discretization we use BDF(2) with \( \Delta t = 1.0, \ rtol = 10^{-3}, \ atol = 10^{-4} \) and intermediate timesteps.

Nonfrozen solution:

Spatial-temporal pattern:
II. Frozen system (with freezing method ([11])): Consider the associated frozen system \((j = 1, 2, 3, 4)\)

\[
v_{j,t} = v_{j,xx} + \sum_{k=1}^{4} \frac{\varphi(\cdot))}{\varphi(-g_k + g_j)} \cdot f \left( \sum_{k=1}^{4} v_k (\cdot - g_k + g_j) \right)
+ \mu_j v_{j,x},
\]

\[
\frac{\partial v_j}{\partial n} = 0,
\]

\[
v_j(0) = u_{j0},
\]

\[
g_j(t) = g_j0,
\]

\[
g_j(0) = g_j0.
\]

where \(f(v) = v(1 - v)(v - \alpha_1)(v - \alpha_2)(v - \alpha_3)\) and \(d = 1\) (i. e. \(B_R(0) = [-R, R]\)). For the numerical computations we use again parameters \(\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{2}, \alpha_3 = \frac{3}{4}, R = 50\), initial data \(u_{10}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) + \frac{1}{4}, u_{20}(x) = u_{10}(x), u_{30}(x) = \frac{1}{4} \tanh \left( \frac{x}{5} \right) - \frac{1}{4}, u_{40}(x) = u_{30}(x)\), initial positions \(\gamma_{10} = -100, \gamma_{20} = -20, \gamma_{30} = 20, \gamma_{40} = 100\), reference functions \(\hat{v}_1(x) = u_{10}(x), \hat{v}_2(x) = u_{20}(x), \hat{v}_3(x) = u_{30}(x), \hat{v}_4(x) = u_{40}(x)\) and bump function \(\varphi(x) = \frac{2}{\exp(\frac{1}{2}x) + \exp(-\frac{1}{2}x)}\). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.4\). For the temporal discretization we use BDF(2) with \(\Delta t = 1.0, rtol = 10^{-2}, atol = 10^{-3}\) and intermediate timesteps.

Frozen solutions, velocities and positions:
Spatial-temporal patterns:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Explicit solutions: not available

Literature: [11], [73]
1.4 FitzHugh-Nagumo model

**Name:** FitzHugh-Nagumo model (FHN) (sometimes called Bonhoeffer-van der Pol model)

**Equations:**

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \left( \begin{array}{c}
  \Delta u_1 + f(u_1) - u_2 + \alpha \\
  D \Delta u_2 + \beta (\gamma u_1 - \delta u_2 + \epsilon)
\end{array} \right)
\]

\(u_1 = u_1(x, t) \in \mathbb{R}, u_2 = u_2(x, t) \in \mathbb{R}, u = (u_1, u_2)^T, x \in \mathbb{R}^d, d \in \{1, 2, 3\}, D, \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}\)

and \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a (cubic nonlinearity) polynomial of third degree (i.e. \(f(w) = w - \zeta w^3\) or \(f(w) = w(1 - w)(w - \zeta)\) with \(0 \neq \zeta \in \mathbb{R}\), cf. Nagumo equation).

**Notations:**

- \(u_1\): membran potential (measure the potential difference across the cell membrane)
- \(u_2\): recovery variable (measure transmembrane currents which affect the ability of the cell to recover before being able to fire again)
- \(D\): diffusion coefficient (usually \(D = 0\))
- \(\alpha\): magnitude of stimulus current
- \(\beta, \gamma, \delta, \epsilon, \zeta\): constant system parameters

**Short description:** The FitzHugh-Nagumo model ([33],[62]), named after Richard FitzHugh (1922-2007) and Jin-Ichi Nagumo (1926-1999), discribes nerve conduction ([83]), propagation of waves and nerve pulses in excitable media (e.g. heart tissue or nerve fiber) and spike generation in squid giant axons. This model exhibits traveling pulses, traveling fronts, traveling multipulses ([64]) in 1D, spiral waves, spiral breakups ([39]), spiral turbulences ([40]), labyrinthine patterns ([40]), spot splittings ([39]) and rotating vortices ([17],[15]) in 2D as well as other phenomena in 3D.

**Phenomena:**

- Traveling 1-front (traveling front)
- Traveling 1-pulse (traveling pulse)
- Traveling 2-pulse (traveling multipulse)

**Set of parameter values:**

<table>
<thead>
<tr>
<th>(d)</th>
<th>(D)</th>
<th>(f(w))</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\delta)</th>
<th>(\epsilon)</th>
<th>(\zeta)</th>
<th>(R)</th>
<th>(\Delta x)</th>
<th>(\Delta t)</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{10})</td>
<td>(w - \zeta w^3)</td>
<td>0</td>
<td>(\frac{2}{25})</td>
<td>1</td>
<td>8</td>
<td>(\frac{7}{10})</td>
<td>(\frac{1}{3})</td>
<td>60</td>
<td>0.1</td>
<td>0.5</td>
<td>(\frac{\partial u}{\partial n}) = 0 on (\partial B_R(0))</td>
<td>1-front</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{10})</td>
<td>(w - \zeta w^3)</td>
<td>0</td>
<td>(\frac{2}{25})</td>
<td>1</td>
<td>4</td>
<td>(\frac{7}{10})</td>
<td>(\frac{1}{3})</td>
<td>60</td>
<td>0.1</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n}) = 0 on (\partial B_R(0))</td>
<td>1-pulse</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{1}{10})</td>
<td>(w - \zeta w^3)</td>
<td>0</td>
<td>(\frac{2}{25})</td>
<td>1</td>
<td>4</td>
<td>(\frac{7}{10})</td>
<td>(\frac{1}{3})</td>
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<td>0.1</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n}) = 0 on (\partial B_R(0))</td>
<td>2-pulse</td>
</tr>
</tbody>
</table>

**Numerical results:**

- Traveling 1-front (traveling front):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \left( \begin{array}{c}
  u_{1,xx} + f(u_1) - u_2 + \alpha \\
  D u_{2,xx} + \beta (\gamma u_1 - \delta u_2 + \epsilon)
\end{array} \right), x \in B_R(0), t \in [0, \infty[\]

\[
\frac{\partial u}{\partial n} = 0, x \in \partial B_R(0), t \in [0, \infty[\]

\[u(0) = u_0, x \in B_R(0), t = 0\]
where \( f(w) = w - \zeta w^3 \) and \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( D = \frac{1}{10}, \alpha = 0, \beta = \frac{2}{25}, \gamma = 1, \delta = 8, \epsilon = \frac{7}{10}, \zeta = \frac{1}{3}, R = 60 \) and initial data \( u_0(x) = (\tanh(x), 1 - \tanh(x))^T \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.5, rtol = 10^{-2}, atol = 10^{-5} \) and intermediate timesteps ([59]).

Nonfrozen solution:

\[
\begin{align*}
\text{Solution } u_1, t = 100 & \quad \text{Solution } u_2, t = 100 \\
\text{Solution } u_3, t = 100 & \quad \text{Solution } u_4, t = 100 \\
\text{Solution } u_5, t = 100 & \quad \text{Solution } u_6, t = 100 \\
\end{align*}
\]

Spatial-temporal pattern:

\[
\begin{align*}
\text{Solution } u_7, t = 100 & \quad \text{Solution } u_8, t = 100 \\
\text{Solution } u_9, t = 100 & \quad \text{Solution } u_{10}, t = 100 \\
\end{align*}
\]

II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{align*}
\begin{bmatrix} v_1 \\ v_2 \\ \frac{\partial v}{\partial n} \\ v(0) \\ \gamma(0) \\ 0 \\
\end{bmatrix}_t & = \begin{bmatrix} v_{1,xx} + f(v_1) - v_2 + \alpha + \lambda_1 v_{1,x} \\ D v_{2,xx} + \beta (\gamma v_1 - \delta v_2 + \epsilon) + \lambda_1 v_{2,x} \\ 0 \\ 0 \\ \langle \tilde{v}_x, v - \tilde{v} \rangle_{L^2(B_R(0), \mathbb{R})} \\
\end{bmatrix}, \quad x \in B_R(0), t \in [0, \infty[ \\
, \quad x \in \partial B_R(0), t \in [0, \infty[ \\
, \quad x \in B_R(0), t = 0 \\
, \quad t \in [0, \infty[ \\
, \quad t = 0 \\
\end{align*}
\]

where \( f(w) = w - \zeta w^3 \) and \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( D = \frac{1}{10}, \alpha = 0, \beta = \frac{2}{25}, \gamma = 1, \delta = 8, \epsilon = \frac{7}{10}, \zeta = \frac{1}{3}, R = 60 \), initial data \( u_0(x) = (\tanh(x), 1 - \tanh(x))^T \).
FitzHugh-Nagumo model

$$(\tanh(x), 1 - \tanh(x))^T$$ and reference function $\hat{v}(x) = \cdot$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\Delta x = 0.1$. For the temporal discretization we use BDF(2) with $\Delta t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps. Frozen solution and velocity:

Spatial-temporal pattern, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 1-pulse (traveling pulse):
  - Nonfrozen solution: Consider the nonfrozen system

$$\begin{align*}
\begin{pmatrix} u_1 \\ u_2 \\ \partial u \\ u \end{pmatrix}_t &= \begin{pmatrix} u_{1,xx} + f(u_1) - u_2 + \alpha \\ Du_{2,xx} + \beta (\gamma u_1 - \delta u_2 + \epsilon) \\ 0 \\ u \end{pmatrix}, \quad x \in B_R(0), \ t \in [0, \infty[, \\
\partial u \bigg|_{\partial B} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[, \\
u(0) &= u_0, \quad x \in B_R(0), \ t = 0,
\end{align*}$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i.e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = 1/10$, $\alpha = 0$, $\beta = 2/25$, $\gamma = 1$, $\delta = 4/5$, $\epsilon = 7/10$, $\zeta = 1/3$, $R = 60$ and initial data $u_0(x) = (\tanh(x), -0.6)^T$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\Delta x = 0.1$. For the temporal discretization we use BDF(2) with $\Delta t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps ([59]).

Nonfrozen solution:

Spatial-temporal pattern:
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}_t = \begin{pmatrix}
v_{1,xx} + f(v_1) - v_2 + \alpha + \lambda_1 v_{1,x} \\
D v_{2,xx} + \beta (\gamma v_1 - \delta v_2 + \epsilon) + \lambda_1 v_{2,x}
\end{pmatrix}, \quad x \in B_R(0), \quad t \in [0, \infty[ \\
\text{where } f(w) = w - \zeta w^3 \text{ and } d = 1 \text{ (i.e. } B_R(0) = [-R, R]). \text{ For the numerical computations we use parameters } D = \frac{1}{10}, \alpha = 0, \beta = \frac{2}{25}, \gamma = 1, \delta = \frac{4}{5}, \epsilon = \frac{7}{10}, \zeta = \frac{1}{3}, R = 60, \text{ initial data } u_0(x) = (\tanh(x), 1 - \tanh(x))^T \text{ and reference function } \hat{v}(x) = \text{?}. \text{ Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with } \Delta x = 0.1. \text{ For the temporal discretization we use BDF(2) with } \Delta t = 0.1, \text{ rtol } = 10^{-2}, \text{ atol } = 10^{-5} \text{ and intermediate timesteps.}
\]

Frozen solution and velocity:

Spatial-temporal pattern, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 2-pulse (traveling multipulse):

I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix}_t = \begin{pmatrix}
u_{1,xx} + f(u_1) - u_2 + \alpha + \lambda_1 u_{1,x} \\
D u_{2,xx} + \beta (\gamma u_1 - \delta u_2 + \epsilon)
\end{pmatrix}, \quad x \in B_R(0), \quad t \in [0, \infty[ \\
\text{where } f(w) = w - \zeta w^3 \text{ and } d = 1 \text{ (i.e. } B_R(0) = [-R, R]). \text{ For the numerical computations we use parameters } D = \frac{1}{10}, \alpha = 0, \beta = \frac{2}{25}, \gamma = 1, \delta = \frac{4}{5}, \epsilon = \frac{7}{10}, \zeta = \frac{1}{3}, R = 100 \text{ and initial data } u_0(x) = \left(-1.2 + \frac{2.5}{1 + \left(\frac{4}{5}\right)^7}, -0.6\right)^T \text{. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with } \Delta x = 0.1. \text{ For the temporal discretization we use BDF(2) with } \Delta t = 0.1, \text{ rtol } = 10^{-2}, \text{ atol } = 10^{-5} \text{ and intermediate timesteps ([12],[11]).}
\]

Nonfrozen solution:
1.4 FitzHugh-Nagumo model

Spatial-temporal pattern:

II. Frozen system (with freezing method ([11])): Consider the associated frozen system \((j = 1, 2)\)

\[
\begin{pmatrix}
  v_{1,j} \\
  v_{2,j}
\end{pmatrix}_t = \begin{pmatrix}
  v_{1,j,xx} \\
  Dv_{2,j,xx}
\end{pmatrix} + \frac{\varphi(\cdot)}{\sum_{k=1}^2 \varphi(-g_k + g_j)} \cdot F \left( \sum_{k=1}^2 v_k(\cdot - g_k + g_j, \cdot) \right) \\
+ \mu_j \begin{pmatrix}
  v_{1,j,x} \\
  v_{2,j,x}
\end{pmatrix},
\]  
\[
\begin{align*}
  \frac{\partial v_j}{\partial n} &= 0, & x &\in B_R(0), \\
v_j(0) &= u_j0, & x &\in \partial B_R(0), \\
g_{j,t} &= \mu_j, & x &\in B_R(0), t = 0, \\
g_{j}(0) &= g_{j0}, & t &\in [0, \infty[,
\end{align*}
\]

with \(0 = (\hat{v}_j)_x, v_j - \hat{v}_j)_{L^2(B_R(0), \mathbb{R})}\)

\[
F \begin{pmatrix}
  w_1 \\
  w_2
\end{pmatrix} = \begin{pmatrix}
  f(w_1) - w_2 + \alpha \\
  \beta (\gamma w_1 - \delta w_2 + \epsilon)
\end{pmatrix}
\]
where \( v_j = (v_{1,j}, v_{2,j})^T \), \( f(w) = w - \zeta w^3 \) and \( d = 1 \) (i. e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( D = \frac{1}{10}, \alpha = 0, \beta = \frac{2}{25}, \gamma = 1, \delta = \frac{4}{5}, \epsilon = \frac{7}{10}, \zeta = \frac{1}{3}, R = 50 \), initial data \( u_0(x) = \left(-1.2 + \frac{2.5}{1+(\frac{x}{5})^2}, -0.6\right)^T \), initial positions \( \gamma_{10} = -50, \gamma_{20} = 50 \), reference functions \( \hat{v}_1(x) =?, \hat{v}_2(x) =? \) and bump function \( \varphi(x) = \frac{2}{\exp(\frac{1}{2}x) + \exp(-\frac{1}{2}x)}. \) Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, rtol = 10^{-2}, atol = 10^{-5} \) and intermediate timesteps.

Frozen solutions:

Velocities and positions:

Spatial-temporal patterns:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

**Explicit solutions:** not available

**Literature:** [33], [62], [32], [35], [34], [68], [12], [11], [40], [59], [64], [45], [17], [15]
1.5 Purwins model

Name: **Purwins model**

Equations:

\[
\begin{align*}
\frac{du}{dt} &= D_u \Delta u + f(u) - \kappa_1 v - \kappa_2 w + \kappa_3 \\
\frac{dv}{dt} &= D_v \Delta v + u - v \\
\frac{dw}{dt} &= D_w \Delta v + u - w
\end{align*}
\]

Notations:

- **\(u\)**: activator
- **\(v, w\)**: inhibitor
- **\(D_u, D_v, D_w\)**: diffusion constants with slow diffusion \(D_v\) and fast diffusion \(D_w\), i.e. \(D_v \ll D_w\)
- **\(\kappa_1, \kappa_2, \kappa_3, \lambda, \tau, \theta\)**: some additional constants

Short description: not available

Set of parameter values: not available

Phenomena: not available

Explicit solutions: not available

Literature: not available
1.6 Barkley model

Name: Barkley model

Equations:

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \begin{pmatrix}
  \Delta u_1 + \frac{1}{\varepsilon} \cdot u_1 \cdot (1 - u_1) \cdot \left( u_1 - \frac{u_2 + b}{a} \right) \\
  D \Delta u_2 + g(u_1) - u_2
\end{pmatrix}
\]

\[u_1 = u_1(x,t) \in \mathbb{R}, u_2 = u_2(x,t) \in \mathbb{R}, u = (u_1,u_2)^T, x \in \mathbb{R}^d, d \in \{1,2,3\}, D,a,b,\varepsilon \in \mathbb{R}, g(w) = w \text{ (}[7])
\]

\[\text{or } g(w) = w^3, g(w) = \begin{cases} 
0 & w \in [0, \frac{1}{7}] \\
1 - 6.75(w - 1)^2 & w \in \left[\frac{1}{8},1\right] \text{ ([20])}. \\
1 & w \in ]1,\infty[
\end{cases}
\]

Notations:

- \(D\) : diffusion constant for the slow species \(u_2\) (0 \(\leq D \ll 1\))
- \(g(u_1)\) : reaction kinetics
- \(\varepsilon\) : sets the timescale separation between the fast \(u_1\)- and the slow \(u_2\)-equation (0 < \(\varepsilon \ll 1\))
- \(a, b\) : other system parameters

Short description: The Barkley model ([7],[8]), named after Dwight Barkley, describes excitable media, oscillatory media ([7]), catalytic surface reactions ([20],[21]), the interaction of a fast activator \(u\) and a slow inhibitor \(v\) (in this case \(g(u)\) describes a delayed production of the inhibitor) and is often used as a qualitative model in pattern forming systems (i.e. Belousov-Zhabotinsky reaction). This model exhibits spiral waves ([78],[12]), rotating spiral and scroll wave solutions. Larger \(a\) gives a longer excitation duration and increasing \(\frac{\varepsilon}{a}\) gives a larger excitability threshold (or equivalently decreasing \(\frac{b}{a}\) produces a spiral with many windings). The standard reaction kinetics \(g(u) = u\) can also be replaced by one of two above mentioned possibilities. In both of these nonstandard cases the model can exhibit spiral breakups followed by spiral turbulences ([21],[70]).

Phenomena:

- Rotating spiral wave (rigidly rotating spiral)
- Meandering spiral wave (meandering spiral)

Set of parameter values:

<table>
<thead>
<tr>
<th>(d)</th>
<th>(D)</th>
<th>(a)</th>
<th>(b)</th>
<th>(\varepsilon)</th>
<th>(g(w))</th>
<th>(R)</th>
<th>(\Delta x)</th>
<th>(\Delta t)</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{1}{10})</td>
<td>(\frac{3}{4})</td>
<td>(\frac{1}{100})</td>
<td>(\frac{1}{30})</td>
<td>(w)</td>
<td>40</td>
<td>0.7</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n}) = 0 on (\partial B_R(0))</td>
<td>Rotating spiral wave</td>
</tr>
</tbody>
</table>

Numerical results:

- Rotating spiral wave (rigidly rotating spiral):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix}_t = \begin{pmatrix}
  \Delta u_1 + \frac{1}{\varepsilon} \cdot u_1 \cdot (1 - u_1) \cdot \left( u_1 - \frac{u_2 + b}{a} \right) \\
  D \Delta u_2 + g(u_1) - u_2
\end{pmatrix}, \ x \in B_R(0), t \in [0,\infty[ \\
\frac{\partial u}{\partial n} = 0, \ x \in \partial B_R(0), t \in [0,\infty[ \\
u(0) = u_0, \ x \in B_R(0), t = 0
\]

where \(g(w) = w\) and \(d = 2\) (i.e. \(B_R(0) = \{ x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leq R \}\)). For the numerical computations we use parameters \(D = \frac{1}{10}, a = \frac{3}{4}, b = \frac{1}{100}, \varepsilon = \frac{1}{30}, R = 40\) and initial data \(u_0(x) = (u_1(x,0),u_2(x,0))^T\).
with \( u_1(x, 0) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \) and \( u_2(x, 0) = \begin{cases} 0 & y \leq 0 \\ \frac{a}{2} & y > 0 \end{cases} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.7 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-3} \), \( atol = 10^{-4} \) and intermediate timesteps ([78],[12]).

Nonfrozen solution:

Spatial-temporal pattern: (with \( x_2 = 0, x_1 \in [-40, 40] \))

- Rotating spiral wave (rigidly rotating spiral): \( d = 2, D_v = \frac{1}{10}, a = \frac{3}{4}, b = \frac{1}{10}, \epsilon = \frac{1}{50}, g(u) = u, R = 40, \Delta x = 0.7, \Delta t = 0.1 \), \( \begin{pmatrix} u \\ v \end{pmatrix} = 0 \) on \( \partial B_R(0) \) (Dirichlet boundary), \( u_0 = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \), \( v_0 = \begin{cases} 0 & y \leq 0 \\ \frac{a}{2} & y > 0 \end{cases} \) ([78],[12]). I. Nonfrozen solution:

Spatial-temporal pattern: (with \( y = 0, x \in [-40, 40] \))
• Meandering rotating spiral wave (Drifting rotating spiral wave): $d = 2, D_v = 0, a = \frac{3}{5}, b = \frac{1}{20}, 
\varepsilon = \frac{1}{50}, g(u) = u, R = 20, \Delta x = 0.5, \Delta t = 0.1, \frac{\partial}{\partial n} \begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary),
\[
u_0 = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}, \quad \nu_0 = \begin{cases} 0 & y \leq 0 \\ \frac{a}{2} & y > 0 \end{cases} \quad ([8]). I. Nonfrozen solution:

Spatial-temporal pattern: (with $y = 0, x \in [-20, 20]$)

• Core breakup (Spiral breakup followed by spiral turbulence): $d = 2, D_v = 0, a = 0.75, b = 0.07, 
\varepsilon = 0.08, R = 50, \Delta x = 1.25, \Delta t = 0.1, g(u) = u^3, \frac{\partial}{\partial n} \begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary),
\[
u_0 \text{ and } \nu_0 \text{ spiral patterns (i.e. solution at time } t = 15 \text{ with parameters } D_v = 0.001, a = 0.75, b = 0.01, 
\varepsilon = 0.02, R = 50, \Delta x = 1.25, \Delta t = 0.1, g(u) = u, \frac{\partial}{\partial n} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \text{ on } \partial B_R(0) \text{ (Neumann boundary)},
\[
u_0 = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}, \quad \nu_0 = \begin{cases} 0 & y \leq 0 \\ \frac{a}{2} & y > 0 \end{cases} \quad ([70]). I. Nonfrozen solution:
1.6 Barkley model

- Core breakup (Spiral breakup followed by spiral turbulence): \( d = 2, D_v = 0, a = 0.84, b = 0.07, \)
  \( \varepsilon = 0.08, R = 50, \Delta x = 1.25, \Delta t = 0.1, g(u) = \begin{cases} 
  0 & u \in \left[0, \frac{1}{3}\right] \\
  1 - 6.75u(u - 1)^2 & u \in \left[\frac{1}{3}, 1\right] \\
  1 & u \in \left]1, \infty\right[ 
\end{cases} \)
  \( \frac{\partial}{\partial n}\left(\frac{u}{v}\right) = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_0 \) and \( v_0 \) spiral patterns (i.e. solution at time \( t = 15 \) with parameters \( D_v = 0.001, a = 0.75, b = 0.01, \varepsilon = 0.02, R = 50, \Delta x = 1.25, \Delta t = 0.1, g(u) = u, \frac{\partial}{\partial n}\left(\frac{u}{v}\right) = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_0 = \begin{cases} 
  0 & x \leq 0 \\
  1 & x > 0 
\end{cases} \) \( v_0 = \begin{cases} 
  0 & y \leq 0 \\
  \frac{a}{2} & y > 0 
\end{cases} \) ([70]). I. Nonfrozen solution:

- Far field breakup (Spiral breakup): \( d = 2, D_v = 0.001, a = 0.75, b = 0.01, \varepsilon = 0.0752, R = 50, \)
  \( \Delta x = 1.25, \Delta t = 0.1, g(u) = \begin{cases} 
  0 & u \in \left[0, \frac{1}{3}\right] \\
  1 - 6.75u(u - 1)^2 & u \in \left[\frac{1}{3}, 1\right] \\
  1 & u \in \left]1, \infty\right[ 
\end{cases} \)
  \( \frac{\partial}{\partial n}\left(\frac{u}{v}\right) = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_0 \) and \( v_0 \) spiral patterns (i.e. solution at time \( t = 15 \) with parameters \( D_v = 0.001, a = 0.75, b = 0.01, \varepsilon = 0.02, R = 50, \Delta x = 1.25, \Delta t = 0.1, g(u) = u, \frac{\partial}{\partial n}\left(\frac{u}{v}\right) = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_0 = \begin{cases} 
  0 & x \leq 0 \\
  1 & x > 0 
\end{cases} \) \( v_0 = \begin{cases} 
  0 & y \leq 0 \\
  \frac{a}{2} & y > 0 
\end{cases} \) ([21],[70]). I. Nonfrozen solution:
• Scroll wave: $d = 3$, $D_v = 0$, $a = \frac{4}{5}$, $b = \frac{1}{100}$, $\varepsilon = \frac{1}{50}$, $g(u) = u$, $R = 15$, $\Delta x = 0.5$, $\Delta t = 0.1$, 
$\frac{\partial}{\partial n} \begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_0 = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$, $v_0 = \begin{cases} 0 & y \leq 0 \\ \frac{a}{2} & y > 0 \end{cases}$.

Explicit solutions: not available

Additional informations: For $D_v = 0$ and $\varepsilon = 0.02$ fixed the following figure shows the dynamics of a single spiral wave as a function of parameters $a$ and $b$ as well as various types of meander. Top: The yellow region denotes periodically rotating spirals and the cyan region denotes meandering spirals. Bottom: Cut through the parameter space at $b = 0.05$ with the different states illustrated by tip paths ([7],[9]).

For their numerical approximation see [44] and [16].

Literature: [78], [20], [7], [19], [9], [70], [21], [8], [44], [16]
1.7 Schrödinger equation

Name: SCHRODINGER EQUATION (sometimes called NONLINEAR SCHRODINGER EQUATION (NLS, NLSE))

Equations: \[ u_t = \alpha \Delta u + \beta |u|^p u \]
\[ u = u(x, t) \in \mathbb{C}, u_1 := \text{Re}(u), u_2 := \text{Im}(u), x \in \mathbb{R}^d, d \in \{1, 2, 3\}, t \in [0, \infty[, \alpha, \beta \in \mathbb{C}, p \in \mathbb{N}_0. \]

Notations: not available

Short description: The Schrödinger equation (71)

Set of parameter values: not available

Phenomena:
• Stationary oscillon (standing oscillating pulse, stable rotating pulse, solitary wave): \( d = 1, \alpha = i, \beta = i, p = 2, R = 25, \Delta x = 0.1, \Delta t = 0.1, \frac{\partial u}{\partial n} = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_{10} = C \sqrt{2} \text{Im}(\beta) \cos(D) \cosh(Cx + E) \), \( u_{20} = C \sqrt{2} \text{Im}(\beta) \sin(D) \cosh(Cx + E) \) with \( C = \frac{1}{2}, D = 1 \) and \( E = 0 \). I. Nonfrozen solution:

Explicit solutions:
• 1d: For \( \alpha = i, \beta = ki \) (with \( k \in \mathbb{R} \)) and \( p = 2 \) we have
  \[ u(x, t) = C \exp \left( i \left( Dx + \left( kC^2 - D^2 \right) t + E \right) \right) \]
  \[ u(x, t) = \pm C \sqrt{2} \exp \left( \frac{i \left( C^2 t + D \right)}{k} \right) \cosh \left( Cx + E \right), \text{ if } k > 0 \]
  \[ u(x, t) = \pm A \sqrt{2} \exp \left( \frac{i Bx + i \left( A^2 + B^2 \right) t + iC}{k} \right) \cosh \left( Ax - 2ABt + D \right), \text{ if } k > 0 \]
  \[ u(x, t) = \frac{C}{\sqrt{t}} \exp \left( \frac{i (x + D)^2}{4t} + i \left( kC^2 \ln(t) + E \right) \right) \]
where \( A, B, C, D, E \) are arbitrary real constants. For a collection of solutions see [29].

Additional informations: With respect to \( p \in \mathbb{N}_0 \) in the literature there are special nomenclature

\[ u_t = \alpha \Delta u + \beta u \quad : \quad \text{LINEAR SCHRODINGER EQUATION} \]
\[ u_t = \alpha \Delta u + \beta |u| u \quad : \quad \text{QUADRATIC NONLINEAR SCHRODINGER EQUATION} \]
\[ u_t = \alpha \Delta u + \beta |u|^2 u \quad : \quad \text{CUBIC NONLINEAR SCHRODINGER EQUATION} \]
\[ u_t = \alpha \Delta u + \beta |u|^3 u \quad : \quad \text{QUARTIC NONLINEAR SCHRODINGER EQUATION} \]
\[ u_t = \alpha \Delta u + \beta |u|^4 u \quad : \quad \text{QUINTIC NONLINEAR SCHRODINGER EQUATION} \]
\[ u_t = \alpha \Delta u + \beta |u|^6 u \quad : \quad \text{SEPTIC NONLINEAR SCHRODINGER EQUATION} \]
Literature: [29], [71]
1.8 Gross-Pitaevskii equation

Name: **GROSS-PITAЕVSKII EQUATION (GPE)**

Equations:

\[ u_t = \alpha \triangle u + \beta V(x)u + \gamma |u|^2 u \]

\[ u = u(x, t) \in \mathbb{C}, \ x \in \mathbb{R}^d, \ d \in \{1, 2, 3\}, \ t \in [0, \infty[, \ \alpha, \beta, \gamma \in \mathbb{C}, \ V : \mathbb{R}^d \to \mathbb{R}. \]

Notations: not available

Short description: The Gross-Pitaevskii equation ([37],[67]), named after Eugene P. Gross and Lev Petrovich Pitaevskii, describes Bose-Einstein condensates (BEC) at zero or very low temperature and the ground states of a quantum system of identical bosons. This model exhibits standing solitary oscillons.

Phenomena:

- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse)

Set of parameter values:

<table>
<thead>
<tr>
<th>d</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(V(x))</th>
<th>(R)</th>
<th>(\Delta x)</th>
<th>(\Delta t)</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{i}{2})</td>
<td>(-i)</td>
<td>(-i)</td>
<td>(\frac{x^2}{2})</td>
<td>10</td>
<td>0.1</td>
<td>0.5</td>
<td>(\frac{\partial u}{\partial n} = 0) on (\partial B_R(0))</td>
<td>rotating pulse</td>
</tr>
</tbody>
</table>

Numerical results:

- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse):
  
  I. Nonfrozen solution: Consider the nonfrozen system

  \begin{align*}
  u_t &= \alpha \triangle u + \beta V(x)u + \gamma |u|^2 u, \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \ t = 0
  \end{align*}

  where \(d = 1\) (i. e. \(B_R(0) = [-R,R]\)). For the numerical computations we use parameters \(\alpha = \frac{i}{2}\), \(\beta = -i\), \(\gamma = -i\), \(V(x) = \frac{x^2}{2}\), \(R = 10\) and initial data \(\text{Re} \ u_0 = \pi^{-\frac{i}{2}} \exp\left(-\frac{x^2}{2}\right), \ \text{Im} \ u_0 = 0\). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.1\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.5, \ rtol = 10^{-2}, \ atol = 10^{-4}\) and intermediate timesteps.

  Nonfrozen solution:

  ![Solution](image1.png)

Spatial-temporal pattern:
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\begin{align*}
v_t &= \alpha \nabla v + \beta V(x)v + \gamma |v|^2 v + i\lambda_1 v, & x \in B_R(0), t \in [0, \infty[ \\
\frac{\partial v}{\partial n} &= 0, & x \in \partial B_R(0), t \in [0, \infty[ \\
v(0) &= u_0, & x \in B_R(0), t = 0 \\
(\gamma_1)_t &= \lambda_1 & t \in [0, \infty[ \\
\gamma_1(0) &= 0, & t = 0 \\
0 &= \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, & t \in [0, \infty[ \\
\end{align*}

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \alpha = \frac{i}{2} \), \( \beta = -i \), \( \gamma = -i \), \( V(x) = \frac{x^2}{2} \), \( R = 10 \) and initial data \( \text{Re} u_0 = \pi^{-\frac{1}{2}} \exp \left(-\frac{x^2}{2}\right) \), \( \text{Im} u_0 = 0 \). As reference functions \( \text{Re} \hat{v}(x) \) and \( \text{Im} \hat{v}(x) \) we choose the real and imaginary part of the nonfrozen solution at time \( t = 1 \), respectively, with the parameters mentioned above with \( \Delta x = 0.05 \) and \( \Delta t = 0.1 \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.05 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-2} \), \( atol = 10^{-4} \) and intermediate timesteps.

Frozen solutions:

Spatial-temporal pattern:
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Explicit solutions: not available

Additional informations: If $\beta = 0$ then we obtain the cubic nonlinear Schrödinger equation.

Literature: [37], [67]
1.9 Complex Ginzburg-Landau equation (CGL)

**Name:** Complex Ginzburg-Landau equation (CGL, CGLE) (sometimes called Cubic Complex Ginzburg-Landau)

**Equations:**

\[
    u_t = \alpha \triangle u + u \left( \mu + \beta |u|^2 \right) + \gamma \overline{u}
\]

\(u = u(x,t) \in \mathbb{C}, u_1 := \text{Re}(u), u_2 := \text{Im}(u), x \in \mathbb{R}^d, d \in \{1, 2, 3\}, t \in [0, \infty[, \alpha, \beta, \gamma, \mu \in \mathbb{C} \text{ with } \gamma = 0, \overline{u} \text{ the complex conjugate of } u.\)

**Notations:** ([41])

- \(u(x,t)\): complex-valued amplitude, slowly varying in space \(x\) and time \(t\)
- \(\text{Im}(\alpha)\): dispersion
- \(\text{Re}(\mu)\): distance from the Hopf bifurcation
- \(\text{Im}(\mu)\): detuning
- \(\text{Im}(\beta)\): nonlinear frequency correction
- \(\gamma\): weak periodic forcing term, forcing amplitude

**Short description:** The complex Ginzburg-Landau equation, named after Vitaly Lazarevich Ginzburg (1916-2009) and Lev Landau (1908-1968), describes nonlinear waves, second-order phase transitions, Rayleigh-Bénard convection, superconductivity, superfluidity and is used in the study of fiber optics ([3]). The equation describes the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation, for which a continuous spectrum of unstable wavenumbers is taken into account. It can be viewed as a highly general normal form for a large class of bifurcations and nonlinear wave phenomena in spatially extended systems.

**Set of parameter values:**

<table>
<thead>
<tr>
<th>(d)</th>
<th>(\alpha)</th>
<th>(\beta)</th>
<th>(\gamma)</th>
<th>(\mu)</th>
<th>(R)</th>
<th>(\Delta x)</th>
<th>(\Delta t)</th>
<th>Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(1 + \frac{1}{2}i)</td>
<td>(-1)</td>
<td>2.02</td>
<td>(1 + 2i)</td>
<td>50</td>
<td>1.25</td>
<td>0.1</td>
<td>(\frac{\partial u}{\partial n} = 0 \text{ or } u = 0 \text{ on } \partial B_R(0))</td>
</tr>
<tr>
<td>2</td>
<td>(1 + \frac{1}{2}i)</td>
<td>(-1)</td>
<td>1.98</td>
<td>(1 + 2i)</td>
<td>50</td>
<td>1.25</td>
<td>0.1</td>
<td>(u = 0 \text{ on } \partial B_R(0))</td>
</tr>
</tbody>
</table>

**Phenomena:**

- **Labyrinthine pattern:** \(d = 2, \alpha = 1 + \frac{1}{2}i, \beta = -1, \gamma = 2.02, \mu = 1 + 2i, R = 50, \Delta x = 1.25, \Delta t = 0.1, \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R(0)\) (Neumann boundary), \(u_{10} = \tanh(x), u_{20} = 1 - u_{10}\) ([41],[38]). I. Nonfrozen solution:

[Images of simulations showing labyrinthine patterns at different time steps]

- **Labyrinthine pattern:** \(d = 2, \alpha = 1 + \frac{1}{2}i, \beta = -1, \gamma = 2.02, \mu = 1 + 2i, R = 50, \Delta x = 1.25, \Delta t = 0.1, u = 0 \text{ on } \partial B_R(0)\) (Dirichlet boundary), \(u_{10} = \tanh(x), u_{20} = 1 - u_{10}\) ([41],[38]). I. Nonfrozen solution:
1.9 Complex Ginzburg-Landau equation (CGL)

- Scroll wave: $d = 3, \mu = 1$ (or $\mu = -1$), $\alpha = 1, \beta = 1 + i, \gamma = 0$ ([87],[50],[14])
- Spiral-vortex nucleation (formation of Bloch-front turbulence): $d = ?, \alpha = 1 + \frac{3}{10}i, \beta = -1, \gamma = \frac{1}{5}, \mu = \frac{1}{2} + \frac{3}{20}i, R = 256, \Delta x = ?, \Delta t = ?, \frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_{10} = \text{front}$, $u_{20} = \text{front}$ ([41]).

**Explicit solutions:** not available

**Additional informations:**

1: (Bloch-)spiral increasing $\gamma$ spiral-vortex nucleation increasing $\gamma$ labyrinthine, where ([41])

$$\frac{|\text{Im}(\mu) + \text{Re}(\mu) \cdot \text{Im}(\beta)|}{\sqrt{1 + (\text{Im}(\beta))^2}} =: \gamma_b < \gamma < \gamma_{NIB} := \sqrt{(\text{Im}(\mu))^2 + \frac{1}{9}(\text{Re}(\mu))^2}$$

(2): If $\gamma \neq 0$ this equation is called **Forced Complex Ginzburg-Landau equation**. In case of $\text{Im}(\alpha) = \text{Im}(\beta) = 0, \gamma = 0$ and $\mu \in \mathbb{R}$ the equation is called **real Ginzburg-Landau equation**.

**Literature:** [42], [83], [6], [41], [38], [87], [50], [14], [3]
1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

**Name:** Quintic Complex Ginzburg-Landau equation (QCGL, QCGLE) (sometimes called Cubic-Quintic Complex Ginzburg-Landau equation (CQCGL))

**Equations:**

\[ u_t = \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right) \]

\[ u = u(x,t) \in \mathbb{C}, \ x \in \mathbb{R}^d, \ d \in \{1, 2, 3\}, \ t \in [0, \infty], \ \alpha, \beta, \gamma, \mu \in \mathbb{C}. \]

**Notations:** ([80])

- \(x\): normalized transversal spatial coordinate (or retarded time, in temporal problems with \(d = 1\))
- \(t\): normalized propagation distance or the normalized number of round tips
- \(u\): complex envelope of the electric field
- \(\text{Re}(\alpha)\): spatial or temporal spectral filtering, diffusion coefficient, \(\text{Re}(\alpha) > 0\)
- \(\text{Im}(\alpha)\): determines the lowest order diffraction, i.e. \(\text{Im}(\alpha) > 0\) corresponds to anomalous dispersion and \(\text{Im}(\alpha) < 0\) corresponds to normal dispersion
- \(\mu\): linear gain (or loss) at the (spatial or temporal) control frequency, \(\mu \in \mathbb{R}\)
- \(\text{Re}(\beta)\): nonlinear gain (absorption processes)
- \(\text{Re}(\gamma)\): a higher-order correction term to the nonlinear amplification (absorption)
- \(\text{Im}(\gamma)\): a higher-order correction term to the nonlinear refractive index

**Short description:** The quintic complex Ginzburg-Landau equation ([36]), named after Vitaly Lazarevich Ginzburg (1916-2009) and Lev Landau (1908-1968), describes different aspects of signal propagation in heart tissue, superconductivity, superfluidity, nonlinear optical systems ([60]), photonics, plasmas, physics of lasers, Bose-Einstein condensation, liquid crystals, fluid dynamics, chemical waves, quantum field theory, granular media and is used in the study of hydrodynamic instabilities ([58]). This model shows a variety of coherent structures like stable and unstable pulses, fronts, sources and sinks in 1D ([47],[79],[2],[80]), vortex solitons ([24]), spinning solitons ([25]), rotating spiral waves, propagating clusters ([69]) and exploding dissipative solitons ([75]) in 2D as well as scroll waves and spinning solitons ([26]) in 3D.

**Phenomena:**

- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse)
- Traveling oscillating front (stable rotating front)
- Pulsating soliton
- Creeping soliton
- Rotating 2-pulse (rotating multipulse)
- Rotating pulse combined with a traveling rotating front (traveling rotating multistructure)
- Traveling rotating 2-front (traveling rotating multifront)
- Spinning soliton (spinning solitary wave, localized vortex solution)
- Rotating spiral wave (rigidly rotating spiral)
- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse, merger of 3 colliding solitons into a single stable pulse)
• Spinning 2-soliton (spinning multisoliton, localized 2-vortex solution):

Set of parameter values:

<table>
<thead>
<tr>
<th>d</th>
<th>µ</th>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>R</th>
<th>Δx</th>
<th>Δt</th>
<th>Boundary</th>
<th>Phenomena</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-\frac{1}{10}$</td>
<td>1</td>
<td>3+i</td>
<td>$-\frac{11}{4}+i$</td>
<td>20</td>
<td>0.1</td>
<td>0.1</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>rotating pulse</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{1}{10}$</td>
<td>1</td>
<td>3+i</td>
<td>$-\frac{11}{4}+i$</td>
<td>20</td>
<td>0.1</td>
<td>0.1</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>rotating pulse</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{1}{10}$</td>
<td>$\frac{8}{10} + \frac{7i}{10}$</td>
<td>$\frac{66}{100} + i$</td>
<td>$-\frac{11}{10} - \frac{10}{100}i$</td>
<td>20</td>
<td>0.1</td>
<td>0.1</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>pulsating soliton</td>
</tr>
<tr>
<td>1</td>
<td>$-\frac{1}{10}$</td>
<td>$\frac{8}{1000} + \frac{7i}{10}$</td>
<td>$\frac{66}{100} + i$</td>
<td>$-\frac{11}{10} - \frac{10}{100}i$</td>
<td>20</td>
<td>0.1</td>
<td>0.3</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>creeping soliton</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{7}{4} + \frac{7i}{4}$</td>
<td>$\frac{3}{4} + i$</td>
<td>$-\frac{1}{4} - \frac{1}{1000}i$</td>
<td>20</td>
<td>0.5</td>
<td>0.1</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>spinning soliton</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{7}{4} + \frac{7i}{4}$</td>
<td>$\frac{3}{4} + i$</td>
<td>$-\frac{1}{4} - \frac{1}{1000}i$</td>
<td>20</td>
<td>0.5</td>
<td>0.1</td>
<td>$\partial u/\partial n = 0$ on $\partial B_R(0)$</td>
<td>rotating spiral wave</td>
</tr>
</tbody>
</table>

Numerical results:
• Standing solitary oscillon (rotating pulse, stable localized oscillating pulse):

I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
\partial_u & = \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \ \\
\partial_u & = 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \ \\
u(0) & = u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where $d = 1$ (i.e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 20$ and initial data $\text{Re } u_0 = \frac{5}{2(1+(\frac{x}{R})^2)}$, $\text{Im } u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\Delta x = 0.1$. For the temporal discretization we use BDF(2) with $\Delta t = 0.1$, $\text{rtol} = 10^{-3}$, $\text{atol} = 10^{-4}$ and intermediate timesteps ([79],[47],[78],[76]).

Nonfrozen solution:

Spatial-temporal pattern:
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \alpha \Delta v + v \left( \mu + \beta |v|^2 + \gamma |v|^4 \right) + \lambda_1 v x + i \lambda_2 v, & x \in B_R(0), & t \in [0, \infty[ \\
\frac{\partial v}{\partial t} &= 0, & x \in \partial B_R(0), & t \in [0, \infty[ \\
v(0) &= u_0, & x \in B_R(0), & t = 0 \\
\langle \gamma_1 \rangle_t &= \lambda_1, & t \in [0, \infty[ \\
\gamma_1(0) &= 0, & t = 0 \\
\langle \gamma_2 \rangle_t &= \lambda_2, & t \in [0, \infty[ \\
\gamma_2(0) &= 0, & t = 0 \\
0 &= \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, & t \in [0, \infty[ \\
0 &= \langle i \hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, & t \in [0, \infty[ 
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3 + i, \gamma = -\frac{11}{4} + i, R = 20 \) and initial data \( \text{Re} u_0 = \frac{5}{2(1+(\frac{x}{R})^2)}, \text{Im} u_0 = 0 \). As reference functions \( \text{Re} \hat{v}(x) \) and \( \text{Im} \hat{v}(x) \) we choose the real and imaginary part of the nonfrozen solution at time \( t = 70 \), respectively, with the parameters mentioned above and \( \text{atol} = 10^{-5} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \triangle x = 0.1 \). For the temporal discretization we use BDF(2) with \( \triangle t = 0.1, \text{rtol} = 10^{-3}, \text{atol} = 10^{-5} \) and intermediate timesteps ([79], [47], [78]).

Frozen solutions:

Spatial-temporal pattern:

![Spatial-temporal pattern](image-url)

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

- Traveling oscillating front (stable rotating front):

I. Nonfrozen solution: Consider the nonfrozen system

\[
u_t = \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \quad t \in [0, \infty[ \\
\frac{\partial u}{\partial n} = 0, \quad x \in \partial B_R(0), \quad t \in [0, \infty[ \\
u(0) = u_0, \quad x \in B_R(0), \quad t = 0
\]

where \(d = 1\) (i.e. \(B_R(0) = [-R, R]\)). For the numerical computations we use parameters \(\mu = -\frac{1}{10}\), \(\alpha = 1\), \(\beta = 3 + i\), \(\gamma = -\frac{11}{4} + i\), \(R = 20\) and initial data \(\text{Re } u_0 = \frac{\sqrt{3 + \sqrt{9 + 11 \mu}}}{1 + \exp \left( -\frac{x}{\sqrt{2}} \right)}, \text{Im } u_0 = 0\). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.1\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.1\), \(rtol = 10^{-3}\), \(atol = 10^{-4}\) and intermediate timesteps ([79],[47],[78],[76]).

Nonfrozen solution:

II. Frozen system (with freezing method ([12])): Consider the associated frozen system

Spatial-temporal pattern:
\[ \begin{align*}
  v_t &= \alpha \Delta v + v \left( \mu + \beta |v|^2 + \gamma |v|^4 \right) + \lambda_1 v_x + i \lambda_2 v, \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial v}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  v(0) &= u_0, \quad x \in B_R(0), \ t = 0 \\
  (\gamma_1)_t &= \lambda_1, \quad t \in [0, \infty[ \\
  \gamma_1(0) &= 0, \quad t = 0 \\
  (\gamma_2)_t &= \lambda_2, \quad t \in [0, \infty[ \\
  \gamma_2(0) &= 0, \quad t = 0 \\
  0 &= \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
  0 &= \langle i \hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
\end{align*} \]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3 + i, \gamma = -\frac{11}{4} + i, R = 20 \) and initial data \( \text{Re } u_0 = \frac{2.5}{1 + \left( \frac{4}{5} \right)^2}, \text{Im } u_0 = 0 \). As reference functions \( \text{Re } \hat{v}(x) \) and \( \text{Im } \hat{v}(x) \) we choose the real and imaginary part of the nonfrozen solution at time \( t = 4 \), respectively, with the parameters mentioned above and \( \text{atol} = 10^{-5} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, rtol = 10^{-3}, atol = 10^{-5} \) and intermediate timesteps ([79],[47],[78]).

Frozen solutions:

Spatial-temporal pattern:

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

- Pulsating soliton:
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10} \), \( \alpha = \frac{8}{10} + \frac{1}{2}i \), \( \beta = \frac{66}{100} + i \), \( \gamma = -\frac{1}{10} - \frac{1}{10}i \), \( R = 20 \) and initial data \( \text{Re} u_0 = \text{sech}(x) \), \( \text{Im} u_0 = 0 \).

Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-3} \), \( atol = 10^{-4} \) and intermediate timesteps (5).

Nonfrozen solution:

II. Frozen system (freezing method still not developed): Consider the associated frozen system

Spatial-temporal pattern:

II. Frozen system (freezing method still not developed): Consider the associated frozen system

\[
\begin{align*}
  u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]
\[
v_t = \alpha \Delta v + v \left( \mu + \beta |v|^2 + \gamma |v|^4 \right) + \lambda_1 v_x + i \lambda_2 v, \quad x \in B_R(0), \ t \in [0, \infty[ \\
\frac{\partial v}{\partial n} = 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
v(0) = u_0, \quad x \in B_R(0), \ t = 0 \\
\gamma_1(t) = \lambda_1, \quad t \in [0, \infty[ \\
\gamma_2(t) = \lambda_2, \quad t = 0 \\
0 = \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
0 = \langle i \hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ 
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = \frac{8}{100} + \frac{2}{5}, \beta = \frac{66}{100} + i, \gamma = -\frac{1}{10} - \frac{1}{10}i, \ R = 20 \) and initial data \( \text{Re} \ u_0 = \text{sech}(x), \text{Im} \ u_0 = 0 \).

As reference functions \( \text{Re} \ \hat{v}(x) \) and \( \text{Im} \ \hat{v}(x) \) we choose the real and imaginary part of the nonfrozen solution at time \( t = 10 \), respectively, with the parameters mentioned above. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \ rtol = 10^{-2}, \ atol = 10^{-5} \) and intermediate timesteps.

Frozen solutions:

Spatial-temporal pattern:

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

- Creeping soliton:
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), t = 0
\end{align*}
\]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10} \), \( \alpha = \frac{101}{1000} + \frac{1}{2} i \), \( \beta = \frac{13}{10} + i \), \( \gamma = -\frac{3}{10} - \frac{101}{1000} i \), \( R = 20 \) and initial data \( \text{Re} u_0 = \text{sech}(x), \text{Im} u_0 = 0 \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.3 \), \( rtol = 10^{-3} \), \( atol = 10^{-4} \) and intermediate timesteps (\[5\]).

Nonfrozen solution:

Spatial-temporal pattern:

II. Frozen system (freezing method still not developed).

- Rotating 2-pulse (rotating multipulse):
  I. Nonfrozen solution: Consider the nonfrozen system
\[ \begin{align*}
  u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \ t = 0
\end{align*} \]

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3+i, \gamma = -\frac{11}{4} + i, R = 50 \) and initial data \( \text{Re} \ u_0 = \frac{5}{2 \left(1+(\frac{e^{20}}{2})^2 \right)} + \frac{\Delta}{5 \left(1+(\frac{e^{20}}{2})^2 \right)} \), \( \text{Im} \ u_0 = 0 \).

Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \ rtol = 10^{-3}, atol = 10^{-4} \) and intermediate timesteps.

Nonfrozen solution:

Spatial-temporal pattern:

II. Frozen system (with freezing method ([73])): Consider the associated frozen system

where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3+i, \gamma = -\frac{11}{4} + i, R = 50 \) and initial data \( \text{Re} v_{10} = \frac{5}{2 \left(1+(\frac{e^{20}}{2})^2 \right)} \), \( \text{Im} v_{10} = 0, \text{Re} v_{20} = \text{Re} v_{10}, \text{Im} v_{20} = 0 \). As reference functions and bump function we use \( \text{Re} \hat{v}_1(x) = \text{Re} v_{10}, \text{Im} \hat{v}_1(x) = \text{Im} v_{10}, \text{Re} \hat{v}_2(x) = \text{Re} v_{20}, \text{Im} \hat{v}_2(x) = \text{Im} v_{20} \) and \( \varphi(x) = \text{sech}(0.5 \cdot x) \), respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.2, \ rtol = 10^{-2}, atol = 10^{-6} \) and intermediate timesteps.

Frozen solutions:
Spatial-temporal pattern:

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
Rotating pulse combined with a traveling rotating front (traveling rotating multistructure):

I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
    u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
    \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
    u(0) &= u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 1 \) (i.e., \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3+i, \gamma = -\frac{11}{4} + i, R = 50 \) and initial data \( \text{Re} u_0 = \frac{5}{2 \left( 1 + \left( \frac{\sqrt{9+11\mu}}{2} \right)^2 \right)} + \frac{(3+\sqrt{9+11\mu})}{1+\exp \left( \frac{-x-20}{\sqrt{2}} \right)}, \text{Im} u_0 = 0. \)

Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \text{rtol} = 10^{-3}, \text{atol} = 10^{-4} \) and intermediate timesteps.

Nonfrozen solution:

Spatial-temporal pattern:

II. Frozen system (with freezing method ([73])): Consider the associated frozen system
where \( d = 1 \) (i.e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = -\frac{1}{10}, \alpha = 1, \beta = 3 + i, \gamma = -\frac{11}{4} + i, R = 50 \) and initial data \( \text{Re} v_{10} = \frac{5}{2(1 + (\frac{3}{4})^2)} \), \( \text{Im} v_{10} = 0, \text{Re} v_{20} = \frac{\sqrt{3}}{1 + \exp\left(-\frac{3 + \sqrt{9 + 11\mu}}{4}\right)}, \text{Im} v_{20} = 0 \). As reference functions and bump function we use \( \text{Re} \hat{v}_1(x) = \text{Re} v_{10}, \text{Im} \hat{v}_1(x) = \text{Im} v_{10}, \text{Re} \hat{v}_2(x) = \text{Re} v_{20}, \text{Im} \hat{v}_2(x) = \text{Im} v_{20} \) and \( \varphi(x) = \text{sech}(0.5 \cdot x) \), respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, rtol = 10^{-2}, atol = 10^{-3} \) and intermediate timesteps.

Frozen solutions:

Spatial-temporal pattern:
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

\begin{itemize}
  \item Traveling rotating 2-front (traveling rotating multifront):
    \begin{enumerate}
      \item Nonfrozen solution: Consider the nonfrozen system
        \begin{align*}
        u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
        \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
        u(0) &= u_0, \quad x \in B_R(0), \ t = 0
        \end{align*}
    \end{enumerate}
\end{itemize}

where \( d = 1 \) (i. e. \( B_R(0) = [-R, R] \)). For the numerical computations we use parameters \( \mu = \frac{-1}{10} \), \( \alpha = 1 \), \( \beta = 3 + i \), \( \gamma = -\frac{11}{4} + i \), \( R = 50 \) and initial data \( \text{Re} \ u_0 = \frac{2}{\Delta} \left( 3 + \sqrt{9 + 11 \mu} \right) - \frac{\frac{2}{\Delta} \left( 3 + \sqrt{9 + 11 \mu} \right)}{1 + \exp \left( -\frac{x - 20}{\sqrt{2}} \right)} \), \( \text{Im} \ u_0 = 0 \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.1 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-3} \), \( atol = 10^{-4} \) and intermediate timesteps.

Nonfrozen solution:

Spatial-temporal pattern:
II. Frozen system (with freezing method ([73])): Consider the associated frozen system

where $d = 1$ (i.e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 50$ and initial data $\Re v_{10} = \frac{2}{\Pi} (3 + \sqrt{9 + \Pi \mu}) - \frac{2}{\Pi} (3 + \sqrt{9 + \Pi \mu}) \frac{1 + \exp \left( -\frac{x}{\sqrt{2}} \right)}{1 + \exp \left( -\frac{x}{\sqrt{2}} \right)}$, $\Im v_{10} = 0$, $\Re v_{20} = \frac{2}{\Pi} (3 + \sqrt{9 + \Pi \mu}) \frac{1 + \exp \left( -\frac{x}{\sqrt{2}} \right)}{1 + \exp \left( -\frac{x}{\sqrt{2}} \right)}$, $\Im v_{20} = 0$. As reference functions and bump function we use $\Re \hat{v}_1(x) = \Re v_{10}$, $\Im \hat{v}_1(x) = \Im v_{10}$, $\Re \hat{v}_2(x) = \Re v_{20}$, $\Im \hat{v}_2(x) = \Im v_{20}$ and $\varphi(x) = \sech(0.5 \cdot x)$, respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.2$, $rtol = 10^{-2}$, $atol = 10^{-4}$ and intermediate timesteps.

Frozen solutions:

Spatial-temporal pattern:
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Spinning soliton (spinning solitary wave, localized vortex solution):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \; t \in [0, \infty[ \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \; t \in [0, \infty[ \\
u(0) &= u_0, \quad x \in B_R(0), \; t = 0
\end{align*}
\]

where \(d = 2\) (i.e. \(B_R(0) = \{ x \in \mathbb{R}^2 \; | \; \|x\|_{\mathbb{R}^2} \leq R \} \)). For the numerical computations we use parameters \(\mu = -\frac{1}{2}, \; \alpha = \frac{1}{2} + \frac{1}{2} i, \; \beta = \frac{5}{2} + i, \; \gamma = -1 - \frac{1}{10} i, \; R = 20\) and initial data \(\text{Re} \; u_0 = \frac{\pi}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right), \; \text{Im} \; u_0 = \frac{\pi}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right)\) (or using polar coordinates \(u_0(r, \phi) = \frac{\pi}{5} \exp (i\phi) \exp \left( -\frac{x_1^2 + x_2^2}{49} \right)\)). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \(\Delta x = 0.5\). For the temporal discretization we use BDF(2) with \(\Delta t = 0.1, \; rtol = 10^{-4}, \; atol = 10^{-5}\) and intermediate timesteps ([25],[24],[78],[14],[12]).

Nonfrozen solution:

Spatial-temporal pattern: (with \(x_1 \in [-20, 20]\) and \(x_2 = 0\))
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
v_t = \alpha \Delta v + v \left( \mu + \beta |v|^2 + \gamma |v|^4 \right) + \lambda_1 v_{x_1} + \lambda_2 v_{x_2} + i \lambda_3 v, \quad x \in B_R(0), \ t \in [0, \infty[ \\
\frac{\partial v}{\partial n}(0) = 0, \quad x \in \partial B_R(0) \cap [0, \infty[ \\
v(0) = u_0, \quad x \in B_R(0), \ t = 0 \\
(\gamma_1)_{t-} = \lambda_1, \quad t \in [0, \infty[ \\
(\gamma_2)_{t-} = \lambda_2, \quad t \in [0, \infty[ \\
(\gamma_3)_{t-} = \lambda_3, \quad t \in [0, \infty[ \\
0 = (\hat{v}_{x_1}, v - \hat{v})_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
0 = (\hat{v}_{x_2}, v - \hat{v})_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
0 = (i \hat{v}, v - \hat{v})_{L^2(B_R(0), \mathbb{C})}, \quad t \in [0, \infty[ \\
\]
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Real parts of the eigenfunctions:
Rotating spiral wave (rigidly rotating spiral):
I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \ t \in [0, \infty[ \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \ t \in [0, \infty[ \\
u(0) &= u_0, \quad x \in B_R(0), \ t = 0
\end{align*}
\]

where \( d = 2 \) (i.e. \( B_R(0) = \{ x \in \mathbb{R}^2 \mid \| x \|_{\mathbb{R}^2} \leq R \} \)). For the numerical computations we use parameters \( \mu = -\frac{1}{4}, \alpha = \frac{1}{2} + \frac{1}{2}i, \beta = \frac{13}{5} + i, \gamma = -1 - \frac{1}{4}i, \ R = 20 \) and initial data \( \text{Re} \ u_0 = \frac{2}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right) \), \( \text{Im} \ u_0 = \frac{2}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right) \) (or using polar coordinates \( u_0(r, \phi) = \frac{2}{5} \exp (i\phi) \exp \left( -\frac{r^2}{49} \right) \) ). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.5 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \ rtol = 10^{-4}, \ atol = 10^{-5} \) and intermediate timesteps ([78]).
Nonfrozen solution:

Spatial-temporal pattern: (with \( x_1 \in [-20, 20] \) and \( x_2 = 0 \))

II. Frozen system (with freezing method ([12])): Consider the associated frozen system
\[ v_t = \alpha \Delta v + v \left( \mu + \beta |v|^2 + \gamma |v|^4 \right) + \lambda_1 v_{x1} + \lambda_2 v_{x2} + i \lambda_3 v, \quad x \in B_R(0), \; t \in [0, \infty[ \]

\[ \frac{\partial v}{\partial n} = 0, \quad x \in \partial B_R(0), \; t \in [0, \infty[ \]

\[ v(0) = u_0 \quad \text{for} \; x \in B_R(0), \; t = 0 \]

\[ \gamma_1(t) = \lambda_1, \quad t \in [0, \infty[ \]

\[ \gamma_2(t) = \lambda_2, \quad t \in [0, \infty[ \]

\[ \gamma_3(t) = \lambda_3, \quad t \in [0, \infty[ \]

\[ 0 = \langle \hat{v}_{x1}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})} \quad \text{for} \; t \in [0, \infty[ \]

\[ 0 = \langle \hat{v}_{x2}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})} \quad \text{for} \; t \in [0, \infty[ \]

\[ 0 = \langle i \hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})} \quad \text{for} \; t \in [0, \infty[ \]

where \( d = 2 \) (i.e. \( B_R(0) = \{ x \in \mathbb{R}^2 \mid \| x \|_{\mathbb{R}^2} \leq R \} \)). For the numerical computations we use parameters \( \mu = -\frac{1}{2}, \alpha = \frac{1}{2} + \frac{1}{2}i, \beta = \frac{12}{5} + i, \gamma = -1 - \frac{1}{10}i, \; R = 20 \) and initial data \( \text{Re} u_0 = \frac{2}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right) \), \( \text{Im} u_0 = \frac{5}{2} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right) \) (or using polar coordinates \( u_0(r, \phi) = \frac{r}{5} \exp (i \phi) \exp \left( -\frac{r^2}{49} \right) \)). As reference functions \( \text{Re} \hat{v}(x) \) and \( \text{Im} \hat{v}(x) \) we choose the real and imaginary part of the nonfrozen solution at time \( t = 150 \), respectively, with the parameters mentioned with \( \text{rtol} = 10^{-2} \) and \( \text{atol} = 10^{-4} \). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.5 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \; \text{rtol} = 10^{-2}, \; \text{atol} = 10^{-5} \) and intermediate timesteps.

Frozen solutions:

Spatial-temporal pattern: \( (\text{with } x_1 \in [-20, 20] \text{ and } x_2 = 0) \)

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:
1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

Real parts of the eigenfunctions:

- Spinning 2-soliton (spinning multisoliton, localized 2-vortex solution):
  I. Nonfrozen solution: Consider the nonfrozen system

\[
\begin{align*}
  u_t &= \alpha \Delta u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad x \in B_R(0), \quad t \in [0, \infty[ \\
  \frac{\partial u}{\partial n} &= 0, \quad x \in \partial B_R(0), \quad t \in [0, \infty[ \\
  u(0) &= u_0, \quad x \in B_R(0), \quad t = 0
\end{align*}
\]

where \( d = 2 \) (i.e. \( B_R(0) = \{ x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leq R \} \)). For the numerical computations we use parameters

\[
\begin{align*}
  \mu &= -\frac{1}{2}, \quad \alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10} i, \quad R = 20 \quad \text{and initial data} \quad \text{Re} u_0 = \frac{x_1}{5} \exp \left( -\frac{x_1^2 + x_2^2}{49} \right)
\end{align*}
\]
Im \( u_0 = \frac{2\pi}{\alpha} \exp \left( -\frac{x_1^2 + x_2^2}{4\beta} \right) \) (or using polar coordinates \( u_0(r,\phi) = \frac{\pi}{\alpha} \exp (i\phi) \exp \left( -\frac{r^2}{2\beta} \right) \)). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 0.5 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1 \), \( rtol = 10^{-4} \), \( atol = 10^{-5} \) and intermediate timesteps.

Nonfrozen solution:

Spatial-temporal pattern: (with \( x_1 \in [-20, 20] \) and \( x_2 = 0 \))

**Explicit solutions**: ([80],[56],[57])

**Additional informations**: In case of \( \text{Im}(\alpha) = \text{Im}(\beta) = \text{Im}(\gamma) = \text{Im}(\mu) = 0 \) (i.e. \( \alpha, \beta, \gamma, \mu \in \mathbb{R} \)) the equation is called **quintic real Ginzburg-Landau equation**. If \( \gamma = 0 \) then we obtain the **cubic Ginzburg-Landau equation**. In the following cases there exists soliton solutions ([80])

1. \( \text{Re}(\alpha) < 2 \cdot \text{Re}(\beta), \mu \in \mathbb{R} \) with \( \mu < 0 \), \( \text{Re}(\gamma) < 0 \)
2. \( \text{Re}(\alpha) > 2 \cdot \text{Re}(\beta), \mu \in \mathbb{R} \) with \( \mu > 0 \), \( \text{Re}(\gamma) < 0 \)
3. \( \text{Re}(\alpha) = \text{Re}(\beta) = \text{Re}(\gamma) = \mu = 0 \)

In case of \( \text{Re}(\beta) > 0 \) there exists soliton-like solutions ([80]). Multi-armed spirals are everywhere unstable, moreover the only stable rotating spirals are the one-armed. The QCGL equation describes a subcritical Hopf bifurcation at \( \text{Re}(\mu) = 0 \) with three branches:

\[
|u| = 0 \text{ and } |u|_{\pm} = \sqrt{\frac{-\text{Re}(\beta) \pm \sqrt{\left(\text{Re}(\beta)\right)^2 - 4\text{Re}(\gamma)\text{Re}(\mu)}}{2\text{Re}(\gamma)}}
\]

The first branch is stable if \( \text{Re}(\mu) < 0 \) and unstable if \( \text{Re}(\mu) > 0 \). The second exists only for \( \frac{\left(\text{Re}(\beta)\right)^2}{4\text{Re}(\gamma)} < \text{Re}(\mu) < 0 \) and is always unstable. The third exists only for \( \frac{\left(\text{Re}(\beta)\right)^2}{4\text{Re}(\gamma)} < \text{Re}(\mu) \) and is always stable ([85]).

**Literature**: [77], [54], [82], [74], [84], [58], [47], [69], [2], [24], [75], [25], [26], [79], [14], [87], [56], [57], [40], [81], [80], [5], [76]
1.11 λ-ω system

Name: λ-ω SYSTEM

Equations:

\[ u_t = D \Delta u + (\lambda(|u|) + i\omega(|u|)) \cdot u \]

\[ u = u(x,t) \in \mathbb{C}, \quad x \in \mathbb{R}^d, \quad d \in \{1, 2, 3\}, \quad t \in [0, \infty[, \quad D \in \mathbb{C}, \quad \lambda : \mathbb{R} \to \mathbb{R} \text{ with } \lambda = \lambda(|u|) \text{ (i.e. } \lambda(|u|) = 1 - |u|^2), \quad \omega : \mathbb{R} \to \mathbb{R} \text{ with } \omega = \omega(|u|) \text{ (i.e. } \omega(|u|) = -\alpha|u|^2 \text{ with } \alpha \in \mathbb{R}). \]

Notations:

\[ \text{Re} \, u(x,t) : \text{concentration at position } x \text{ and time } t \text{ of the first reactant} \]
\[ \text{Im} \, u(x,t) : \text{concentration at position } x \text{ and time } t \text{ of the second reactant} \]
\[ D : \text{diffusion coefficient} \]
\[ \text{Re} \, f(u), \text{Im} \, f(u) : f(u) = (\lambda(|u|) + i\omega(|u|)) \cdot u, \text{reaction kinetics} \]
\[ \alpha : \text{reaction parameter (positive constant)} \]

Short description: The λ-ω system ([53],[61]) describes chemical reaction processes ([53],[52]), physiological processes in the study of cardiac arrhythmias ([66]), time evolution of biological systems ([61]) and is often used to analyse the mechanism of pattern formation ([46]) as well as to study the onset of turbulent behavior ([55]). An example of an emerging technological application based on pattern forming systems is given by memory devices using magnetic domain patterns ([27]). This model exhibits rotating spirals as well as scroll wave and scroll ring solutions ([10],[30]).

Phenomena:

• Rotating spiral wave (rigidly rotating spiral)

Set of parameter values:

| d | D | \lambda(|u|) | \omega(|u|) | R | \Delta x | \Delta t | Boundary | Phenomena |
|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 - |u|^2 | -|u|^2 | 50 | 1.0 | 0.1 | \frac{\partial u}{\partial n} = 0 \text{ on } \partial B_R(0) | rotating spiral wave |

Numerical results:

• Rotating spiral wave (rigidly rotating spiral):

I. Nonfrozen solution: Consider the nonfrozen system

\[ u_t = D \Delta u + (\lambda(|u|) + i\omega(|u|)) \cdot u, \quad x \in B_R(0), \quad t \in [0, \infty[ \]
\[ \frac{\partial u}{\partial n} = 0, \quad x \in \partial B_R(0), \quad t \in [0, \infty[ \]
\[ u(0) = u_0, \quad x \in B_R(0), \quad t = 0 \]

where \( d = 2 \) (i.e. \( B_R(0) = \{ x \in \mathbb{R}^2 \mid \| x \|_{\mathbb{R}^2} \leq R \} \)). For the numerical computations we use parameters \( D = 1, \lambda(|u|) = 1 - |u|^2, \omega(|u|) = -|u|^2, \quad R = 50 \text{ and initial data } u_{10} = \frac{\pi}{25}, u_{20} = \frac{\pi}{25} \text{ (or using polar coordinates } u_0(r, \phi) = \frac{\pi}{25} (\cos(\phi) + i \sin(\phi)) \text{). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with \( \Delta x = 1.0 \). For the temporal discretization we use BDF(2) with \( \Delta t = 0.1, \quad rtol = 10^{-3}, \quad atol = 10^{-8} \text{ and intermediate timesteps ([13],[53],[12])}. Nonfrozen solution:
Spatial-temporal pattern: (with $x_1 \in [-50, 50]$ and $x_2 = 0$)

II. Frozen system (with freezing method ([12])): Consider the associated frozen system

\[
\begin{align*}
  v_t &= D \Delta v + (\lambda(|v|) + i\omega(|v|)) \cdot v + \lambda_1 v_{x_1} + \lambda_2 v_{x_2} + i\lambda_3 v, &\quad x \in B_R(0), t \in [0, \infty[ \\
  \frac{\partial v}{\partial n} &= 0, &\quad x \in \partial B_R(0), t \in [0, \infty[ \\
  v(0) &= u_0, &\quad x \in B_R(0), t = 0 \\
  \gamma_1(0) &= 0, &\quad t \in [0, \infty[ \\
  \gamma_2(0) &= 0, &\quad t \in [0, \infty[ \\
  \gamma_3(0) &= 0, &\quad t \in [0, \infty[ \\
  0 &= \langle \hat{v}_{x_1}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, &\quad t \in [0, \infty[ \\
  0 &= \langle \hat{v}_{x_2}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, &\quad t \in [0, \infty[ \\
  0 &= \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0), \mathbb{C})}, &\quad t \in [0, \infty[ 
\end{align*}
\]

where $d = 2$ (i.e. $B_R(0) = \{ x \in \mathbb{R}^2 \mid \| x \|_{\mathbb{R}^2} \leq R \}$). For the numerical computations we use parameters $D = 1$, $\lambda(|v|) = 1 - |v|^2$, $\omega(|v|) = -|v|^2$, $R = 50$, and initial data $u_{10} = \frac{x}{20}$, $u_{20} = \frac{y}{20}$ (or using polar coordinates $u_0(r, \phi) = \frac{r}{2} (\cos(\phi) + i\sin(\phi))$). As reference functions $\text{Re} \hat{v}(x)$ and $\text{Im} \hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 150$, respectively, with the parameters mentioned above and $\Delta x = 2.0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\Delta x = 2.0$. For the temporal discretization we use BDF(2) with $\Delta t = 1.0$, $rtol = 10^{-3}$, $atol = 10^{-5}$ and intermediate timesteps.

Frozen solutions:
Spatial-temporal pattern: (with $x_1 \in [-50, 50]$ and $x_2 = 0$)

Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Real parts of the eigenfunctions:
Explicit solutions: not available

Literature: [12], [61], [18], [52], [46], [55], [27], [66], [53], [10], [13], [30]
1.12 Autocatalysis model

Name: Autocatalysis model

Equations:
\begin{align*}
  u_t &= D_u \Delta u - uv^m \\
  v_t &= D_v \Delta v + uv^m
\end{align*}

\[ u = u(x,t) \in \mathbb{R}, \ v = v(x,t) \in \mathbb{R}, \ x \in \mathbb{R}^d, \ d \in \{1, 2, 3\}, \ t \in [0, \infty[, \ D_u, D_v \in \mathbb{R}, \ m \in \mathbb{N}. \]

Notations:
\begin{itemize}
  \item \( u(x,t) \) : concentration of the reactant at position \( x \) and time \( t \)
  \item \( v(x,t) \) : concentration of the autocatalyst at position \( x \) and time \( t \)
  \item \( D_u, D_v \) : diffusion coefficients (positive constant)
  \item \( m \) : other system parameter (positive integer)
\end{itemize}

Short description: not available

Set of parameters:

<table>
<thead>
<tr>
<th>( d )</th>
<th>( D_u )</th>
<th>( D_v )</th>
<th>( m )</th>
<th>( R )</th>
<th>( \Delta x )</th>
<th>( \Delta t )</th>
<th>Boundary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>1</td>
<td>9</td>
<td>180</td>
<td>0.1</td>
<td>0.1</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial[-180, 180] )</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>200</td>
<td>5.0</td>
<td>0.1</td>
<td>( \frac{\partial u}{\partial n} = 0 ) on ( \partial B_R(0) )</td>
</tr>
</tbody>
</table>

Phenomena:
- Traveling wave front (traveling wave, traveling front): \( d = 1, \ D_u = 0.1, \ D_v = 1, \ m = 9, \ R = 180, \)
  \( \Delta x = 0.1, \ \Delta t = 0.1, \ \frac{\partial u}{\partial n} = 0 \) on \( \partial[-180, 180] \) (Neumann boundary), \( u_0 = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}, \ v_0 = 1 - u_0. \)

- Traveling wave front (traveling wave, traveling front): \( d = 2, \ D_u = 2, \ D_v = 1, \ m = 2, \ R = 200, \)
  \( \Delta x = 5.0, \ \Delta t = 0.1, \ \frac{\partial u}{\partial m} = 0 \) on \( \partial B_R(0) \) (Neumann boundary), \( u_0 = \frac{1}{1 + \exp\left(-\frac{y - 50 - 5 \cos\left(\frac{\pi x}{4}\right)}{4}\right)}, \ v_0 = 1 - u_0. \)
**Explicit solutions:** not available

**Literature:** not available
Bibliography


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