Mathematical Models of Reaction Diffusion Systems, their Numerical Solutions and the Freezing Method with Comsol Multiphysics

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1 Mathematical Models

In this chapter I want to give a large assortment of examples for reaction-diffusion equations, their phenomena and some background stuff. The focus of this investigations will be the freezing method ([12]) and their numerical computations with Comsol Multiphysics 3.5a. For this purpose we use the following notions:

d	spatial dimension, $d \in \{1, 2, 3\}$
$\mathbb{R}^d$	d-dimensional euclidean space
x	spatial variable, $x \in \mathbb{R}^d$
t	temporal variable, $t \in [0, \infty[$
R	ratio, $R \in \mathbb{R}$ with $R > 0$
$B_R(0)$	$B_R(0) = \{x \in \mathbb{R}^d \mid \|x\|_2 \leqslant R\}$, ball in $\mathbb{R}^d$ with ratio R and center 0
$\partial B_R(0)$	$\partial B_R(0) = \{x \in \mathbb{R}^d \mid \|x\|_2 = R\}$, boundary of $B_R(0)$
$\triangle x$	spatial discretization parameter, spatial stepsize
$\triangle t$	temporal discretization parameter, temporal stepsize
$\mathrm{Re}(u)$	real part of u
$\mathrm{Im}(u)$	imaginary part of u
i	imaginary unit
$\frac{\partial}{\partial n}$	normal derivative, derivative in outer direction

1.1 Fisher's equation

Name: Fisher's equation (sometimes called Kolmogorov-Petrovsky-Piskounov equation (KPP) or Fisher-Kolmogorov equation)

Equations:

$$u_t = \triangle u + au\,(1-u)$$

where $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$ and $a \in \mathbb{R}$.

Notations:

u : frequency of the mutant gene
a : system parameter

Short description: The Fisher's equation ([31]), named after Ronald Aylmer Fisher (1890-1962) and Andrey Nicolaevich Kolmogorov (1903-1987), describes the spreading of biological populations, chemical reaction processes, heat and mass transfer ([28]) and is also used in genetics. This very simple model exhibits traveling front solutions.

Phenomena:

- Traveling 1-front (traveling front)

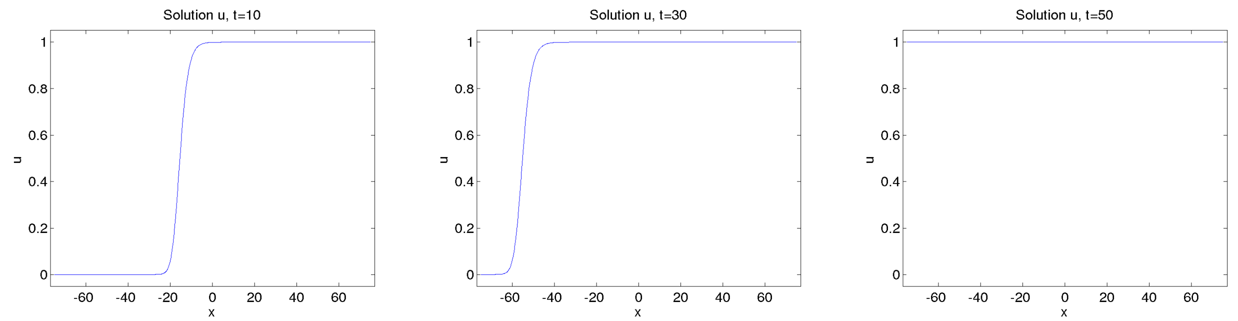
Set of parameter values:

d	a	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	1	75	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	1-front

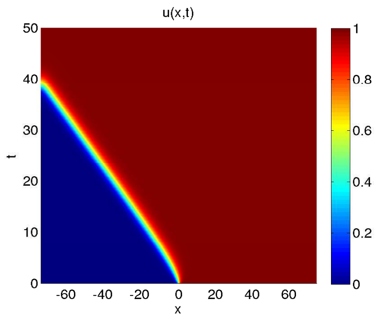
Numerical results:
• Traveling 1-front (traveling front):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= u_{xx} + au(1-u) && , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 && , x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 && , x \in B_R(0),\ t = 0 \end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use parameters $a = 1$, $R = 75$ and initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.
Nonfrozen solution:



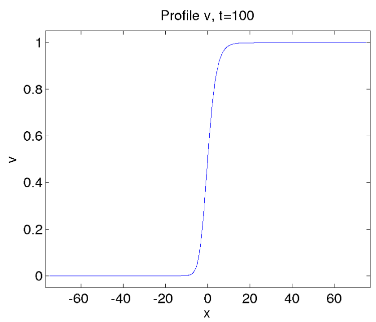
Spatial-temporal pattern:

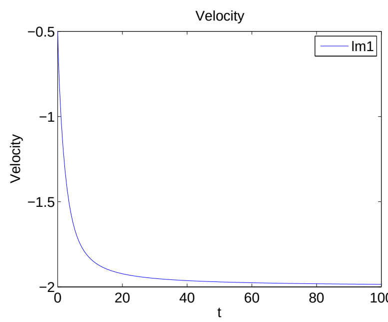


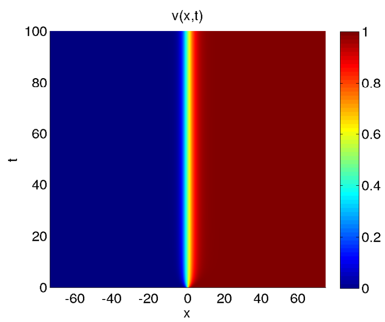
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$\begin{array}{lll} v_t & = & v_{xx} + av(1-v) + \lambda_1 v_x & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\ \gamma_t & = & \lambda_1 & , t \in [0,\infty[\\ \gamma(0) & = & 0 & , t = 0 \\ 0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[\end{array}$$

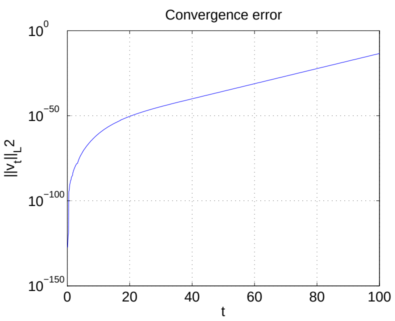
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $a = 1$, $R = 75$, initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$ and reference function $\hat{v}(x) = u_0(x)$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.01$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solution, velocity and spatial-temporal pattern:

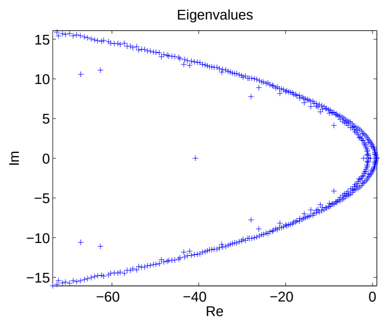






Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:





The increasing convergence error indicates that Fisher's-Front cannot be frozen by the freezing method. This can be recognized by the essential spectra. Because to ensure the stability of the front the essential spectra must exhibit a spectral gab, which doesn't exist.

Explicit solutions:
• 1d:

$$u(x,t) = \frac{1}{\left(1 + C\exp\left(-\frac{5}{6}\lambda t \pm \frac{x}{6}\sqrt{6\lambda}\right)\right)^2}, \text{ if } \lambda > 0$$

$$u(x,t) = \frac{1 + 2C\exp\left(-\frac{5}{6}\lambda t \pm \frac{x}{6}\sqrt{-6\lambda}\right)}{\left(1 + C\exp\left(-\frac{5}{6}\lambda t \pm \frac{x}{6}\sqrt{-6\lambda}\right)\right)^2}, \text{ if } \lambda < 0$$

are two possible solutions of the nonfrozen system with parameter $\lambda \in \mathbb{R}$ and an arbitrary constant $C \in \mathbb{R}$ ([1],[51],[48]).

Literature: [28], [1], [51], [22], [31], [48]

1.2 Nagumo equation

Name: NAGUMO EQUATION

Equations:

$$u_t = \triangle u + u\,(1-u)\,(u-\alpha)$$

$u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha \in]0,1[$.

Notations: not available

Short description: The Nagumo equation ([62]), named after Jin-Ichi Nagumo (1926-1999), describes propagation of nerve pulses in a nerve axon ([62]), spread of genetic traits, shape and speed of pulses in the nerve and is widely used in biology, circuit theory, heat and mass transfer and other fields. This model exhibits traveling front and traveling multifront solutions as well as sources and sinks ([4]).

Phenomena:

- Traveling 1-front (traveling front)
- Traveling 2-front (traveling multifront)
- Front interaction (3-front collision)

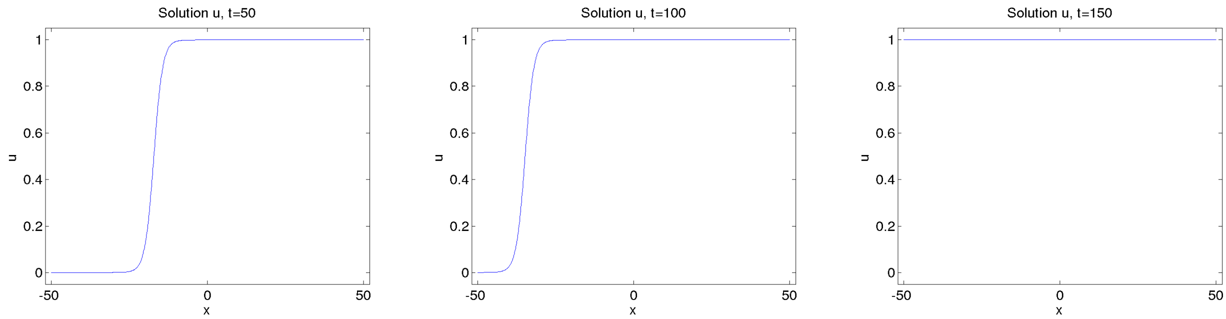
Set of parameter values:

d	α	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	$\frac{1}{4}$	75	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	1-front
1	$\frac{1}{4}$	100	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	2-front
1	$\frac{1}{4}$	150	0.3	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	3-front collision

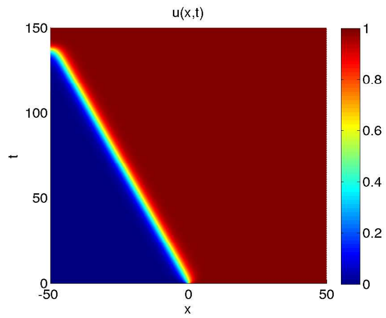
Numerical results:
• Traveling 1-front (traveling front):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{lcll} u_t & = & u_{xx} + u(1-u)(u-\alpha) & , x \in B_R(0),\, t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & , x \in \partial B_R(0),\, t \in [0,\infty[\\ u(0) & = & u_0 & , x \in B_R(0),\, t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use parameters $\alpha = \frac{1}{4}$, $R = 50$ and initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$ (also $u_0 = \begin{cases} 0 & x < 0 \\ \frac{x}{R} & x \geqslant 0 \end{cases}$ possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps ([13],[12]).
Nonfrozen solution:



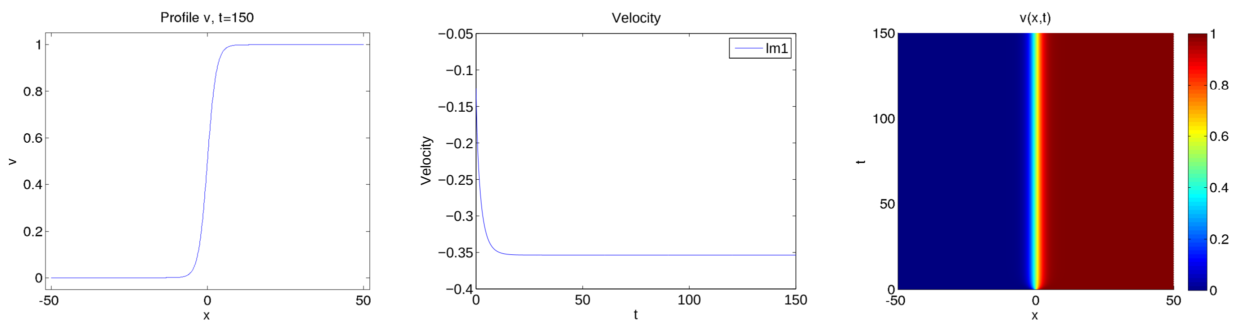
Spatial-temporal pattern:



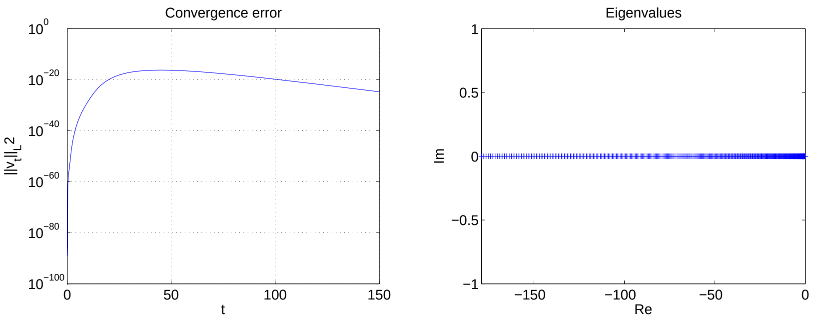
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lcll}
v_t & = & v_{xx} + v(1-v)(v-\alpha) + \lambda_1 v_x & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\
\gamma_t & = & \lambda_1 & , t \in [0,\infty[\\
\gamma(0) & = & 0 & , t = 0 \\
0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha = \frac{1}{4}$, $R = 50$, initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$ and reference function $\hat{v}(x) = u_0(x)$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solution, velocity and spatial-temporal pattern:



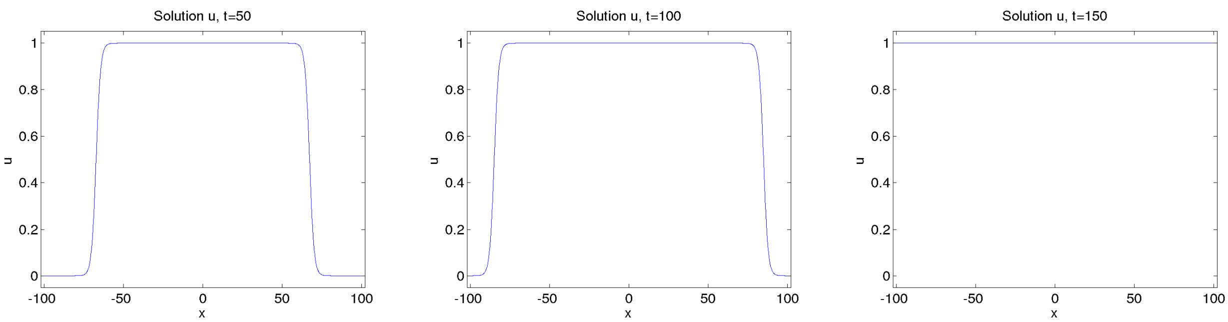
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



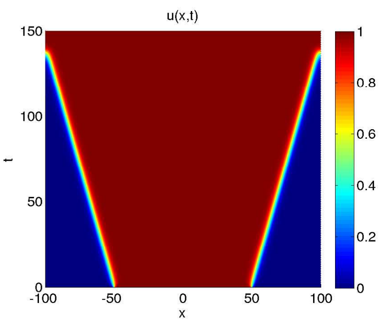
• Traveling 2-front (traveling multifront):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{lcll} u_t & = & u_{xx} + u(1-u)(u-\alpha) & , x \in B_R(0),\, t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & , x \in \partial B_R(0),\, t \in [0,\infty[\\ u(0) & = & u_0 & , x \in B_R(0),\, t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha = \frac{1}{4}$, $R = 100$ and initial data $u_0(x) = -\frac{1}{2}\tanh(x-50) + \frac{1}{2} + \begin{cases} \frac{1}{2}\tanh(x+50) - \frac{1}{2} & x \leqslant 0 \\ 0 & x > 0 \end{cases}$ (also $u_0 = \exp\left(-\frac{x^2}{R}\right)$ possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps ([11])).
Nonfrozen solution:



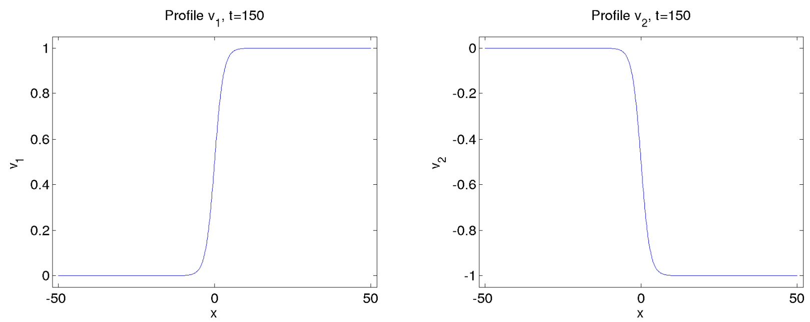
Spatial-temporal pattern:



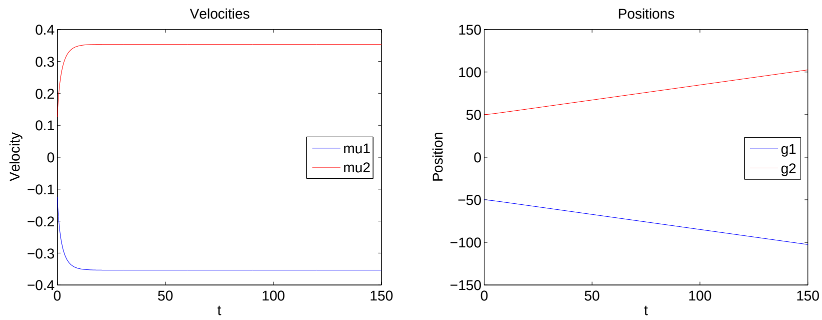
II. Frozen system (with freezing method ([11],[73])): Consider the associated frozen system ($j = 1, 2$)

$$
\begin{array}{lcll}
v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{2}\varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{2} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & \quad + \mu_j v_{j,x} & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0,\infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

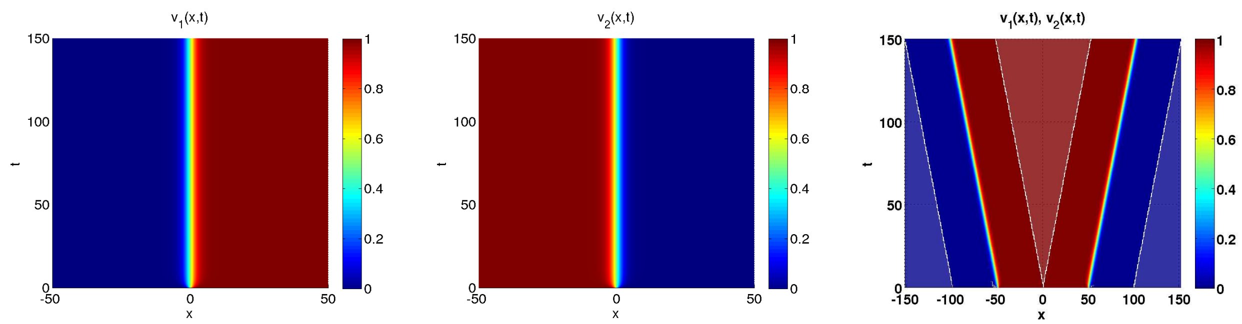
where $f(v) = v(1-v)(v-\alpha)$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha = \frac{1}{4}$, $R = 50$, initial data $u_{10}(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$, $u_{20}(x) = -\frac{1}{2}\tanh(x) - \frac{1}{2}$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 50$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right) + \exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:



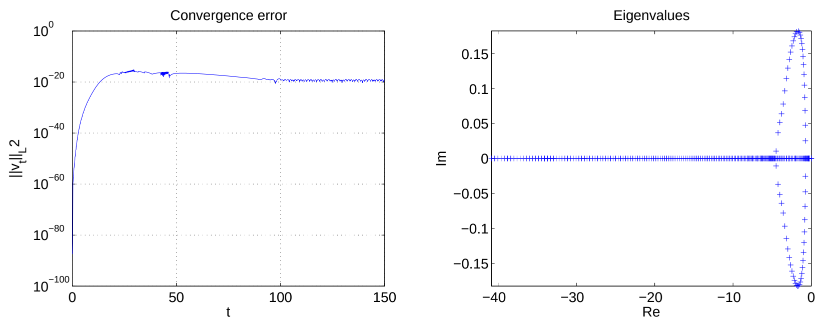
Velocities and positions:



Spatial-temporal patterns:



Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



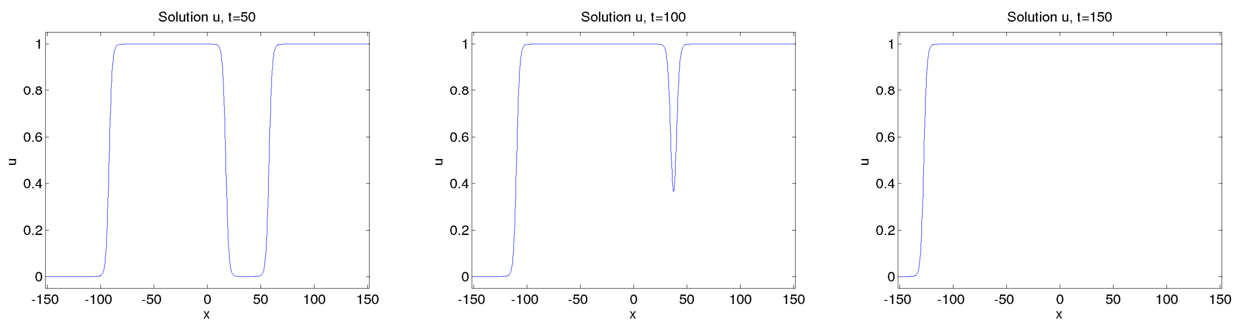
• Front interaction (3-front collision):

I. Nonfrozen solution: Consider the nonfrozen system

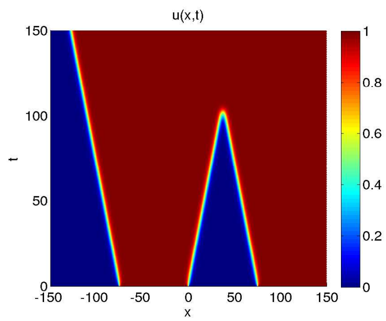
$$\begin{array}{lll} u_t &= u_{xx} + u(1-u)(u-\alpha) & , x \in B_R(0),\, t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 & , x \in \partial B_R(0),\, t \in [0,\infty[\\ u(0) &= u_0 & , x \in B_R(0),\, t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha = \frac{1}{4}$, $R = 150$ and initial data $u_0 = \begin{cases} \frac{1}{2}\tanh(x+75) + \frac{1}{2} & x \leqslant -50 \\ -\frac{1}{2}\tanh(x) + \frac{1}{2} & x \in]-50, 50[\\ \frac{1}{2}\tanh(x+75) + \frac{1}{2} & x \geqslant 50 \end{cases}$ possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps ([13],[12]).

Nonfrozen solution:



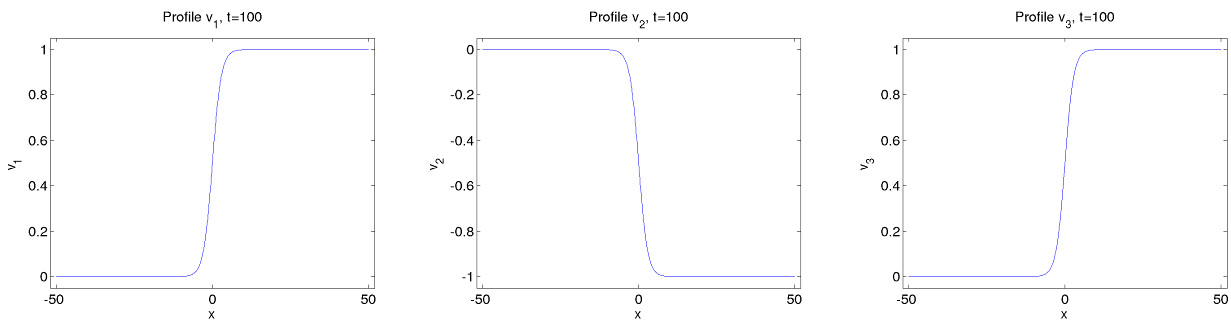
Spatial-temporal pattern:



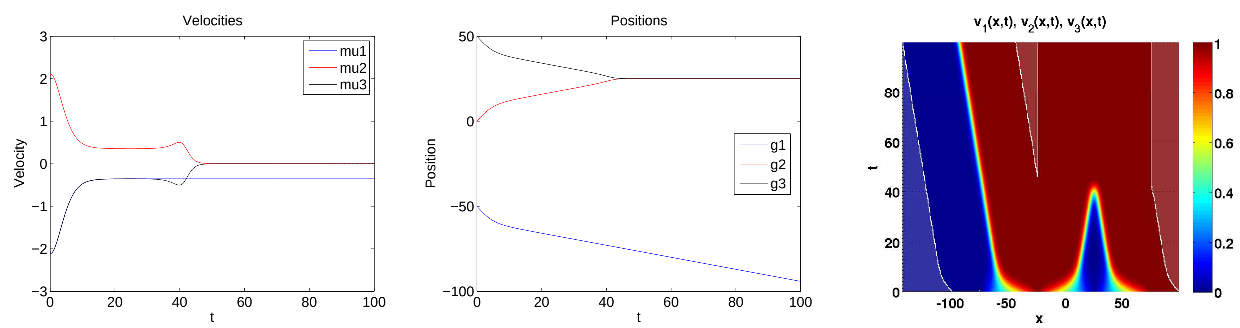
II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2, 3$)

$$\begin{array}{lcll}
v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{3} \varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{3} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & +\mu_j v_{j,x} & , x \in B_R(0),\ t \in [0, \infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0, \infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0, \infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0, \infty[
\end{array}$$

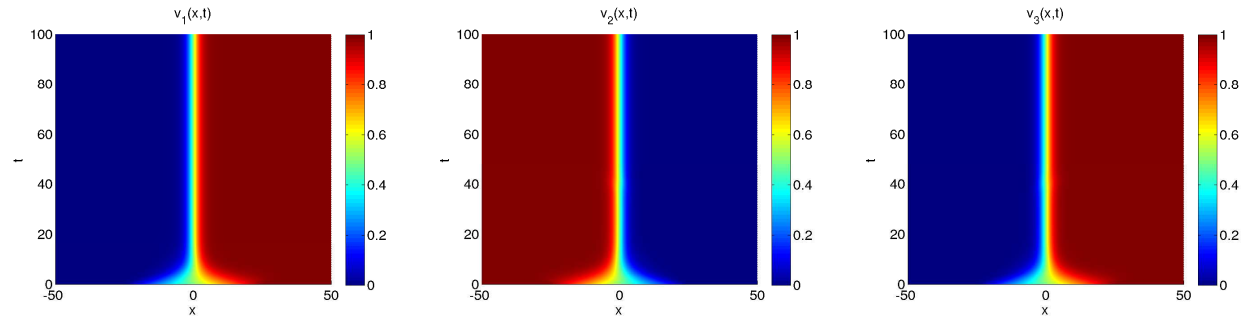
where $f(v) = v(1-v)(v-\alpha)$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha = \frac{1}{4}$, $R = 50$, initial data $u_{10}(x) = \begin{cases} 1 & x > 25 \\ \frac{1}{50}x + \frac{1}{2} & x \in [-25, 25] \\ 0 & x < 25 \end{cases}$, $u_{20}(x) = \begin{cases} -1 & x > 25 \\ -\frac{1}{50}x - \frac{1}{2} & x \in [-25, 25] \\ 0 & x < 25 \end{cases}$, $u_{30}(x) = u_{10}(x)$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 0$, $\gamma_{30} = 50$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$, $\hat{v}_3(x) = u_{30}(x)$ and bump function $\varphi(x) = \frac{2}{\exp(\frac{1}{2}x) + \exp(-\frac{1}{2}x)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-2}$ and intermediate timesteps.
Frozen solutions:



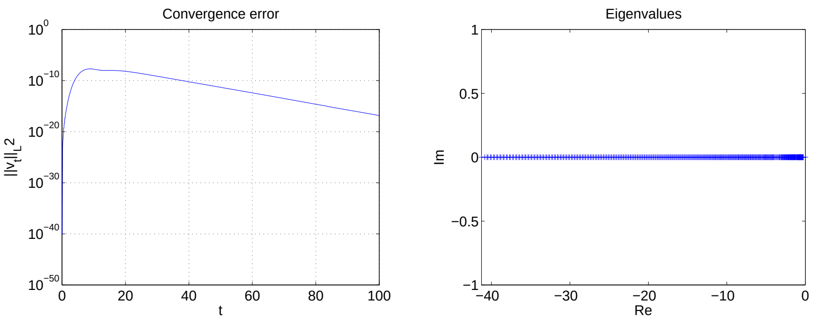
Velocities, positions and Spatial-temporal patterns:



Spatial-temporal patterns:



Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



Explicit solutions:
- 1d: $u(x) = \overline{u}(\xi)$ with $\xi = x - \mu t$ where

$$\overline{u}(\xi) = \left(1 + \exp\left(-\frac{\xi}{\sqrt{2}}\right)\right)^{-1} \text{ with speed } \mu = -\sqrt{2}\left(\frac{1}{2} - \alpha\right)$$
$$\overline{u}(\xi) = \left(1 + \exp\left(\frac{\xi}{\sqrt{2}}\right)\right)^{-1} \text{ with speed } \mu = \sqrt{2}\left(\frac{1}{2} - \alpha\right)$$

are two possible profiles $\overline{u}$ with their velocity μ and parameter $\alpha \in]0, \frac{1}{2}[$ ([23],[4]).
- 1d:

$$u(x,t) = \frac{A\exp\left(\pm\frac{\sqrt{2}}{2}x + \left(\frac{1}{2} - \alpha\right)t\right) + \alpha B\exp\left(\pm\frac{\sqrt{2}}{2}\alpha x + \alpha\left(\frac{\alpha}{2} - 1\right)t\right)}{A\exp\left(\pm\frac{\sqrt{2}}{2}x + \left(\frac{1}{2} - \alpha\right)t\right) + B\exp\left(\pm\frac{\sqrt{2}}{2}\alpha x + \alpha\left(\frac{\alpha}{2} - 1\right)t\right) + C}$$

where A, B, C are arbitrary constants ([49]).

Additional informations: The Nagumo equation (sometimes called Allen-Cahn model or Zeldovich equation arising in combustion theory ([86])) is a simplified form of the FitzHugh-Nagumo equations (to this later). Especially for $\alpha = -1$ we receive a special case of the Chafee-Infante equation (sometimes called Newell-Whitehead-Segel equation describing Rayleigh-Benard convections ([63],[72]))

$$u_t = \triangle u + \lambda(u - u^3), \quad \lambda \in \mathbb{R}$$

or more general

$$u_t = \triangle u + au - bu^3, \quad a, b \in \mathbb{R}$$

which in general possesses an attractor as well as a pitchfork bifurcation ([65],[43]).

Literature: [62], [4], [13], [11], [23], [49], [12]

1.3 Quintic Nagumo equation

Name: Quintic Nagumo equation

Equations:

$$u_t = \triangle u + u\,(1-u)\,(u-\alpha_1)\,(u-\alpha_2)\,(u-\alpha_3)$$

$u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha_1, \alpha_2, \alpha_3 \in]0,1[$ with $\alpha_1 < \alpha_2 < \alpha_3$.

Notations: not available

Short description: The Quintic Nagumo equation, named after Jin-Ichi Nagumo (1926-1999) describes an extension of the Nagumo equation and was developed only for mathematical investigations ([11],[73]). This model exhibits traveling front and traveling multifront solutions.

Phenomena:

- Traveling 1-front (traveling front)
- Traveling 2-front (traveling multifront)
- Front interaction (2-front collision)
- Traveling 3-front (traveling multifront)
- Traveling 4-front (traveling multifront)
- Front interaction (4-front double collision)

Set of parameter values:

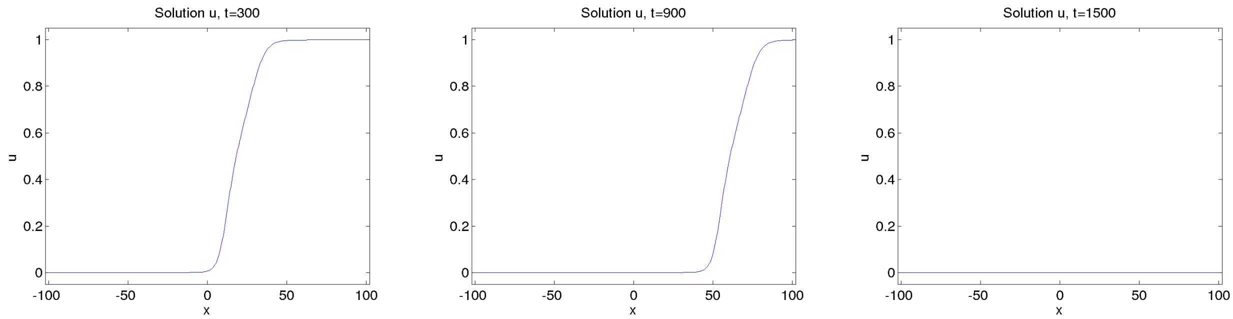
d	α_1	α_2	α_3	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{17}{20}$	100	0.3	0.3	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	1-front
1	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{10}$	100	0.3	0.3	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	2-front
1	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{10}$	250	0.4	1.0	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	3-front
1	$\frac{1}{16}$	$\frac{1}{2}$	$\frac{7}{10}$	150	0.3	0.5	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	4-front

Numerical results:
- Traveling 1-front (traveling front):

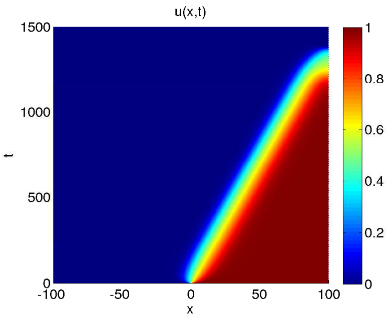
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned}
u_t &= u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) &&, x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\
u(0) &= u_0 &&, x \in B_R(0),\ t = 0
\end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use parameters $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{17}{20}$, $R = 100$ and initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.
Nonfrozen solution:



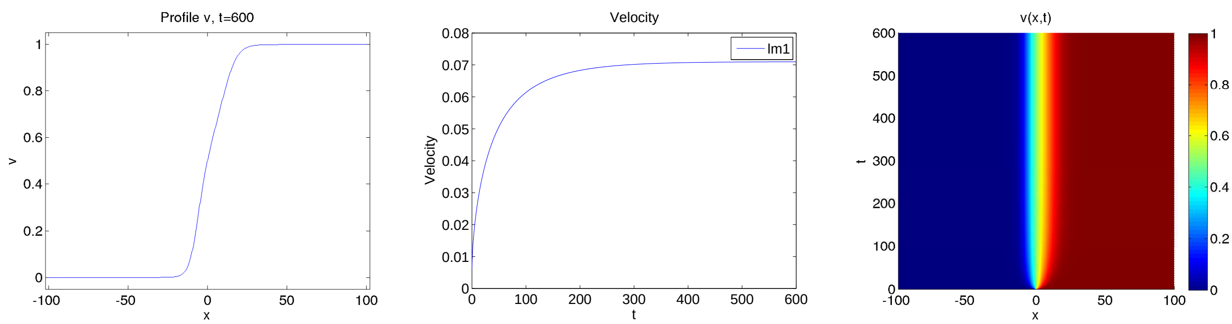
Spatial-temporal pattern:



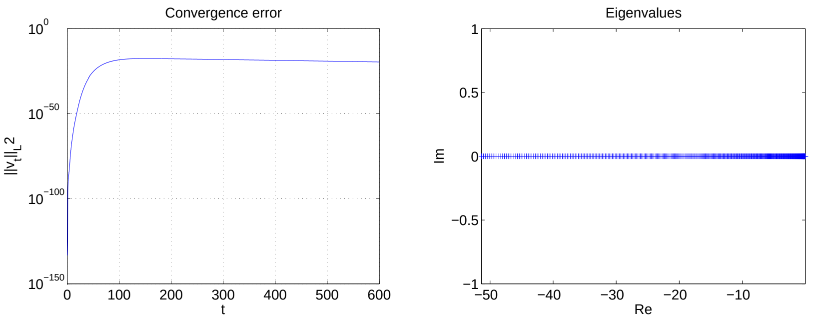
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lcll}
v_t & = & v_{xx} + v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3) + \lambda_1 v_x & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\
\gamma_t & = & \lambda_1 & , t \in [0,\infty[\\
\gamma(0) & = & 0 & , t = 0 \\
0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{2}{5}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{17}{20}$, $R = 100$, initial data $u_0(x) = \frac{1}{2}\tanh(x) + \frac{1}{2}$ and reference function $\hat{v}(x) = u_0(x)$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.2$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solution, velocity and spatial-temporal pattern:



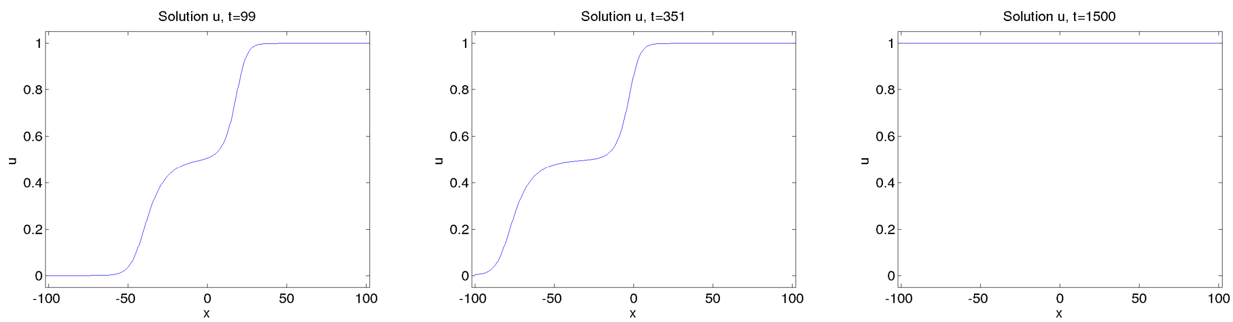
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



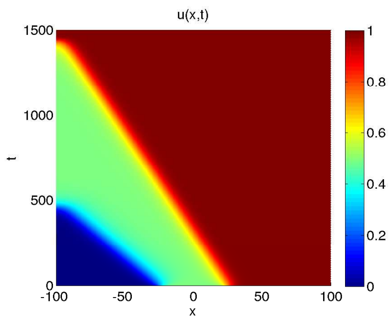
• Traveling 2-front (traveling multifront):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) &&, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\ t = 0 \end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 100$ and initial data $u_0(x) = \frac{1}{4}\tanh\left(\frac{x+25}{5}\right) + \frac{1}{4}\tanh\left(\frac{x-25}{5}\right) + \frac{1}{2}$ (also $u_0 = \frac{1}{2}\tanh\left(\frac{x}{5}\right) + \frac{1}{2}$ possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.
Nonfrozen solution:



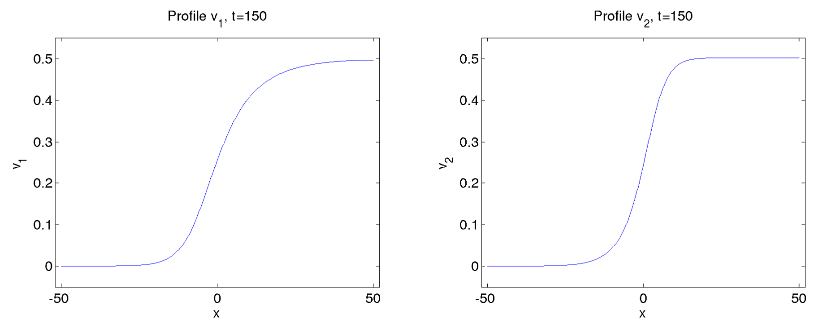
Spatial-temporal pattern:



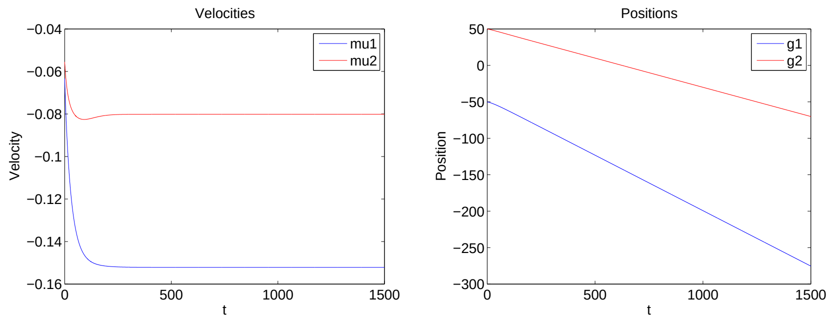
II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2$)

$$\begin{array}{rcll} v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{2}\varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{2} v_k(\cdot - g_k + g_j, \cdot)\right) & \\ & & +\mu_j v_{j,x} & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\ g_{j,t} & = & \mu_j & , t \in [0,\infty[\\ g_j(0) & = & g_{j0} & , t = 0 \\ 0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[\end{array}$$

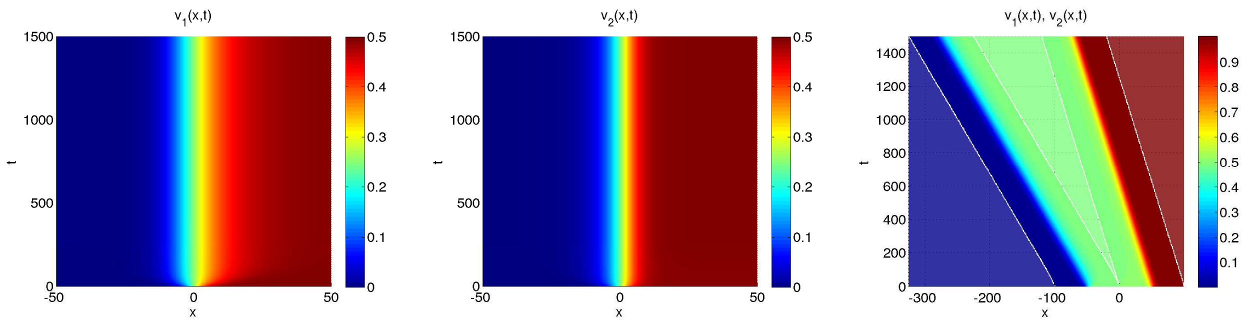
where $f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$ and $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 50$, initial data $u_{10}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, $u_{20}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 50$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right)+\exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:



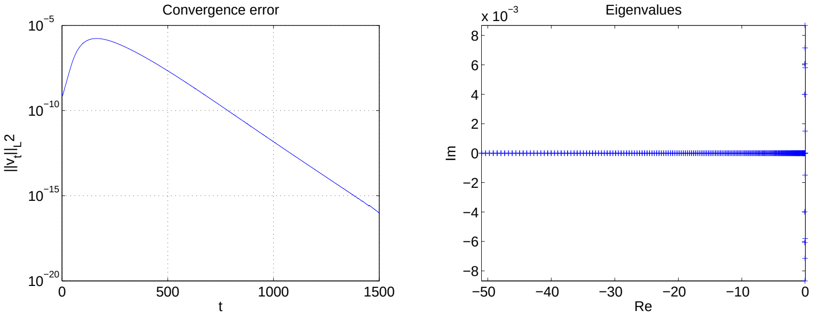
Velocities and positions:



Spatial-temporal patterns:



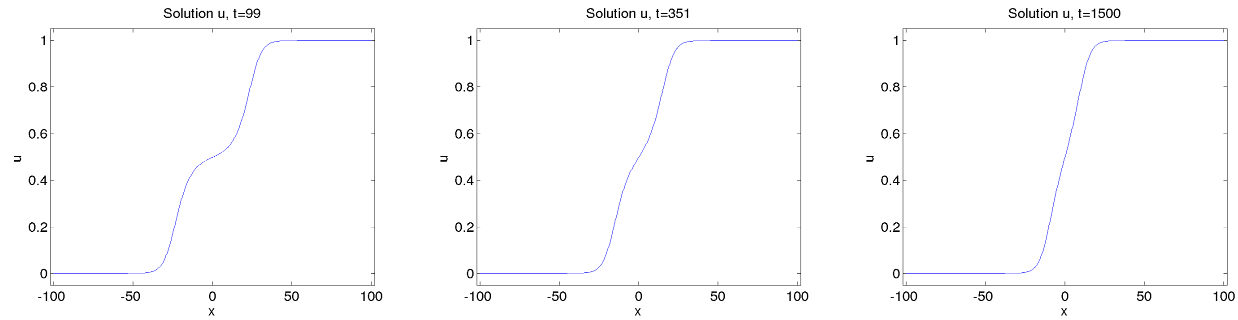
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



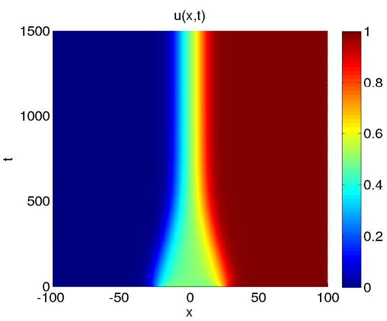
• Front interaction (2-front collision):
I. Nonfrozen solution: Consider the nonfrozen system

$$
\begin{array}{lll}
u_t & = & u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) \quad , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial u}{\partial n} & = & 0 \quad , x \in \partial B_R(0),\ t \in [0,\infty[\\
u(0) & = & u_0 \quad , x \in B_R(0),\ t = 0
\end{array}
$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{3}{4}$, $R = 100$ and initial data $u_0(x) = \frac{1}{4}\tanh\left(\frac{x+25}{5}\right) + \frac{1}{4}\tanh\left(\frac{x-25}{5}\right) + \frac{1}{2}$ (also $u_0 = \frac{1}{2}\tanh\left(\frac{x}{5}\right) + \frac{1}{2}$ possible). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.
Nonfrozen solution:



Spatial-temporal pattern:

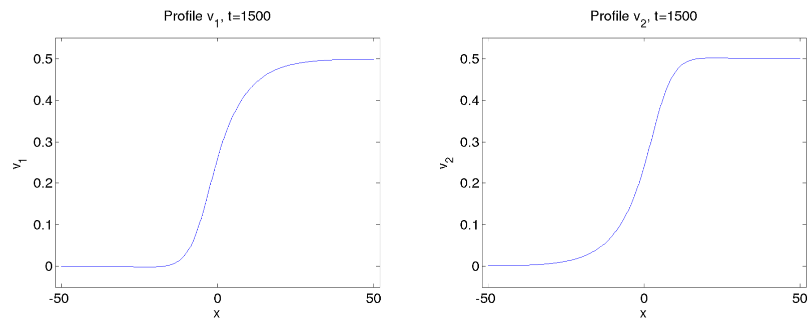


II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2$)

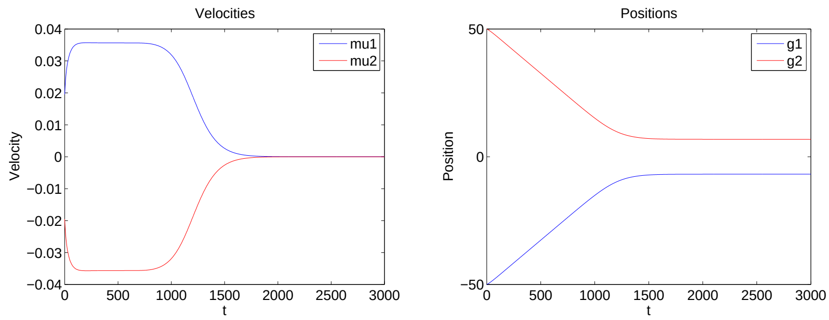
$$
\begin{array}{rcll}
v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{2}\varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{2} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & +\mu_j v_{j,x} & , x \in B_R(0),\, t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\, t \in [0,\infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\, t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0,\infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

where $f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{3}{4}$, $R = 50$, initial data $u_{10}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, $u_{20}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 50$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right)+\exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 1.5$, $rtol = 10^{-3}$, $atol = 5 \cdot 10^{-4}$ and intermediate timesteps.

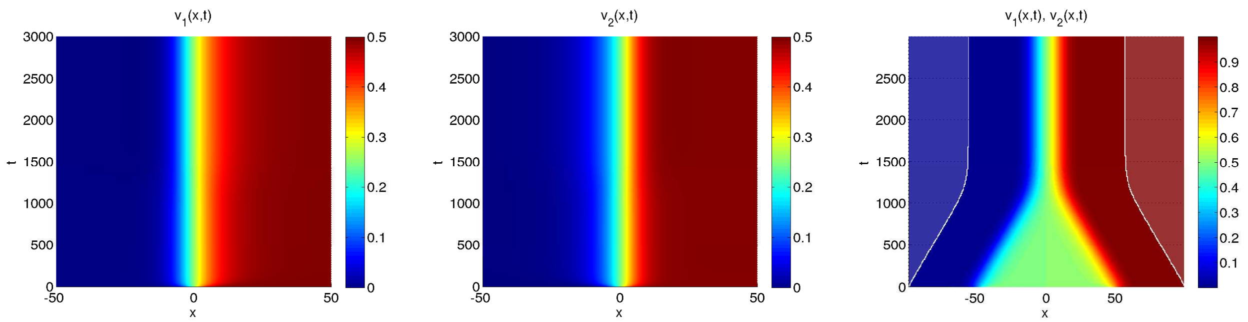
Frozen solutions:



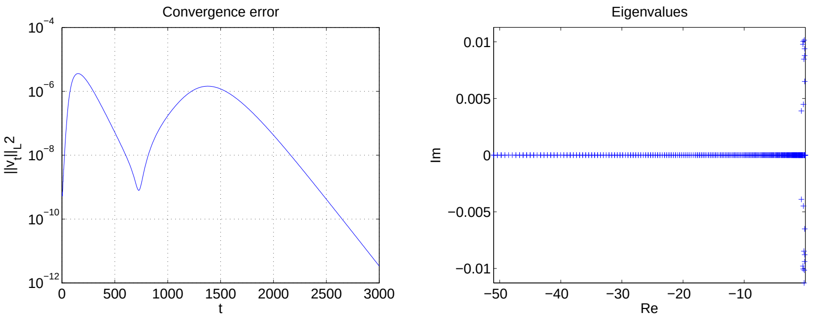
Velocities and positions:



Spatial-temporal patterns:



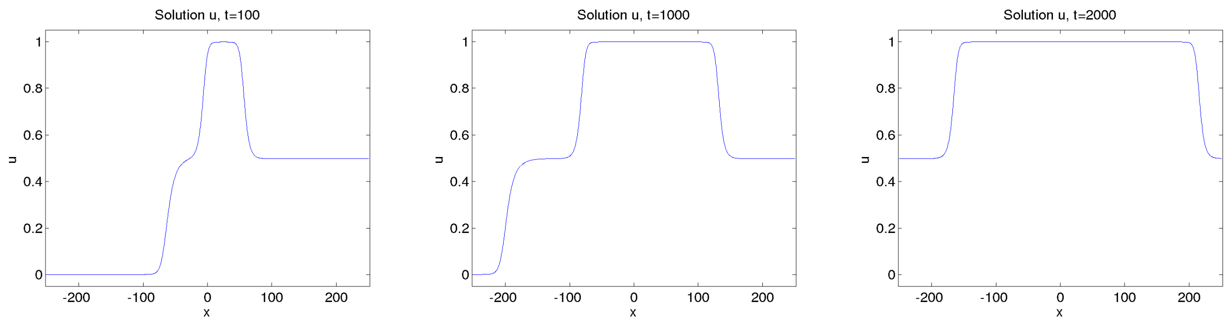
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



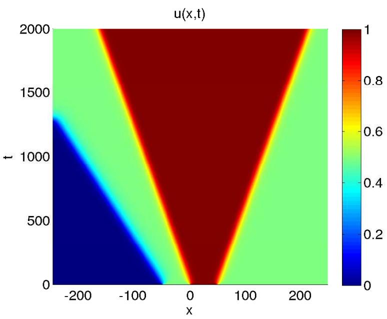
• Traveling 3-front (traveling multifront):
I. Nonfrozen solution: Consider the nonfrozen system

$$
\begin{array}{lcll}
u_t & = & u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial u}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
u(0) & = & u_0 & , x \in B_R(0),\ t = 0
\end{array}
$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 250$ and initial data $u_0(x) = \frac{1}{4}\tanh(x+50) + \frac{1}{4}\tanh(x) - \frac{1}{4}\tanh(x-50) + \frac{1}{4}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.4$. For the temporal discretization we use BDF(2) with $\triangle t = 1.0$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:



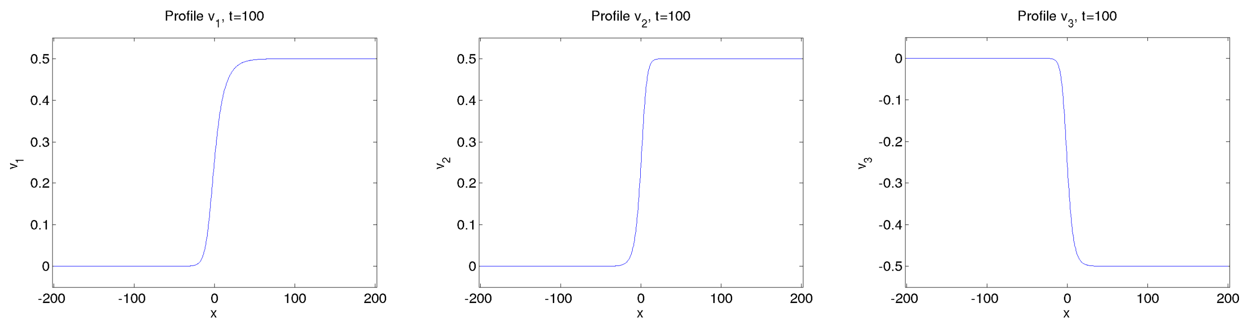
Spatial-temporal pattern:



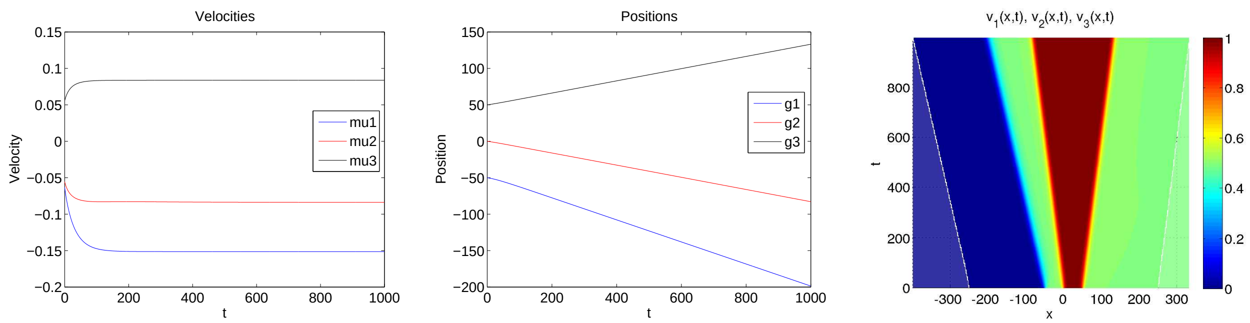
II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2, 3$)

$$
\begin{array}{rcll}
v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{3}\varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{3} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & +\mu_j v_{j,x} & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0,\infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

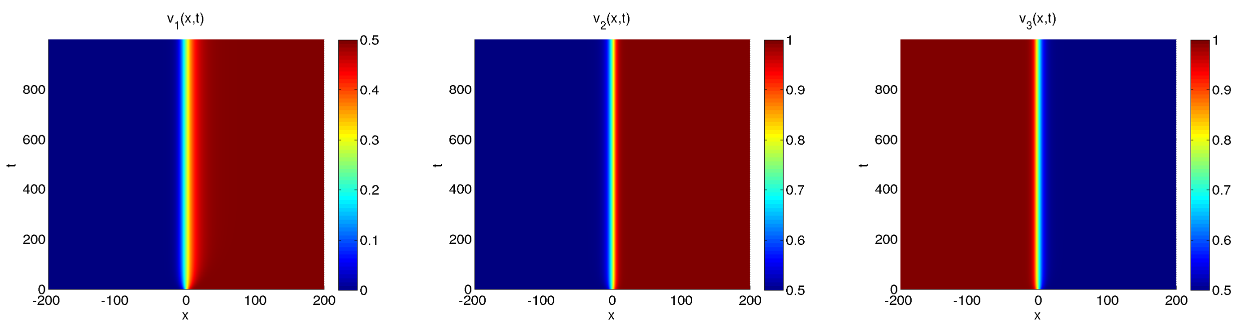
where $f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 200$, initial data $u_{10}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, $u_{20}(x) = u_{10}(x)$, $u_{30}(x) = -\frac{1}{4}\tanh\left(\frac{x}{5}\right) - \frac{1}{4}$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 0$, $\gamma_{30} = 50$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$, $\hat{v}_3(x) = u_{30}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right) + \exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.4$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:



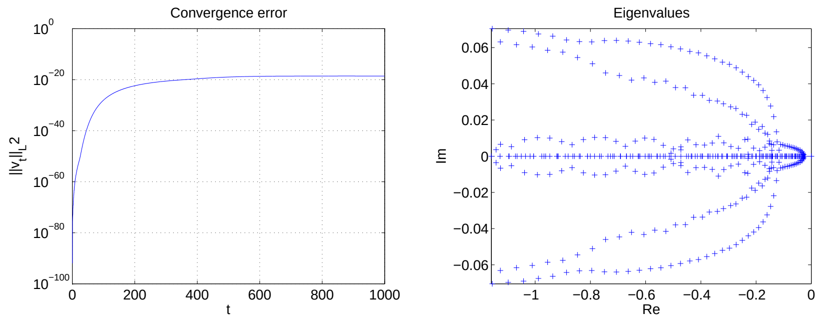
Velocities, positions and Spatial-temporal patterns:



Spatial-temporal patterns:



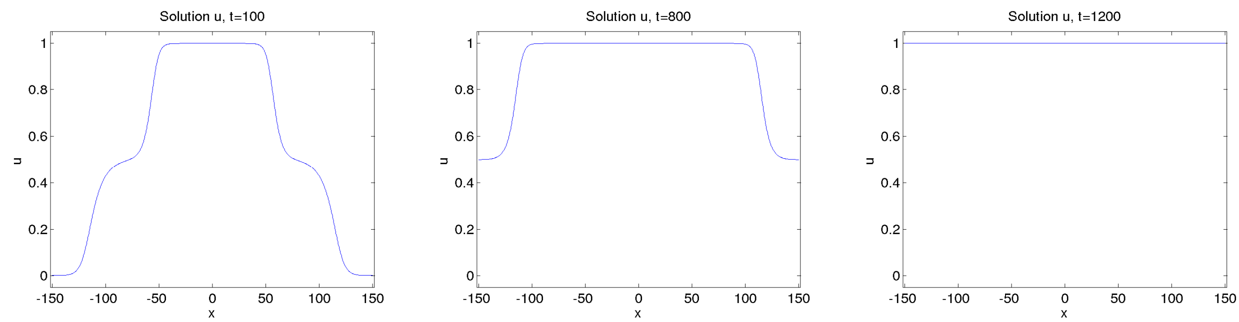
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



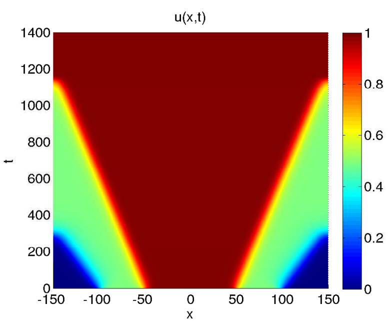
• Traveling 4-front (traveling multifront):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{lll} u_t &= u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 & , x \in B_R(0),\ t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 150$ and initial data $u_0(x) = \frac{1}{4}\tanh\left(\frac{x+100}{5}\right) + \frac{1}{4}\tanh\left(\frac{x+50}{5}\right) - \frac{1}{4}\tanh\left(\frac{x-50}{5}\right) - \frac{1}{4}\tanh\left(\frac{x-100}{5}\right)$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 0.5$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:



Spatial-temporal pattern:

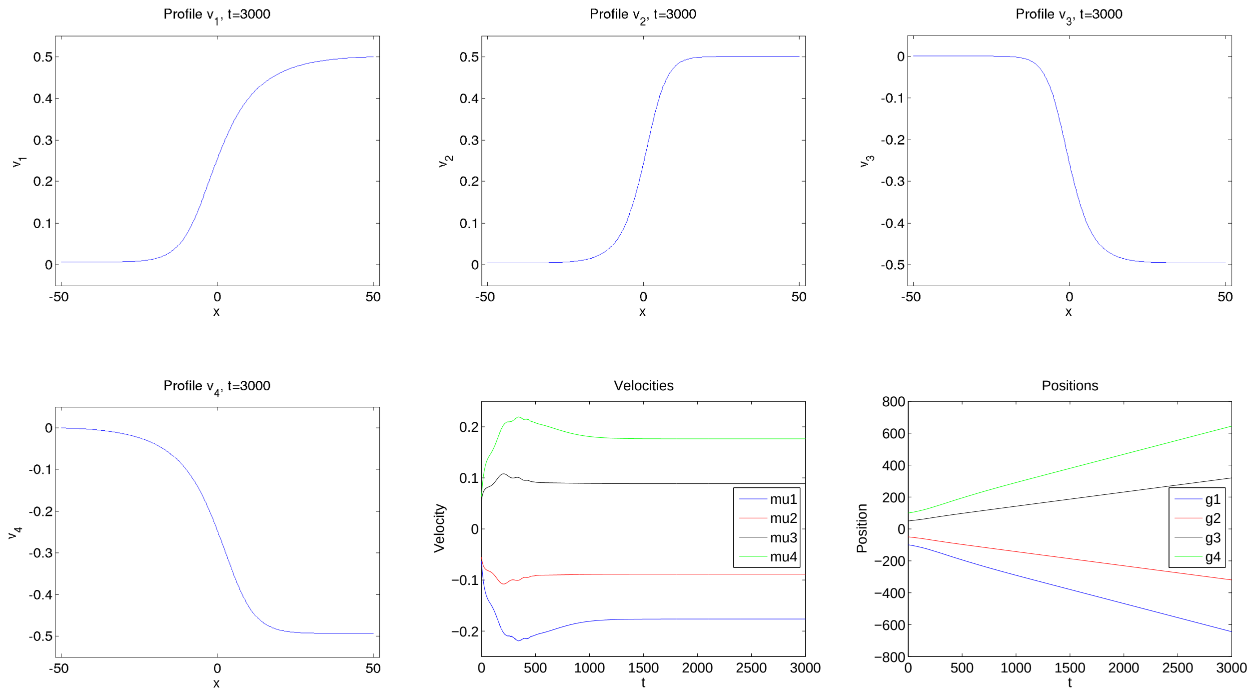


II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2, 3, 4$)

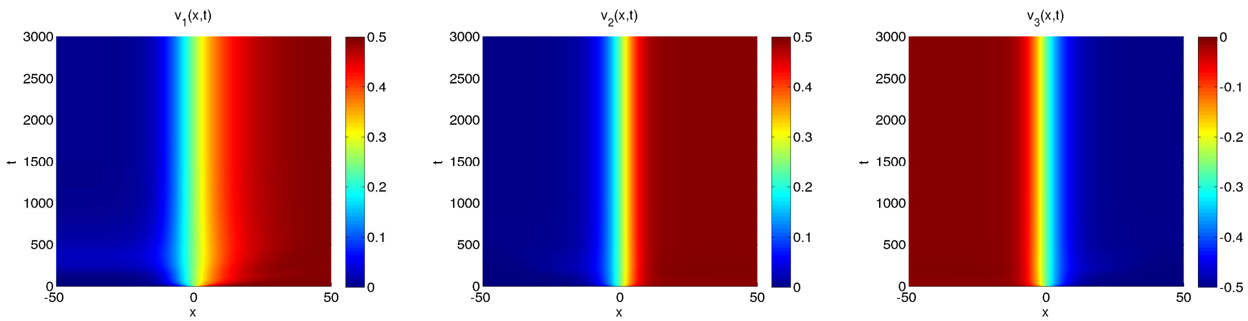
$$
\begin{aligned}
v_{j,t} &= v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{4}\varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{4} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
&\quad + \mu_j v_{j,x} & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} &= 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v_j(0) &= u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} &= \mu_j & , t \in [0,\infty[\\
g_j(0) &= g_{j0} & , t = 0 \\
0 &= \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{aligned}
$$

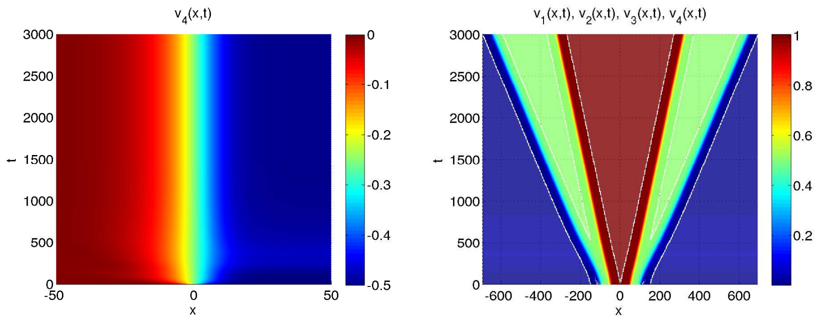
where $f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$ and $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{1}{16}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{7}{10}$, $R = 50$, initial data $u_{10}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, $u_{20}(x) = u_{10}(x)$, $u_{30}(x) = -\frac{1}{4}\tanh\left(\frac{x}{5}\right) - \frac{1}{4}$, $u_{40}(x) = u_{30}(x)$, initial positions $\gamma_{10} = -100$, $\gamma_{20} = -50$, $\gamma_{30} = 50$, $\gamma_{40} = 100$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$, $\hat{v}_3(x) = u_{30}(x)$, $\hat{v}_4(x) = u_{40}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right) + \exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.5$. For the temporal discretization we use BDF(2) with $\triangle t = 1.0$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.

Frozen solutions, velocities and positions:

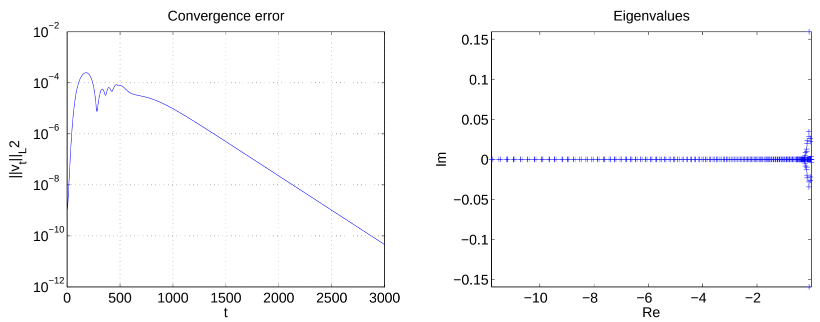


Spatial-temporal patterns:





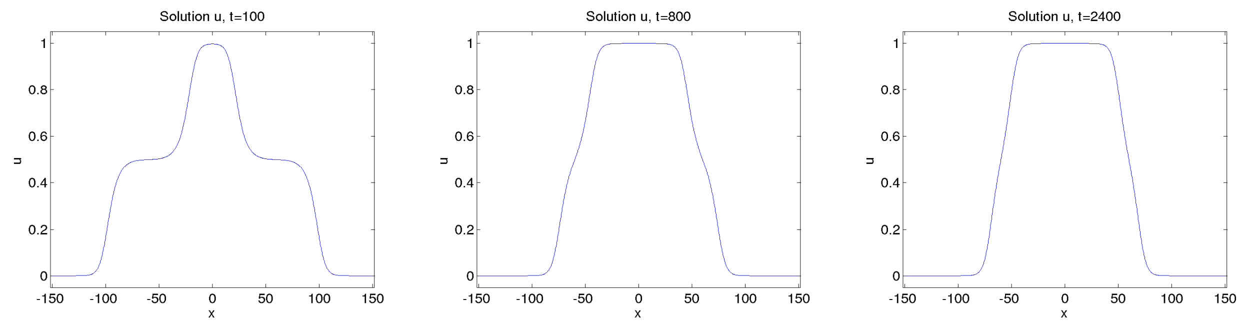
Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



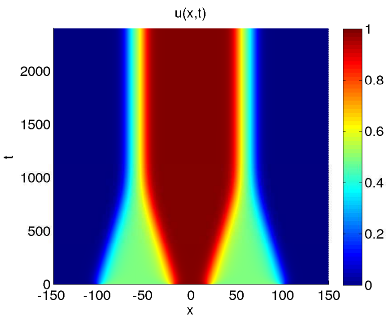
• Front interaction (4-front double collision):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{lcll} u_t & = & u_{xx} + u(1-u)(u-\alpha_1)(u-\alpha_2)(u-\alpha_3) & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) & = & u_0 & , x \in B_R(0),\ t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{3}{4}$, $R = 150$ and initial data $u_0(x) = \frac{1}{4}\tanh\left(\frac{x+100}{5}\right) + \frac{1}{4}\tanh\left(\frac{x+20}{5}\right) - \frac{1}{4}\tanh\left(\frac{x-20}{5}\right) - \frac{1}{4}\tanh\left(\frac{x-100}{5}\right)$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.3$. For the temporal discretization we use BDF(2) with $\triangle t = 1.0$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:



Spatial-temporal pattern:

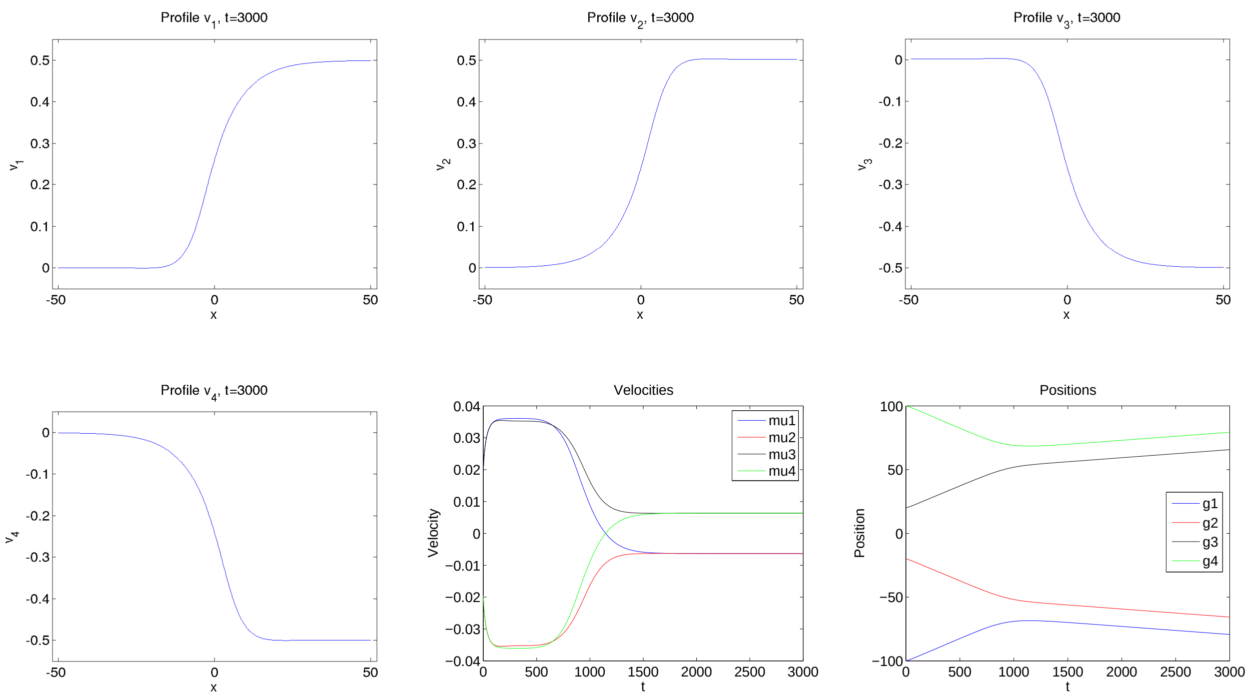


II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1,2,3,4$)

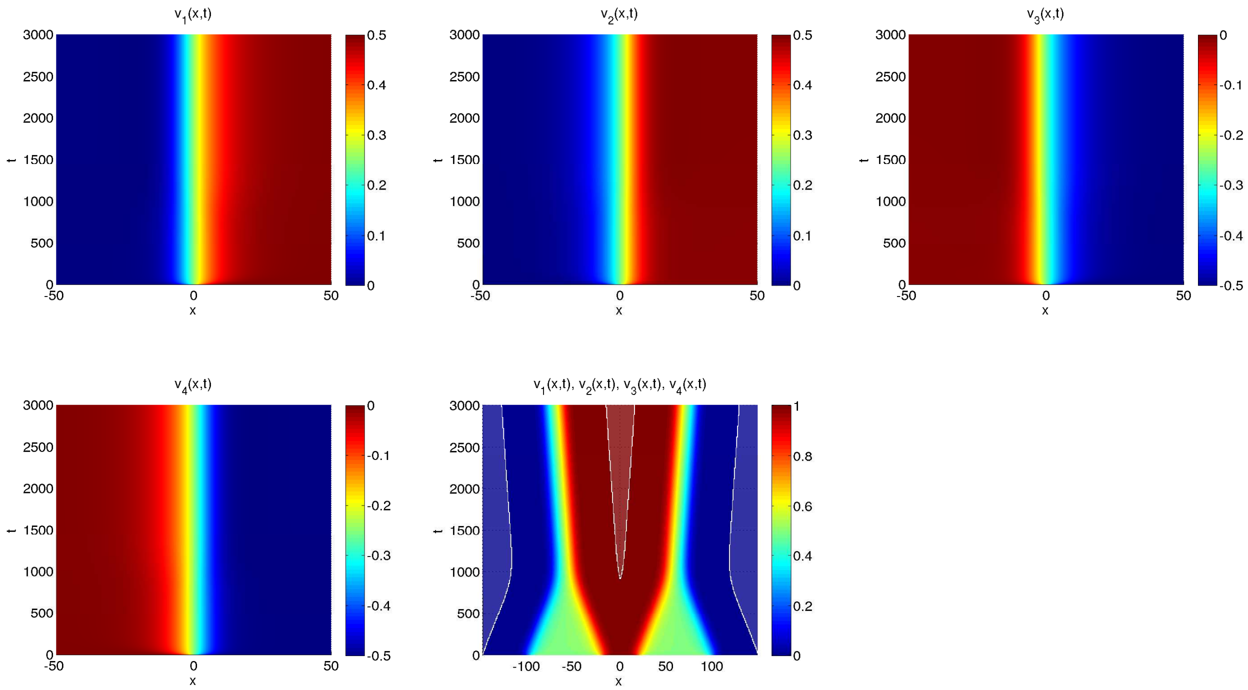
$$
\begin{array}{lcll}
v_{j,t} & = & v_{j,xx} + \frac{\varphi(\cdot)}{\sum_{k=1}^{4} \varphi(\cdot - g_k + g_j)} \cdot f\left(\sum_{k=1}^{4} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & \quad + \mu_j v_{j,x} & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0,\infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

where $f(v) = v(1-v)(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)$ and $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use again parameters $\alpha_1 = \frac{1}{4}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{3}{4}$, $R = 50$, initial data $u_{10}(x) = \frac{1}{4}\tanh\left(\frac{x}{5}\right) + \frac{1}{4}$, $u_{20}(x) = u_{10}(x)$, $u_{30}(x) = -\frac{1}{4}\tanh\left(\frac{x}{5}\right) - \frac{1}{4}$, $u_{40}(x) = u_{30}(x)$, initial positions $\gamma_{10} = -100$, $\gamma_{20} = -20$, $\gamma_{30} = 20$, $\gamma_{40} = 100$, reference functions $\hat{v}_1(x) = u_{10}(x)$, $\hat{v}_2(x) = u_{20}(x)$, $\hat{v}_3(x) = u_{30}(x)$, $\hat{v}_4(x) = u_{40}(x)$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right) + \exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.4$. For the temporal discretization we use BDF(2) with $\triangle t = 1.0$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.

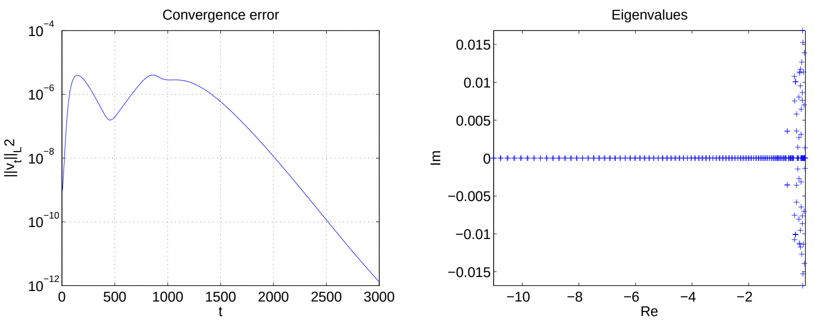
Frozen solutions, velocities and positions:



Spatial-temporal patterns:



Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



Explicit solutions: not available

Literature: [11], [73]

1.4 FitzHugh-Nagumo model

Name: FitzHugh-Nagumo model (FHN) (sometimes called Bonhoeffer-van der Pol model)

Equations:
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} \triangle u_1 + f(u_1) - u_2 + \alpha \\ D\triangle u_2 + \beta(\gamma u_1 - \delta u_2 + \epsilon) \end{pmatrix}$$

$u_1 = u_1(x,t) \in \mathbb{R}$, $u_2 = u_2(x,t) \in \mathbb{R}$, $u = (u_1, u_2)^T$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $D, \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a (cubic nonlinearity) polynomial of third degree (i. e. $f(w) = w - \zeta w^3$ or $f(w) = w(1-w)(w-\zeta)$ with $0 \neq \zeta \in \mathbb{R}$, cf. Nagumo equation).

Notations:

u_1	:	membran potential (measure the potential difference across the cell membrane)
u_2	:	recovery variable (measure transmembrane currents which affect the ability of the cell to recover before being able to fire again)
D	:	diffusion coefficient (usually $D = 0$)
α	:	magnitude of stimulus current
$\beta, \gamma, \delta, \epsilon, \zeta$	:	constant system parameters

Short description: The FitzHugh-Nagumo model ([33],[62]), named after Richard FitzHugh (1922-2007) and Jin-Ichi Nagumo (1926-1999), discribes nerve conduction ([83]), propagation of waves and nerve pulses in excitable media (e.g. heart tissue or nerve fiber) and spike generation in squid giant axons. This model exhibits traveling pulses, traveling fronts, traveling multipulses ([64]) in 1D, spiral waves, spiral breakups ([39]), spiral turbulences ([40]), labyrinthine patterns ([40]), spot splittings ([39]) and rotating vortices ([17],[15]) in 2D as well as other phenomena in 3D.

Phenomena:

- Traveling 1-front (traveling front)
- Traveling 1-pulse (traveling pulse)
- Traveling 2-pulse (traveling multipulse)

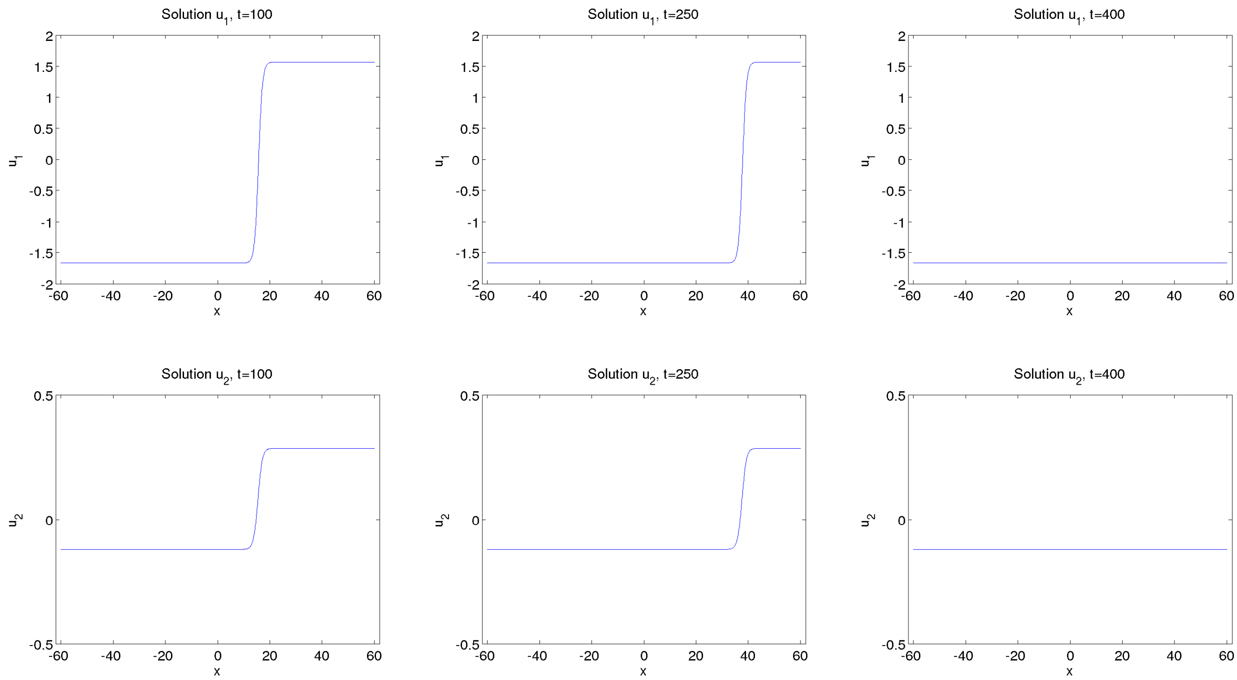
Set of parameter values:

d	D	$f(w)$	α	β	γ	δ	ϵ	ζ	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	$\frac{1}{10}$	$w - \zeta w^3$	0	$\frac{2}{25}$	1	8	$\frac{7}{10}$	$\frac{1}{3}$	60	0.1	0.5	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	1-front
1	$\frac{1}{10}$	$w - \zeta w^3$	0	$\frac{2}{25}$	1	$\frac{4}{5}$	$\frac{7}{10}$	$\frac{1}{3}$	60	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	1-pulse
1	$\frac{1}{10}$	$w - \zeta w^3$	0	$\frac{2}{25}$	1	$\frac{4}{5}$	$\frac{7}{10}$	$\frac{1}{3}$	100	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	2-pulse

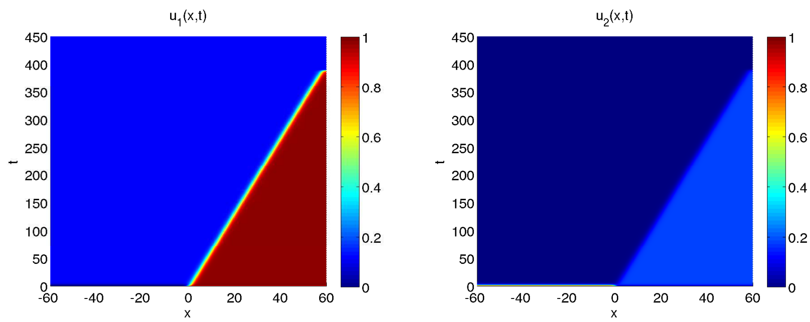
Numerical results:
• Traveling 1-front (traveling front):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t &= \begin{pmatrix} u_{1,xx} + f(u_1) - u_2 + \alpha \\ Du_{2,xx} + \beta(\gamma u_1 - \delta u_2 + \epsilon) \end{pmatrix} &&, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\ t = 0 \end{aligned}$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = 8$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 60$ and initial data $u_0(x) = (\tanh(x), 1 - \tanh(x))^T$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.5$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps ([59]).
Nonfrozen solution:



Spatial-temporal pattern:



II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{aligned}
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t &= \begin{pmatrix} v_{1,xx} + f(v_1) - v_2 + \alpha + \lambda_1 v_{1,x} \\ D v_{2,xx} + \beta(\gamma v_1 - \delta v_2 + \epsilon) + \lambda_1 v_{2,x} \end{pmatrix} && , x \in B_R(0),\ t \in [0, \infty[\\
\frac{\partial v}{\partial n} &= 0 && , x \in \partial B_R(0),\ t \in [0, \infty[\\
v(0) &= u_0 && , x \in B_R(0),\ t = 0 \\
\gamma_t &= \lambda_1 && , t \in [0, \infty[\\
\gamma(0) &= 0 && , t = 0 \\
0 &= \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{R})} && , t \in [0, \infty[
\end{aligned}
$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = 8$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 60$, initial data $u_0(x) =$

$(\tanh(x), 1-\tanh(x))^T$ and reference function $\hat{v}(x) =?$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps.
Frozen solution and velocity:

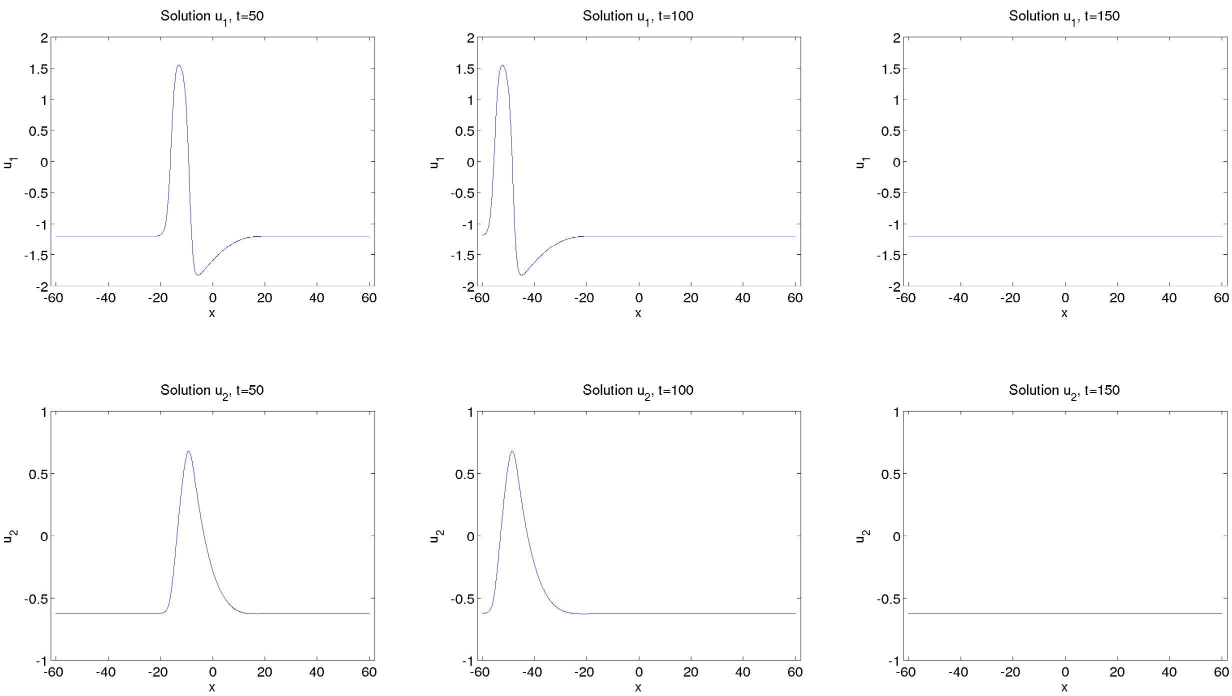
Spatial-temporal pattern, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 1-pulse (traveling pulse):

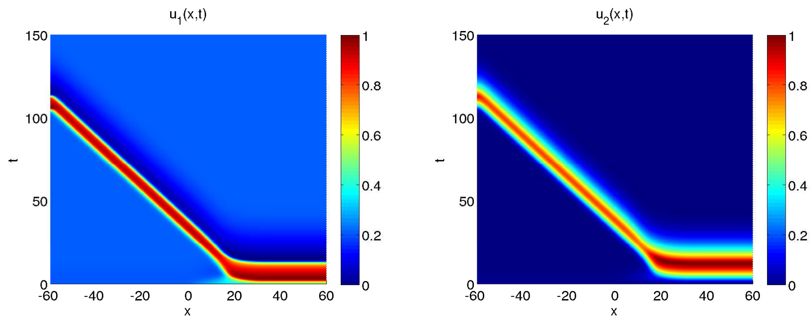
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned}
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t &= \begin{pmatrix} u_{1,xx} + f(u_1) - u_2 + \alpha \\ Du_{2,xx} + \beta(\gamma u_1 - \delta u_2 + \epsilon) \end{pmatrix} &&, x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\
u(0) &= u_0 &&, x \in B_R(0),\ t = 0
\end{aligned}$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = \frac{4}{5}$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 60$ and initial data $u_0(x) = (\tanh(x), -0.6)^T$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps ([59]).
Nonfrozen solution:



Spatial-temporal pattern:



II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$\begin{array}{rcll} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t & = & \begin{pmatrix} v_{1,xx} + f(v_1) - v_2 + \alpha + \lambda_1 v_{1,x} \\ D v_{2,xx} + \beta(\gamma v_1 - \delta v_2 + \epsilon) + \lambda_1 v_{2,x} \end{pmatrix} & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\ \gamma_t & = & \lambda_1 & , t \in [0,\infty[\\ \gamma(0) & = & 0 & , t = 0 \\ 0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[\end{array}$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = \frac{4}{5}$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 60$, initial data $u_0(x) = (\tanh(x), 1 - \tanh(x))^T$ and reference function $\hat{v}(x) =?$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps.
Frozen solution and velocity:

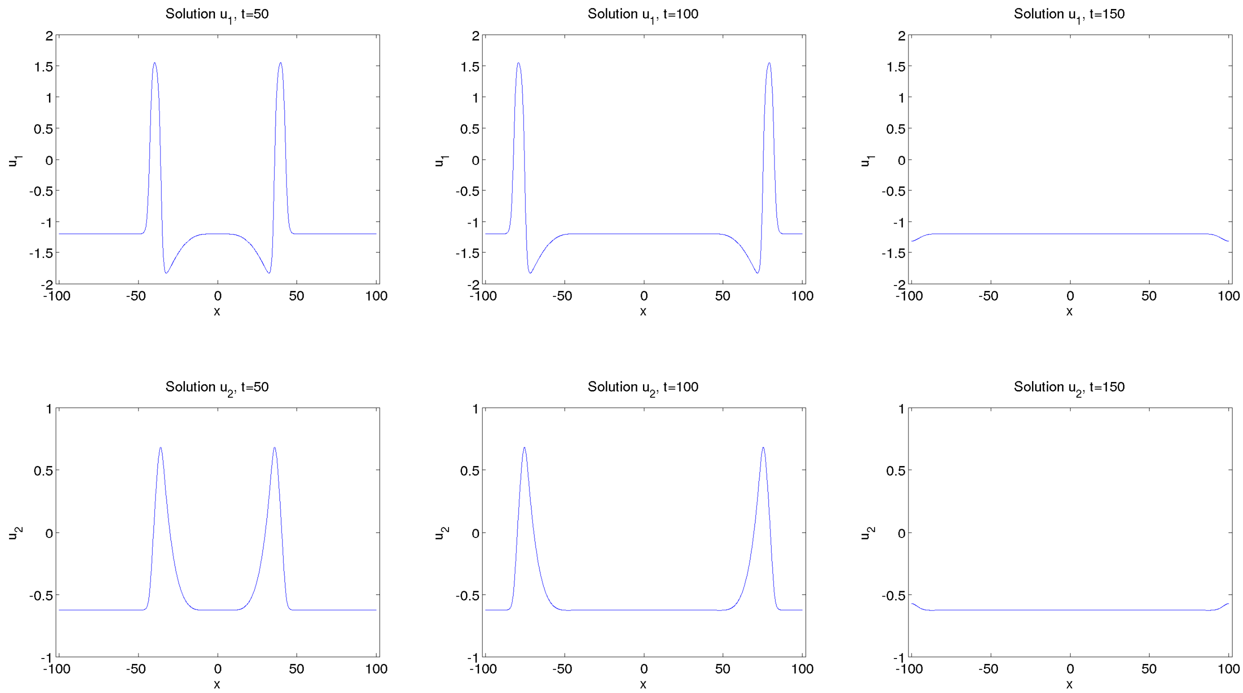
Spatial-temporal pattern, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

- Traveling 2-pulse (traveling multipulse):

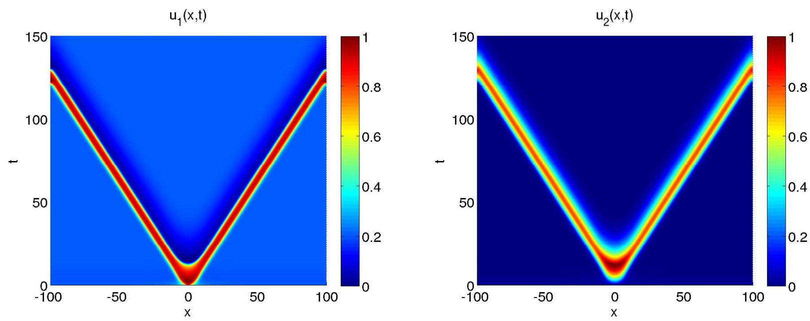
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{rcll} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t & = & \begin{pmatrix} u_{1,xx} + f(u_1) - u_2 + \alpha \\ D u_{2,xx} + \beta(\gamma u_1 - \delta u_2 + \epsilon) \end{pmatrix} & , x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) & = & u_0 & , x \in B_R(0),\ t = 0 \end{array}$$

where $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = \frac{4}{5}$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 100$ and initial data $u_0(x) = \left(-1.2 + \frac{2.5}{1+\left(\frac{x}{3}\right)^2}, -0.6\right)^T$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps ([12],[11]).
Nonfrozen solution:



Spatial-temporal pattern:



II. Frozen system (with freezing method ([11])): Consider the associated frozen system ($j = 1, 2$)

$$
\begin{array}{rcll}
\begin{pmatrix} v_{1,j} \\ v_{2,j} \end{pmatrix}_t & = & \begin{pmatrix} v_{1,j,xx} \\ Dv_{2,j,xx} \end{pmatrix} + \frac{\varphi(\cdot)}{\sum_{k=1}^{2}\varphi(\cdot - g_k + g_j)} \cdot F\left(\sum_{k=1}^{2} v_k(\cdot - g_k + g_j, \cdot)\right) & \\
 & & +\mu_j \begin{pmatrix} v_{1,j,x} \\ v_{2,j,x} \end{pmatrix} & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v_j}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v_j(0) & = & u_{j0} & , x \in B_R(0),\ t = 0 \\
g_{j,t} & = & \mu_j & , t \in [0,\infty[\\
g_j(0) & = & g_{j0} & , t = 0 \\
0 & = & \langle (\hat{v}_j)_x, v_j - \hat{v}_j \rangle_{L^2(B_R(0),\mathbb{R})} & , t \in [0,\infty[
\end{array}
$$

with

$$
F\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} f(w_1) - w_2 + \alpha \\ \beta\left(\gamma w_1 - \delta w_2 + \epsilon\right) \end{pmatrix}
$$

where $v_j = (v_{1,j}, v_{2,j})^T$, $f(w) = w - \zeta w^3$ and $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $D = \frac{1}{10}$, $\alpha = 0$, $\beta = \frac{2}{25}$, $\gamma = 1$, $\delta = \frac{4}{5}$, $\epsilon = \frac{7}{10}$, $\zeta = \frac{1}{3}$, $R = 50$, initial data $u_0(x) = \left(-1.2 + \frac{2.5}{1+\left(\frac{x}{3}\right)^2}, -0.6\right)^T$, initial positions $\gamma_{10} = -50$, $\gamma_{20} = 50$, reference functions $\hat{v}_1(x) =?$, $\hat{v}_2(x) =?$ and bump function $\varphi(x) = \frac{2}{\exp\left(\frac{1}{2}x\right)+\exp\left(-\frac{1}{2}x\right)}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps.
Frozen solutions:

Velocities and positions:

Spatial-temporal patterns:

Convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

Explicit solutions: not available

Literature: [33], [62], [32], [35], [34], [68], [12], [11], [40], [59], [64], [45], [17], [15]

1.5 Purwins model

Name: Purwins model

Equations:

$$\begin{aligned} u_t &= D_u \triangle u + f(u) - \kappa_1 v - \kappa_2 w + \kappa_3 \\ \tau v_t &= D_v \triangle v + u - v \\ \theta w_t &= D_w \triangle v + u - w \end{aligned}$$

$u = u(x,t) \in \mathbb{R}$, $v = v(x,t) \in \mathbb{R}$, $w = w(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $D_u, D_v, D_w, \alpha, \beta, \gamma, \tau, \theta \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a (cubic nonlinearity) polynomial of third degree (i.e. $f(u) = \lambda u - u^3$).

Notations:

u	:	activator
v, w	:	inhibitor
D_u, D_v, D_w	:	diffusion constants with slow diffusion D_v and fast diffusion D_w, i.e. $D_v << D_w$
$\kappa_1, \kappa_2, \kappa_3, \lambda, \tau, \theta$	:	some additional constants

Short description: not available

Set of parameter values: not available

Phenomena: not available

Explicit solutions: not available

Literature: not available

1.6 Barkley model

Name: Barkley model

Equations:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} \triangle u_1 + \frac{1}{\varepsilon} \cdot u_1 \cdot (1-u_1) \cdot \left(u_1 - \frac{u_2+b}{a}\right) \\ D\triangle u_2 + g(u_1) - u_2 \end{pmatrix}$$

$u_1 = u_1(x,t) \in \mathbb{R}$, $u_2 = u_2(x,t) \in \mathbb{R}$, $u = (u_1, u_2)^T$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $D, a, b, \varepsilon \in \mathbb{R}$, $g(w) = w$ ([7])

(or $g(w) = w^3$, $g(w) = \begin{cases} 0 & , w \in [0, \frac{1}{3}[\\ 1 - 6.75w(w-1)^2 & , w \in [\frac{1}{3}, 1] \\ 1 & , w \in]1, \infty[\end{cases}$ ([20])).

Notations:

D	:	diffusion constant for the slow species u_2 $(0 \leqslant D << 1)$
$g(u_1)$	:	reaction kinetics
ε	:	sets the timescale separation between the fast u_1- and the slow u_2-equation $(0 < \varepsilon << 1)$
a, b	:	other system parameters

Short description: The Barkley model ([7],[8]), named after Dwight Barkley, discribes excitable media, oscillatory media ([7]), catalytic surface reactions ([20],[21]), the interaction of a fast activator u and a slow inhibitor v (in this case $g(u)$ discribes a delayed production of the inhibitor) and is often used as a qualitative model in pattern forming systems (i.e. Belousov-Zhabotinsky reaction). This model exhibits spiral wave ([78], [12]), rotating spiral and scroll wave solutions. Larger a gives a longer excitation duration and increasing $\frac{b}{a}$ gives a larger excitability threshold (or equivalently decreasing $\frac{b}{a}$ produce a spiral with many windings). The standard reaction kinetics $g(u) = u$ can also be replaced by one of two above mentioned possibilities. In both of these nonstandard cases the model can exhibit spiral breakups followed by spiral turbulences ([21],[70]).

Phenomena:

- Rotating spiral wave (rigidly rotating spiral)
- Meandering spiral wave (meandering spiral)

Set of parameter values:

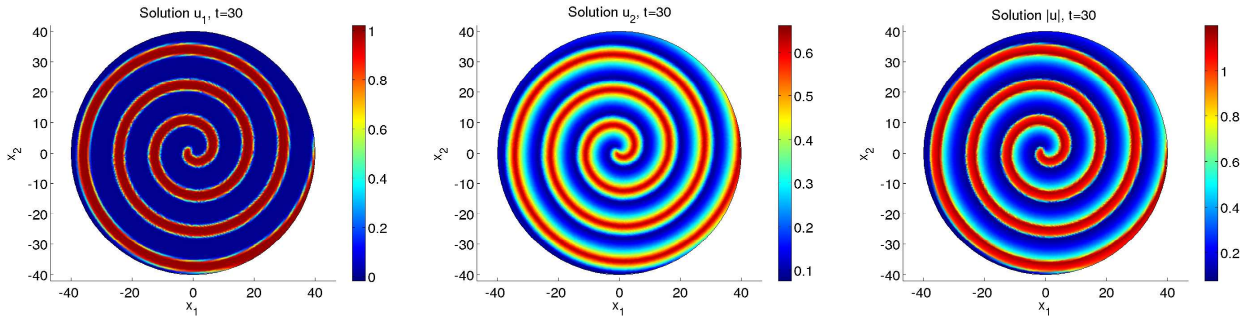
d	D	a	b	ε	$g(w)$	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
2	$\frac{1}{10}$	$\frac{3}{4}$	$\frac{1}{100}$	$\frac{1}{50}$	w	40	0.7	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	Rotating spiral wave

Numerical results:
• Rotating spiral wave (rigidly rotating spiral):
I. Nonfrozen solution: Consider the nonfrozen system

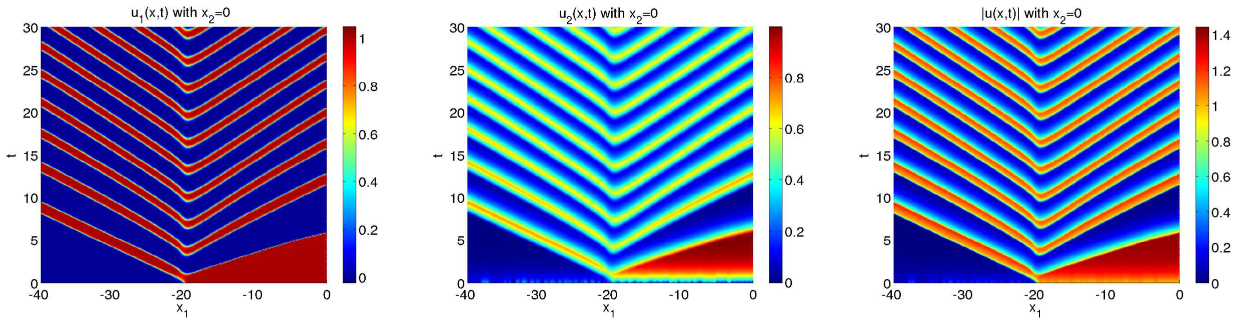
$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t &= \begin{pmatrix} \triangle u_1 + \frac{1}{\varepsilon} \cdot u_1 \cdot (1-u_1) \cdot \left(u_1 - \frac{u_2+b}{a}\right) \\ D\triangle u_2 + g(u_1) - u_2 \end{pmatrix} &&, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\ t = 0 \end{aligned}$$

where $g(w) = w$ and $d = 2$ (i. e. $B_R(0) = \{x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leqslant R\}$). For the numerical computations we use parameters $D = \frac{1}{10}$, $a = \frac{3}{4}$, $b = \frac{1}{100}$, $\varepsilon = \frac{1}{50}$, $R = 40$ and initial data $u_0(x) = (u_1(x,0), u_2(x,0))^T$

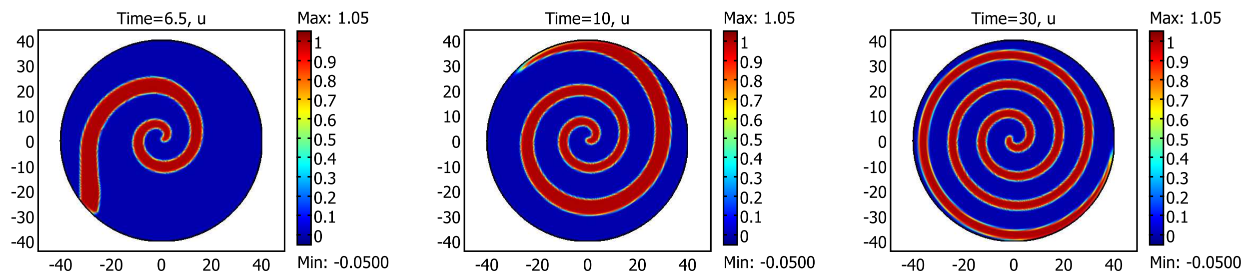
with $u_1(x,0) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$ and $u_2(x,0) = \begin{cases} 0 & y \leqslant 0 \\ \frac{a}{2} & y > 0 \end{cases}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.7$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps ([78],[12]).
Nonfrozen solution:



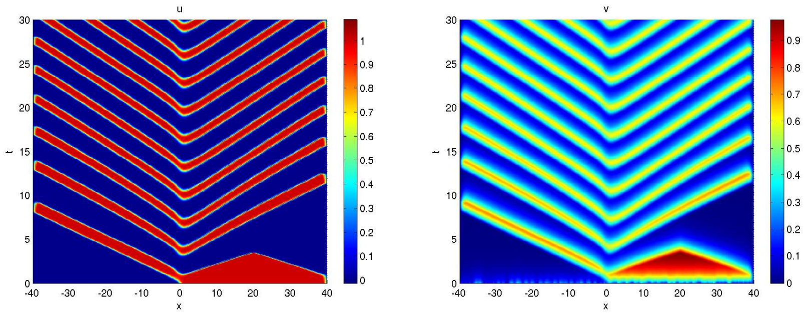
Spatial-temporal pattern: (with $x_2 = 0$, $x_1 \in [-40, 40]$)



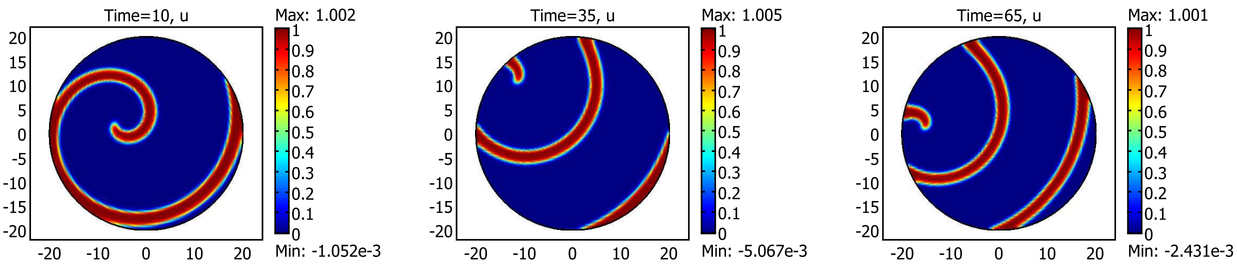
- Rotating spiral wave (rigidly rotating spiral): $d = 2$, $D_v = \frac{1}{10}$, $a = \frac{3}{4}$, $b = \frac{1}{100}$, $\varepsilon = \frac{1}{50}$, $g(u) = u$, $R = 40$, $\triangle x = 0.7$, $\triangle t = 0.1$, $\begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Dirichlet boundary), $u_0 = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$, $v_0 = \begin{cases} 0 & y \leqslant 0 \\ \frac{a}{2} & y > 0 \end{cases}$ ([78],[12]). I. Nonfrozen solution:



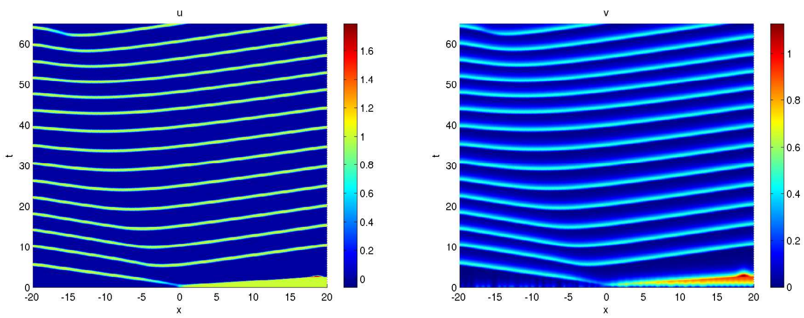
Spatial-temporal pattern: (with $y = 0$, $x \in [-40, 40]$)



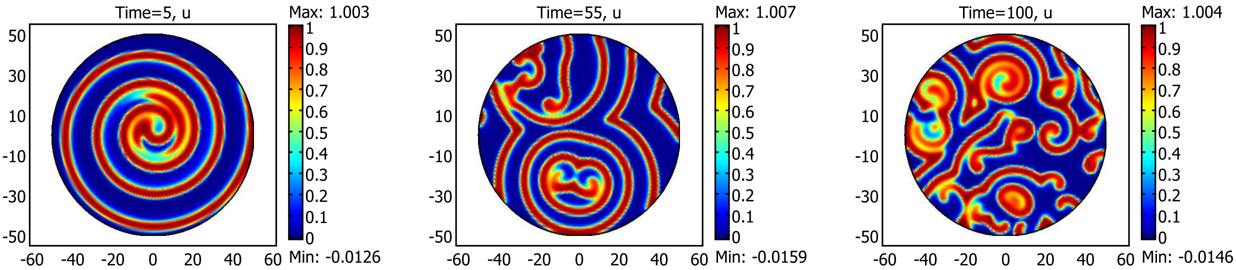
• Meandering rotating spiral wave (Drifting rotating spiral wave): $d = 2$, $D_v = 0$, $a = \frac{3}{5}$, $b = \frac{1}{20}$, $\varepsilon = \frac{1}{50}$, $g(u) = u$, $R = 20$, $\triangle x = 0.5$, $\triangle t = 0.1$, $\frac{\partial}{\partial n}\begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_0 = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$, $v_0 = \begin{cases} 0 & y \leqslant 0 \\ \frac{a}{2} & y > 0 \end{cases}$ ([8]). I. Nonfrozen solution:



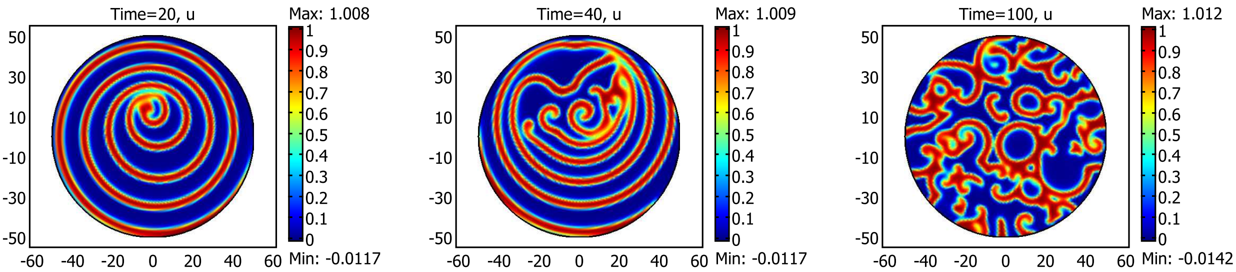
Spatial-temporal pattern: (with $y = 0$, $x \in [-20, 20]$)



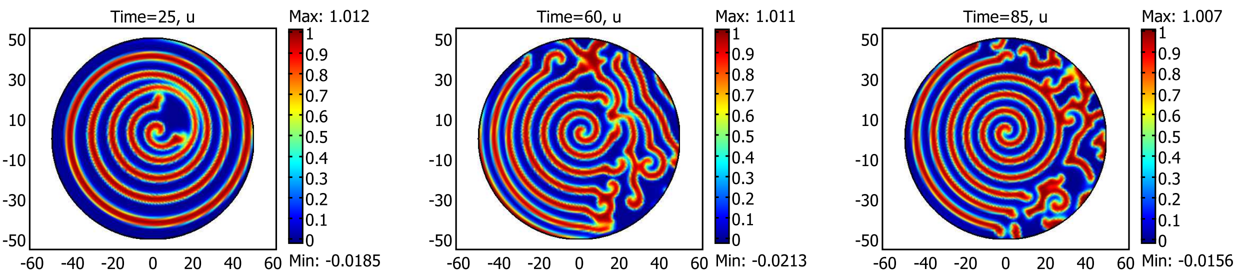
• Core breakup (Spiral breakup followed by spiral turbulence): $d = 2$, $D_v = 0$, $a = 0.75$, $b = 0.07$, $\varepsilon = 0.08$, $R = 50$, $\triangle x = 1.25$, $\triangle t = 0.1$, $g(u) = u^3$, $\frac{\partial}{\partial n}\begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary), u_0 and v_0 spiral patterns (i.e. solution at time $t = 15$ with parameters $D_v = 0.001$, $a = 0.75$, $b = 0.01$, $\varepsilon = 0.02$, $R = 50$, $\triangle x = 1.25$, $\triangle t = 0.1$, $g(u) = u$, $\frac{\partial}{\partial n}\begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_0 = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$, $v_0 = \begin{cases} 0 & y \leqslant 0 \\ \frac{a}{2} & y > 0 \end{cases}$) ([70]). I. Nonfrozen solution:



- Core breakup (Spiral breakup followed by spiral turbulence): $d=2$, $D_v=0$, $a=0.84$, $b=0.07$, $\varepsilon=0.08$, $R=50$, $\triangle x=1.25$, $\triangle t=0.1$, $g(u)=\begin{cases}0 & u\in[0,\frac{1}{3}[\\ 1-6.75u(u-1)^2 & u\in[\frac{1}{3},1]\\ 1 & u\in]1,\infty[\end{cases}$, $\frac{\partial}{\partial n}\begin{pmatrix}u\\ v\end{pmatrix}=0$ on $\partial B_R(0)$ (Neumann boundary), u_0 and v_0 spiral patterns (i.e. solution at time $t=15$ with parameters $D_v=0.001$, $a=0.75$, $b=0.01$, $\varepsilon=0.02$, $R=50$, $\triangle x=1.25$, $\triangle t=0.1$, $g(u)=u$, $\frac{\partial}{\partial n}\begin{pmatrix}u\\ v\end{pmatrix}=0$ on $\partial B_R(0)$ (Neumann boundary), $u_0=\begin{cases}0 & x\leqslant 0\\ 1 & x>0\end{cases}$, $v_0=\begin{cases}0 & y\leqslant 0\\ \frac{a}{2} & y>0\end{cases}$) ([70]). I. Nonfrozen solution:



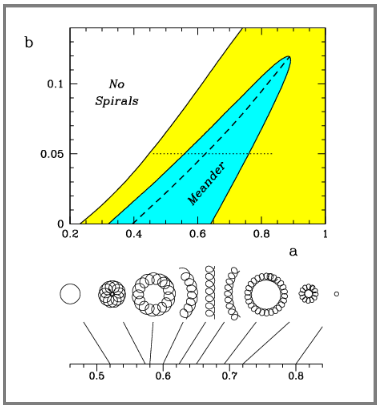
- Far field breakup (Spiral breakup): $d=2$, $D_v=0.001$, $a=0.75$, $b=0.01$, $\varepsilon=0.0752$, $R=50$, $\triangle x=1.25$, $\triangle t=0.1$, $g(u)=\begin{cases}0 & u\in[0,\frac{1}{3}[\\ 1-6.75u(u-1)^2 & u\in[\frac{1}{3},1]\\ 1 & u\in]1,\infty[\end{cases}$, $\frac{\partial}{\partial n}\begin{pmatrix}u\\ v\end{pmatrix}=0$ on $\partial B_R(0)$ (Neumann boundary), u_0 and v_0 spiral patterns (i.e. solution at time $t=15$ with parameters $D_v=0.001$, $a=0.75$, $b=0.01$, $\varepsilon=0.02$, $R=50$, $\triangle x=1.25$, $\triangle t=0.1$, $g(u)=u$, $\frac{\partial}{\partial n}\begin{pmatrix}u\\ v\end{pmatrix}=0$ on $\partial B_R(0)$ (Neumann boundary), $u_0=\begin{cases}0 & x\leqslant 0\\ 1 & x>0\end{cases}$, $v_0=\begin{cases}0 & y\leqslant 0\\ \frac{a}{2} & y>0\end{cases}$) ([21],[70]). I. Nonfrozen solution:



- Scroll wave: $d = 3$, $D_v = 0$, $a = \frac{4}{5}$, $b = \frac{1}{100}$, $\varepsilon = \frac{1}{50}$, $g(u) = u$, $R = 15$, $\triangle x = 0.5$, $\triangle t = 0.1$, $\frac{\partial}{\partial n}\begin{pmatrix} u \\ v \end{pmatrix} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_0 = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$, $v_0 = \begin{cases} 0 & y \leqslant 0 \\ \frac{a}{2} & y > 0 \end{cases}$.

Explicit solutions: not available

Additional informations: For $D_v = 0$ and $\varepsilon = 0.02$ fixed the following figure shows the dynamics of a single spiral wave as a function of parameters a and b as well as various types of meander. Top: The yellow region denotes periodically rotating spirals and the cyan region denotes meandering spirals. Bottom: Cut through the parameter space at $b = 0.05$ with the different states illustrated by tip paths ([7],[9]).



For their numerical approximation see [44] and [16].

Literature: [78], [20], [7], [19], [9], [70], [21], [8], [44], [16]

1.7 Schrödinger equation

Name: SCHRÖDINGER EQUATION (sometimes called NONLINEAR SCHRÖDINGER EQUATION (NLS, NLSE))

Equations:

$$u_t = \alpha \triangle u + \beta \left| u \right|^p u$$

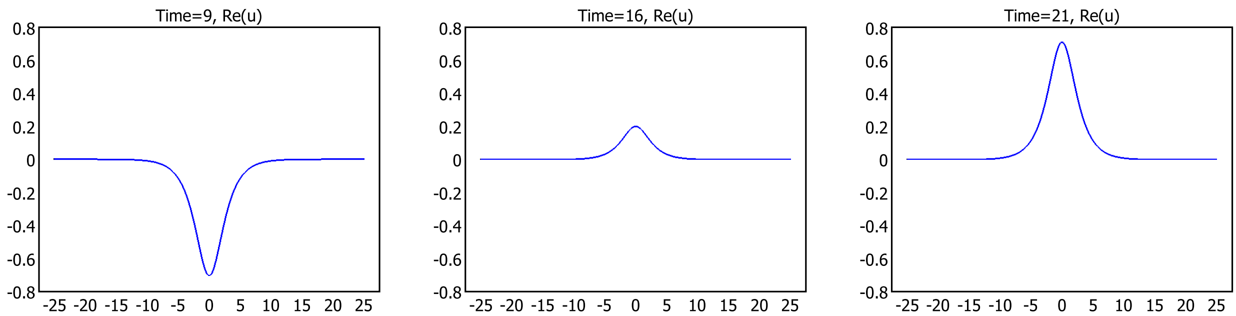
$u = u(x,t) \in \mathbb{C}$, $u_1 := \mathrm{Re}(u)$, $u_2 := \mathrm{Im}(u)$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha, \beta \in \mathbb{C}$, $p \in \mathbb{N}_0$.

Notations: not available

Short description: The Schrödinger equation ([71])

Set of parameter values: not available

Phenomena:
• Stationary oscillon (standing oscillating pulse, stable rotating pulse, solitary wave): $d = 1$, $\alpha = i$, $\beta = i$, $p = 2$, $R = 25$, $\triangle x = 0.1$, $\triangle t = 0.1$, $\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_{10} = C\sqrt{\frac{2}{\mathrm{Im}(\beta)}}\frac{\cos(D)}{\cosh(Cx+E)}$, $u_{20} = C\sqrt{\frac{2}{\mathrm{Im}(\beta)}}\frac{\sin(D)}{\cosh(Cx+E)}$ with $C = \frac{1}{2}$, $D = 1$ and $E = 0$. I. Nonfrozen solution:



Explicit solutions:
• 1d: For $\alpha = i$, $\beta = ki$ (with $k \in \mathbb{R}$) and $p = 2$ we have

$$u(x,t) = C\exp\left(i\left(Dx + \left(kC^2 - D^2\right)t + E\right)\right)$$

$$u(x,t) = \pm C\sqrt{\frac{2}{k}}\frac{\exp\left(i\left(C^2 t + D\right)\right)}{\cosh\left(Cx + E\right)}, \text{ if } k > 0$$

$$u(x,t) = \pm A\sqrt{\frac{2}{k}}\frac{\exp\left(iBx + i(A^2 + B^2)t + iC\right)}{\cosh\left(Ax - 2ABt + D\right)}, \text{ if } k > 0$$

$$u(x,t) = \frac{C}{\sqrt{t}}\exp\left(i\frac{(x+D)^2}{4t} + i\left(kC^2\ln(t) + E\right)\right)$$

where A, B, C, D, E are arbitrary real constants. For a collection of solutions see [29].

Additional informations: With respect to $p \in \mathbb{N}_0$ in the literature there are special nomenclature

$u_t = \alpha\triangle u + \beta u$	:	LINEAR SCHRÖDINGER EQUATION
$u_t = \alpha\triangle u + \beta\left\|u\right\| u$	:	QUADRATIC NONLINEAR SCHRÖDINGER EQUATION
$u_t = \alpha\triangle u + \beta\left\|u\right\|^2 u$	:	CUBIC NONLINEAR SCHRÖDINGER EQUATION
$u_t = \alpha\triangle u + \beta\left\|u\right\|^3 u$	:	QUARTIC NONLINEAR SCHRÖDINGER EQUATION
$u_t = \alpha\triangle u + \beta\left\|u\right\|^4 u$	:	QUINTIC NONLINEAR SCHRÖDINGER EQUATION
$u_t = \alpha\triangle u + \beta\left\|u\right\|^6 u$	:	SEPTIC NONLINEAR SCHRÖDINGER EQUATION

Literature: [29], [71]

1.8 Gross-Pitaevskii equation

Name: GROSS-PITAEVSKII EQUATION (GPE)

Equations:

$$u_t = \alpha \triangle u + \beta V(x) u + \gamma \left|u\right|^2 u$$

$u = u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha, \beta, \gamma \in \mathbb{C}$, $V : \mathbb{R}^d \to \mathbb{R}$.

Notations: not available

Short description: The Gross-Pitaevskii equation ([37],[67]), named after Eugene P. Gross and Lev Petrovich Pitaevskii, describes Bose-Einstein condensates (BEC) at zero or very low temperature and the ground states of a quantum system of identical bosons. This model exhibits standing solitary oscillons.

Phenomena:

- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse)

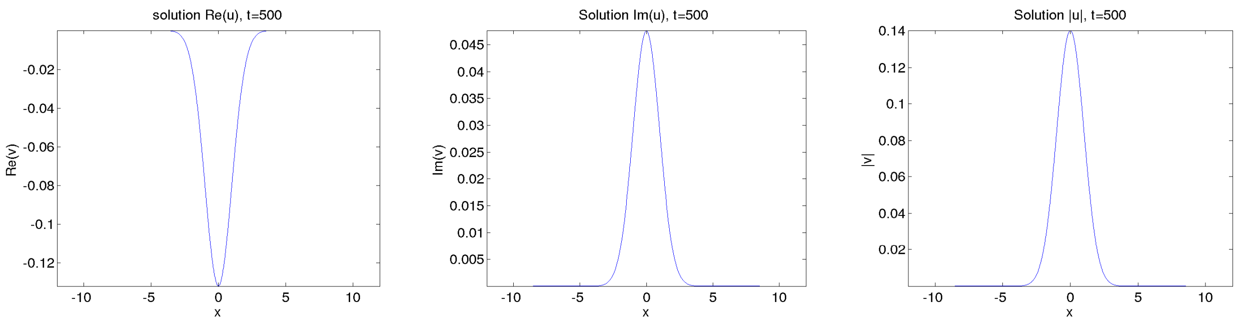
Set of parameter values:

d	α	β	γ	$V(x)$	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	$\frac{i}{2}$	$-i$	$-i$	$\frac{x^2}{2}$	10	0.1	0.5	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	rotating pulse

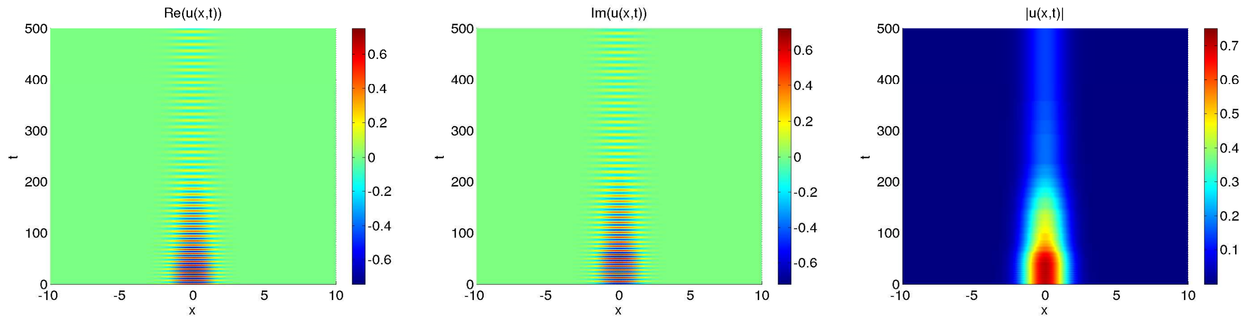
Numerical results:
• Standing solitary oscillon (rotating pulse, stable localized oscillating pulse):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= \alpha \triangle u + \beta V(x) u + \gamma \left|u\right|^2 u &&, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\ t = 0 \end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R,R]$). For the numerical computations we use parameters $\alpha = \frac{i}{2}$, $\beta = -i$, $\gamma = -i$, $V(x) = \frac{x^2}{2}$, $R = 10$ and initial data $\operatorname{Re} u_0 = \pi^{-\frac{1}{4}} \exp\left(-\frac{x^2}{2}\right)$, $\operatorname{Im} u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.5$, $rtol = 10^{-2}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:



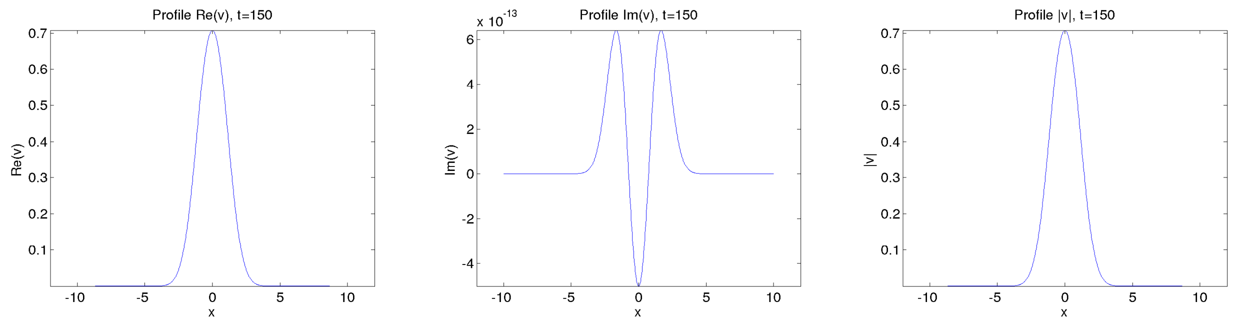
Spatial-temporal pattern:



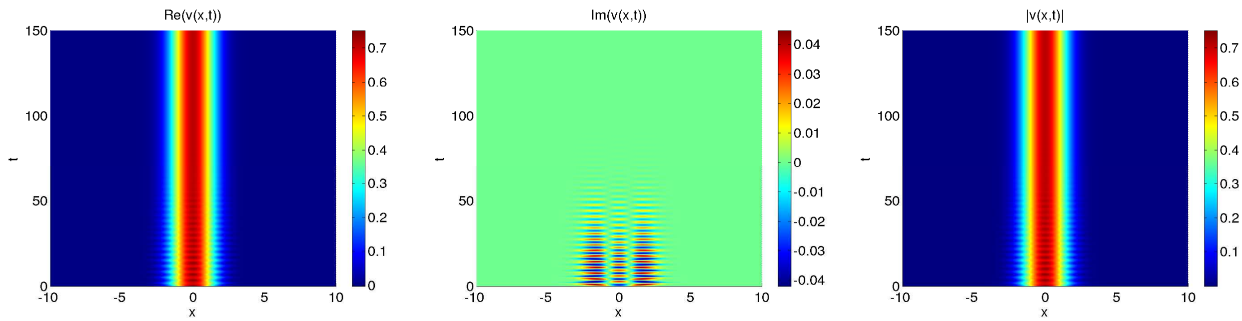
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lll}
v_t & = \alpha\triangle v + \beta V(x)v + \gamma\left|v\right|^2 v + i\lambda_1 v & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = u_0 & , x \in B_R(0),\ t = 0 \\
(\gamma_1)_t & = \lambda_1 & , t \in [0,\infty[\\
\gamma_1(0) & = 0 & , t = 0 \\
0 & = \langle i\hat{v}, v - \hat{v}\rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[
\end{array}
$$

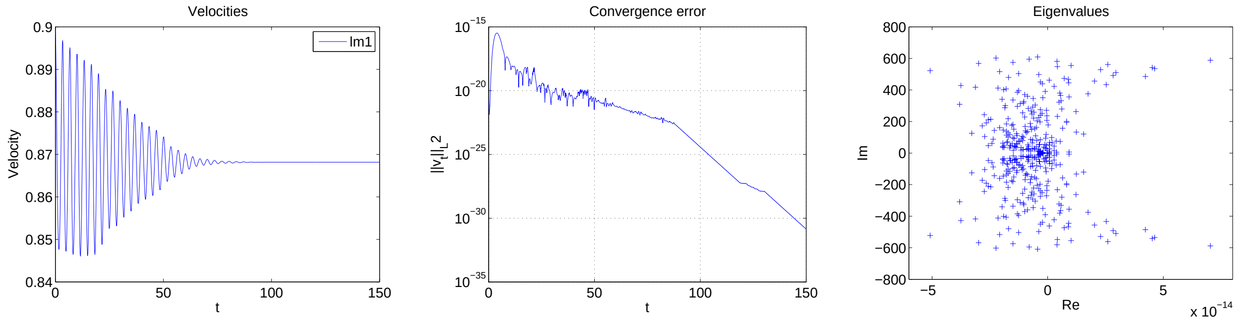
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\alpha = \frac{i}{2}$, $\beta = -i$, $\gamma = -i$, $V(x) = \frac{x^2}{2}$, $R = 10$ and initial data $\operatorname{Re} u_0 = \pi^{-\frac{1}{4}}\exp\left(-\frac{x^2}{2}\right)$, $\operatorname{Im} u_0 = 0$. As reference functions $\operatorname{Re}\hat{v}(x)$ and $\operatorname{Im}\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 1$, respectively, with the parameters mentioned above with $\triangle x = 0.05$ and $\triangle t = 0.1$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.05$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:



Spatial-temporal pattern:



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



Explicit solutions: not available

Additional informations: If $\beta = 0$ then we obtain the CUBIC NONLINEAR SCHRÖDINGER EQUATION.

Literature: [37], [67]

1.9 Complex Ginzburg-Landau equation (CGL)

Name: Complex Ginzburg-Landau equation (CGL, CGLE) (sometimes called Cubic Complex Ginzburg-Landau)

Equations:

$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2\right) + \gamma \overline{u}$$

$u = u(x,t) \in \mathbb{C}$, $u_1 := \mathrm{Re}(u)$, $u_2 := \mathrm{Im}(u)$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha, \beta, \gamma, \mu \in \mathbb{C}$ with $\gamma = 0$, $\overline{u}$ the complex conjugate of u.

Notations: ([41])

$u(x,t)$	:	complex-valued amplitude, slowly varying in space x and time t
$\mathrm{Im}(\alpha)$	:	dispersion
$\mathrm{Re}(\mu)$	:	distance from the Hopf bifurcation
$\mathrm{Im}(\mu)$	:	detuning
$\mathrm{Im}(\beta)$	:	nonlinear frequency correction
γ	:	weak periodic forcing term, forcing amplitude

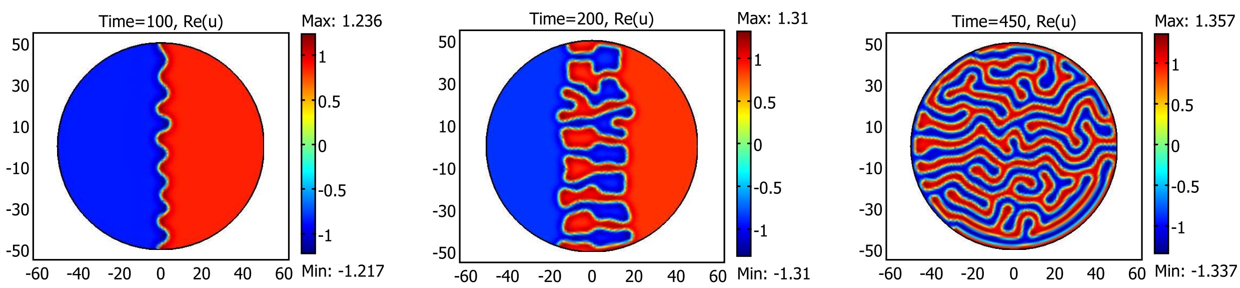
Short description: The complex Ginzburg-Landau equation, named after Vitaly Lazarevich Ginzburg (1916-2009) and Lev Landau (1908-1968), discribes nonlinear waves, second-order phase transitions, Rayleigh-Bénard convection, superconductivity, superfluidity and is used in the study of fiber optics ([3]). The equation describes the evolution of amplitudes of unstable modes for any process exhibiting a Hopf bifurcation, for which a continuous spectrum of unstable wavenumbers is taken into account. It can be viewed as a highly general normal form for a large class of bifurcations and nonlinear wave phenomena in spatially extended systems.

Set of parameter values:

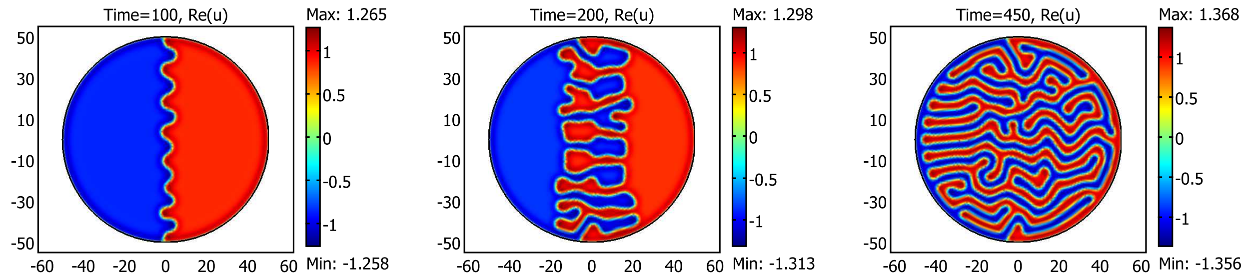
d	α	β	γ	μ	R	$\triangle x$	$\triangle t$	Boundary
2	$1+\frac{1}{2}i$	-1	2.02	$1+2i$	50	1.25	0.1	$\frac{\partial u}{\partial n} = 0$ or $u = 0$ on $\partial B_R(0)$
2	$1+\frac{1}{2}i$	-1	1.98	$1+2i$	50	1.25	0.1	$u = 0$ on $\partial B_R(0)$

Phenomena:

• Labyrinthine pattern: $d = 2$, $\alpha = 1 + \frac{1}{2}i$, $\beta = -1$, $\gamma = 2.02$, $\mu = 1 + 2i$, $R = 50$, $\triangle x = 1.25$, $\triangle t = 0.1$, $\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_{10} = \tanh(x)$, $u_{20} = 1 - u_{10}$ ([41],[38]). I. Nonfrozen solution:



• Labyrinthine pattern: $d = 2$, $\alpha = 1+\frac{1}{2}i$, $\beta = -1$, $\gamma = 2.02$, $\mu = 1+2i$, $R = 50$, $\triangle x = 1.25$, $\triangle t = 0.1$, $u = 0$ on $\partial B_R(0)$ (Dirichlet boundary), $u_{10} = \tanh(x)$, $u_{20} = 1 - u_{10}$ ([41],[38]). I. Nonfrozen solution:



- Scroll wave: $d = 3$, $\mu = 1$ (or $\mu = -1$), $\alpha = 1$, $\beta = 1 + i$, $\gamma = 0$ ([87],[50],[14])
- Spiral-vortex nucleation (formation of Bloch-front turbulence): $d =?$, $\alpha = 1 + \frac{3}{10}i$, $\beta = -1$, $\gamma = \frac{1}{5}$, $\mu = \frac{1}{2} + \frac{3}{20}i$, $R = 256$, $\triangle x =?$, $\triangle t =?$, $\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_{10} = front$, $u_{20} = front$ ([41]).

Explicit solutions: not available

Additional informations: (1): (Bloch-)spiral $\overset{\text{increasing } \gamma}{\longrightarrow}$ spiral-vortex nucleation $\overset{\text{increasing } \gamma}{\longrightarrow}$ labyrinthine, where ([41])

$$\frac{|\mathrm{Im}(\mu) + \mathrm{Re}(\mu) \cdot \mathrm{Im}(\beta)|}{\sqrt{1 + (\mathrm{Im}(\beta))^2}} =: \gamma_b < \gamma << \gamma_{NIB} := \sqrt{(\mathrm{Im}(\mu))^2 + \frac{1}{9}(\mathrm{Re}(\mu))^2} \text{ (for spiral)}$$

In case of $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta) = \mu = 0$ there exists soliton solutions ([80]). If $\mu = \gamma = 0$ then we obtain the CUBIC NONLINEAR SCHRÖDINGER EQUATION.
(2): If $\gamma \neq 0$ this equation is called FORCED COMPLEX GINZBURG-LANDAU EQUATION. In case of $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta) = 0$, $\gamma = 0$ and $\mu \in \mathbb{R}$ the equation is called REAL GINZBURG-LANDAU EQUATION.

Literature: [42], [83], [6], [41], [38], [87], [50], [14], [3]

1.10 Quintic Complex Ginzburg-Landau equation (QCGL)

Name: Quintic Complex Ginzburg-Landau equation (QCGL, QCGLE) (sometimes called Cubic-Quintic Complex Ginzburg-Landau equation (CQCGL))

Equations:

$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right)$$

$u = u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $\alpha, \beta, \gamma, \mu \in \mathbb{C}$.

Notations: ([80])

x	:	normalized transversal spatial coordinate (or retarded time, in temporal problems with $d = 1$)
t	:	normalized propagation distance or the normalized number of round tips
u	:	complex anvelope of the electric field
$\mathrm{Re}(\alpha)$	:	spatial or temporal spectral filtering, diffusion coefficient, $\mathrm{Re}(\alpha) > 0$
$\mathrm{Im}(\alpha)$	:	determines the lowest order diffraction, i.e. $\mathrm{Im}(\alpha) > 0$ corresponds to anomalous dispersion and $\mathrm{Im}(\alpha) < 0$ corresponds to normal dispersion
μ	:	linear gain (or loss) at the (spatial or temporal) control frequency, $\mu \in \mathbb{R}$
$\mathrm{Re}(\beta)$	:	nonlinear gain (absorption processes)
$\mathrm{Re}(\gamma)$	:	a higher-order correction term to the nonlinear amplification (absorption)
$\mathrm{Im}(\gamma)$	:	a higher-order correction term to the nonlinear refractive index

Short description: The quintic complex Ginzburg-Landau equation ([36]), named after Vitaly Lazarevich Ginzburg (1916-2009) and Lev Landau (1908-1968), describes different aspects of signal propagation in heart tissue, superconductivity, superfluidity, nonlinear optical systems ([60]), photonics, plasmas, physics of lasers, Bose-Einstein condensation, liquid crystals, fluid dynamics, chemical waves, quantum field theory, granular media and is used in the study of hydrodynamic instabilities ([58]). This model shows a variety of coherent structures like stable and unstable pulses, fronts, sources and sinks in 1D ([47],[79],[2],[80]), vortex solitons ([24]), spinning solitons ([25]), rotating spiral waves, propagating clusters ([69]) and exploding dissipative solitons ([75]) in 2D as well as scroll waves and spinning solitons ([26]) in 3D.

Phenomena:

- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse)
- Traveling oscillating front (stable rotating front)
- Pulsating soliton
- Creeping soliton
- Rotating 2-pulse (rotating multipulse)
- Rotating pulse combined with a traveling rotating front (traveling rotating multistructure)
- Traveling rotating 2-front (traveling rotating multifront)
- Spinning soliton (spinning solitary wave, localized vortex solution)
- Rotating spiral wave (rigidly rotating spiral)
- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse, merger of 3 colliding solitons into a single stable pulse)

- Spinning 2-soliton (spinning multisoliton, localized 2-vortex solution):

Set of parameter values:

d	μ	α	β	γ	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
1	$-\frac{1}{10}$	1	$3+i$	$-\frac{11}{4}+i$	20	0.1	0.1	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	rotating pulse
1	$-\frac{1}{10}$	1	$3+i$	$-\frac{11}{4}+i$	20	0.1	0.1	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	rotating front
1	$-\frac{1}{10}$	$\frac{8}{10}+\frac{1}{2}i$	$\frac{66}{100}+i$	$-\frac{1}{10}-\frac{1}{10}i$	20	0.1	0.1	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	pulsating soliton
1	$-\frac{1}{10}$	$\frac{101}{1000}+\frac{1}{2}i$	$\frac{13}{10}+i$	$-\frac{3}{10}-\frac{101}{1000}i$	20	0.1	0.3	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	creeping soliton
2	$-\frac{1}{2}$	$\frac{1}{2}+\frac{1}{2}i$	$\frac{5}{2}+i$	$-1-\frac{1}{10}i$	20	0.5	0.1	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	spinning soliton
2	$-\frac{1}{2}$	$\frac{1}{2}+\frac{1}{2}i$	$\frac{13}{5}+i$	$-1-\frac{1}{10}i$	20	0.5	0.1	$\frac{\partial u}{\partial n}=0$ on $\partial B_R(0)$	rotating spiral wave

Numerical results:

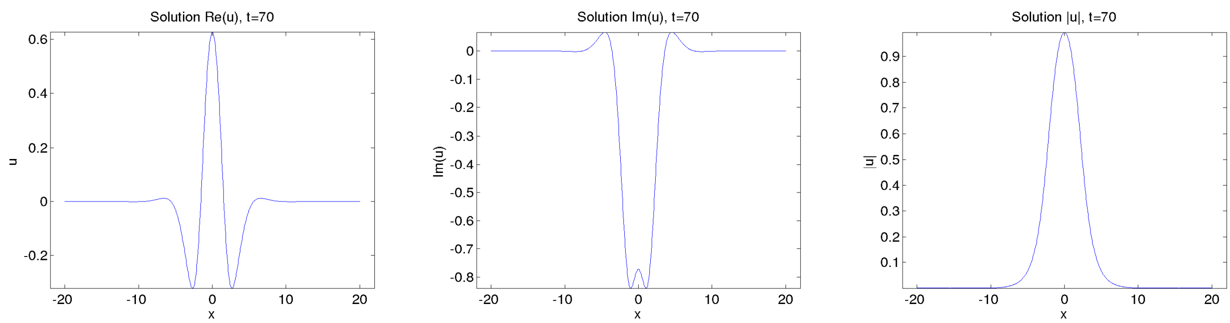
- Standing solitary oscillon (rotating pulse, stable localized oscillating pulse):

I. Nonfrozen solution: Consider the nonfrozen system

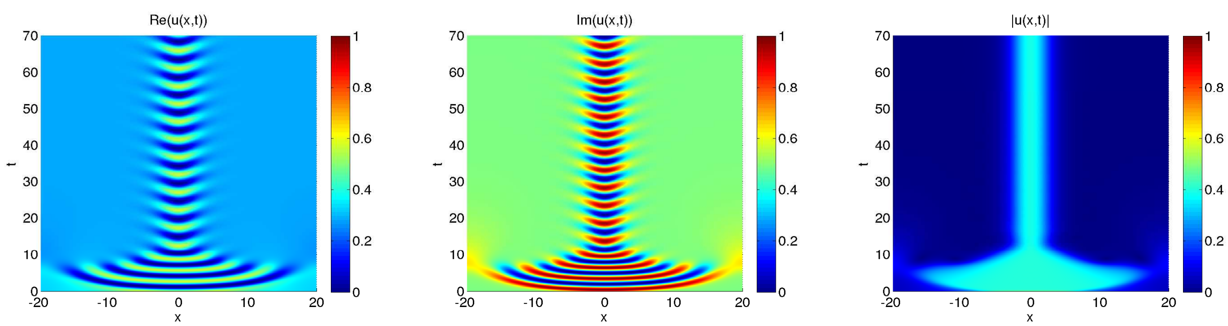
$$\begin{array}{rcll} u_t & = & \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right) & ,\ x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & ,\ x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) & = & u_0 & ,\ x \in B_R(0),\ t = 0 \end{array}$$

where $d=1$ (i. e. $B_R(0)=[-R,R]$). For the numerical computations we use parameters $\mu=-\frac{1}{10}$, $\alpha=1$, $\beta=3+i$, $\gamma=-\frac{11}{4}+i$, $R=20$ and initial data $\mathrm{Re}\,u_0=\frac{5}{2\left(1+\left(\frac{x}{5}\right)^2\right)}$, $\mathrm{Im}\,u_0=0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x=0.1$. For the temporal discretization we use BDF(2) with $\triangle t=0.1$, $rtol=10^{-3}$, $atol=10^{-4}$ and intermediate timesteps ([79],[47],[78],[76]).

Nonfrozen solution:



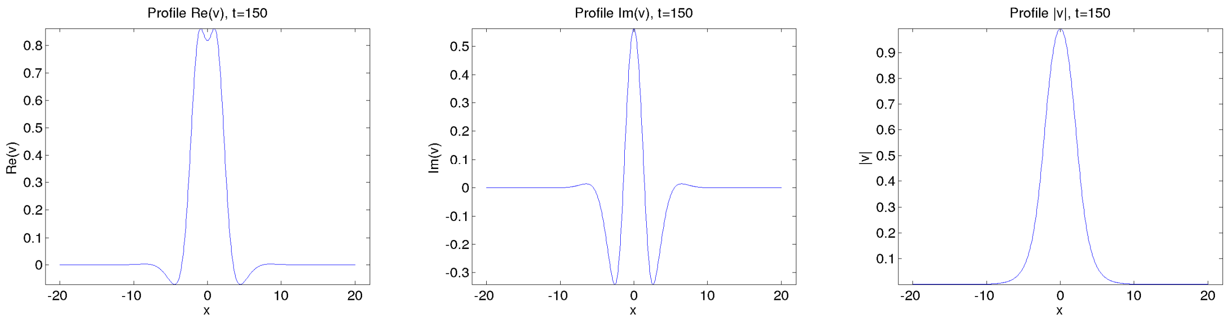
Spatial-temporal pattern:



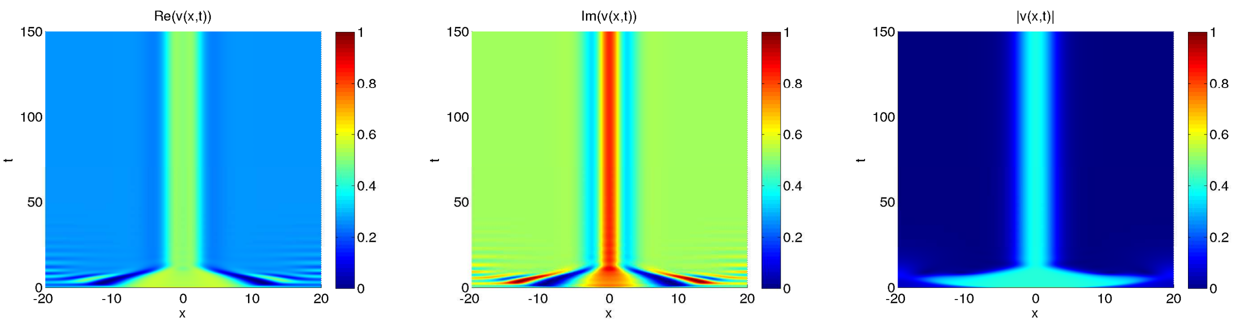
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lcll}
v_t & = & \alpha \triangle v + v\left(\mu + \beta\,|v|^2 + \gamma\,|v|^4\right) + \lambda_1 v_x + i\lambda_2 v & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\
(\gamma_1)_t & = & \lambda_1 & , t \in [0,\infty[\\
\gamma_1(0) & = & 0 & , t = 0 \\
(\gamma_2)_t & = & \lambda_2 & , t \in [0,\infty[\\
\gamma_2(0) & = & 0 & , t = 0 \\
0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[\\
0 & = & \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[
\end{array}
$$

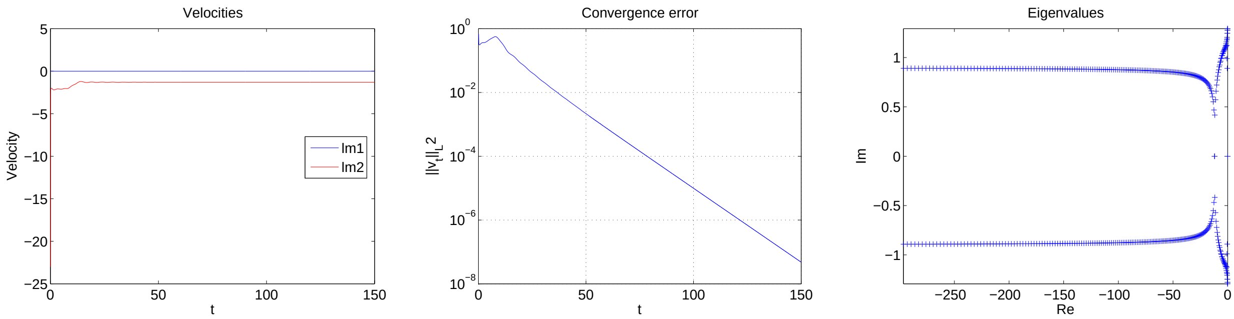
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 20$ and initial data $\mathrm{Re}\, u_0 = \frac{5}{2\left(1+\left(\frac{x}{5}\right)^2\right)}$, $\mathrm{Im}\, u_0 = 0$. As reference functions $\mathrm{Re}\,\hat{v}(x)$ and $\mathrm{Im}\,\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 70$, respectively, with the parameters mentioned above and $atol = 10^{-5}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-5}$ and intermediate timesteps ([79],[47],[78]).
Frozen solutions:



Spatial-temporal pattern:



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



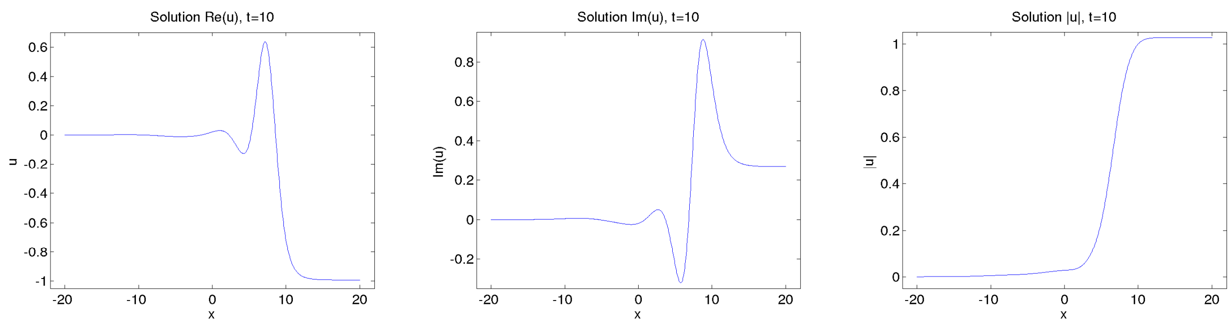
- Traveling oscillating front (stable rotating front):

I. Nonfrozen solution: Consider the nonfrozen system

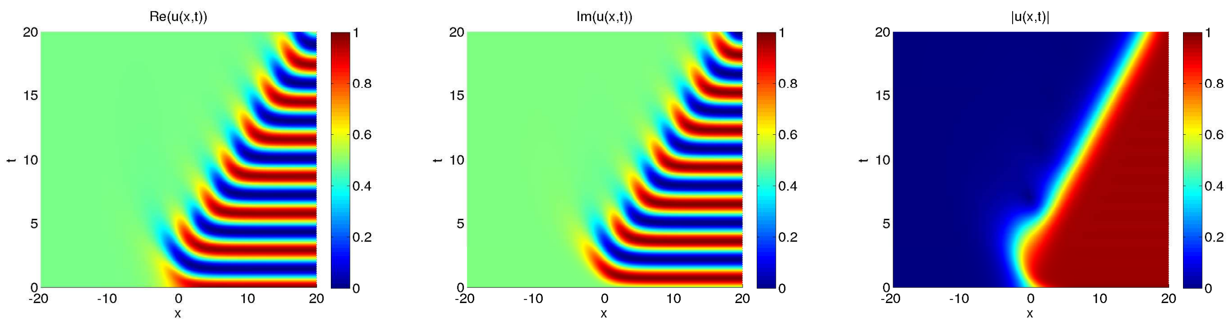
$$\begin{aligned} u_t &= \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right) &&, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\ t = 0 \end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3+i$, $\gamma = -\frac{11}{4}+i$, $R = 20$ and initial data $\mathrm{Re}\, u_0 = \frac{\frac{2}{11}\cdot\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x}{\sqrt{2}}\right)}$, $\mathrm{Im}\, u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps ([79],[47],[78],[76]).

Nonfrozen solution:



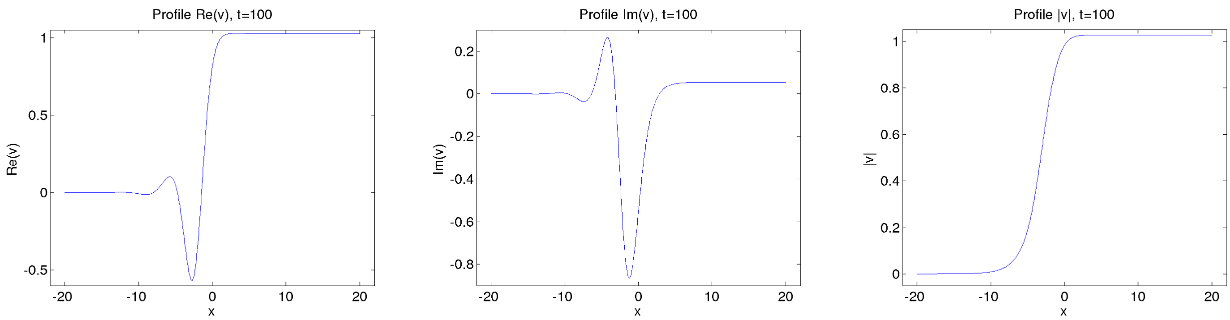
Spatial-temporal pattern:



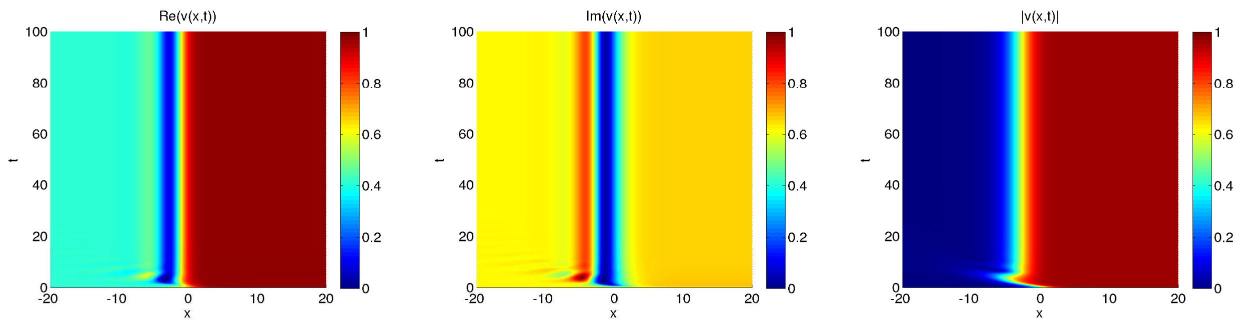
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lcll}
v_t & = & \alpha \triangle v + v\left(\mu + \beta\,|v|^2 + \gamma\,|v|^4\right) + \lambda_1 v_x + i\lambda_2 v & ,\, x \in B_R(0),\, t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & ,\, x \in \partial B_R(0),\, t \in [0,\infty[\\
v(0) & = & u_0 & ,\, x \in B_R(0),\, t = 0 \\
(\gamma_1)_t & = & \lambda_1 & ,\, t \in [0,\infty[\\
\gamma_1(0) & = & 0 & ,\, t = 0 \\
(\gamma_2)_t & = & \lambda_2 & ,\, t \in [0,\infty[\\
\gamma_2(0) & = & 0 & ,\, t = 0 \\
0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & ,\, t \in [0,\infty[\\
0 & = & \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & ,\, t \in [0,\infty[
\end{array}
$$

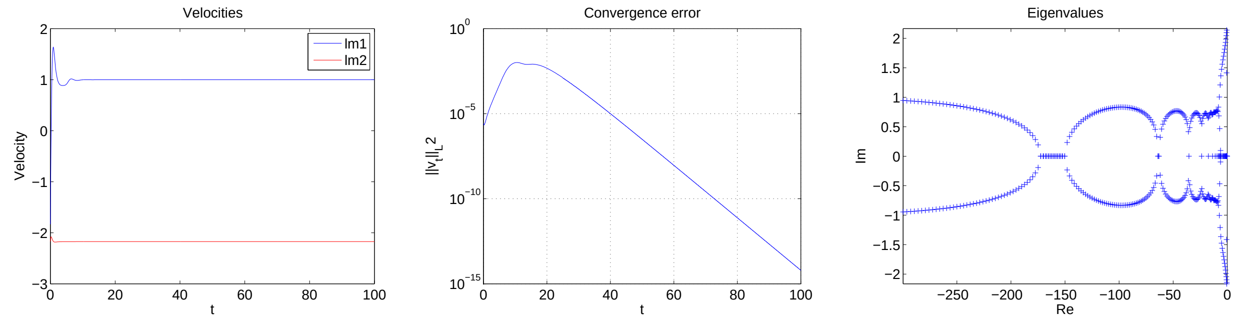
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 20$ and initial data $\mathrm{Re}\, u_0 = \frac{2.5}{1+\left(\frac{x}{5}\right)^2}$, $\mathrm{Im}\, u_0 = 0$. As reference functions $\mathrm{Re}\, \hat{v}(x)$ and $\mathrm{Im}\, \hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 4$, respectively, with the parameters mentioned above and $atol = 10^{-5}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-5}$ and intermediate timesteps ([79],[47],[78]).
Frozen solutions:



Spatial-temporal pattern:



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



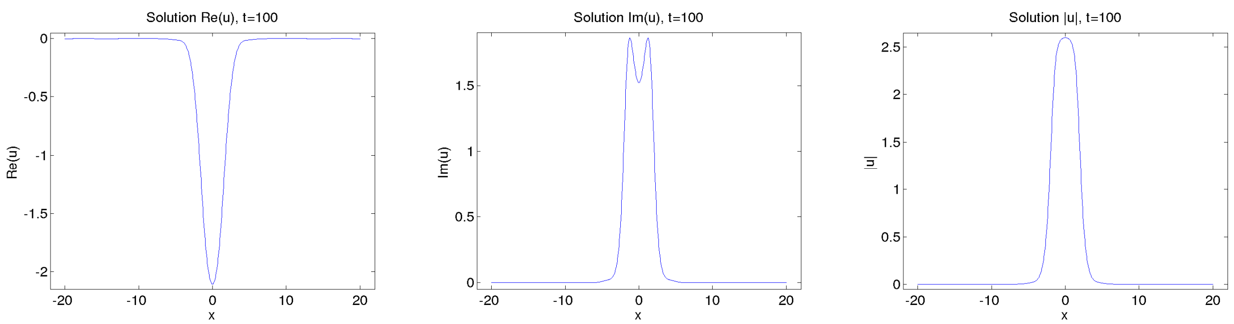
• Pulsating soliton:

I. Nonfrozen solution: Consider the nonfrozen system

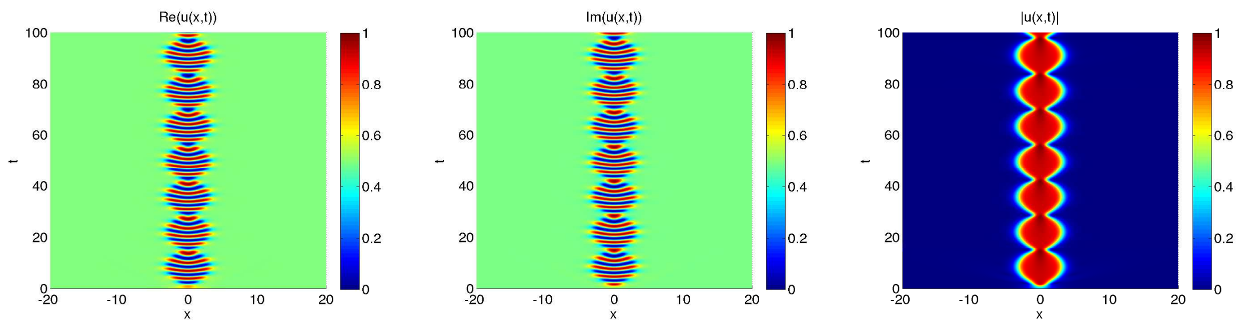
$$\begin{array}{rcll} u_t & = & \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right) & ,\ x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & ,\ x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) & = & u_0 & ,\ x \in B_R(0),\ t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = \frac{8}{10} + \frac{1}{2}i$, $\beta = \frac{66}{100} + i$, $\gamma = -\frac{1}{10} - \frac{1}{10}i$, $R = 20$ and initial data $\operatorname{Re} u_0 = \operatorname{sech}(x)$, $\operatorname{Im} u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps ([5]).

Nonfrozen solution:



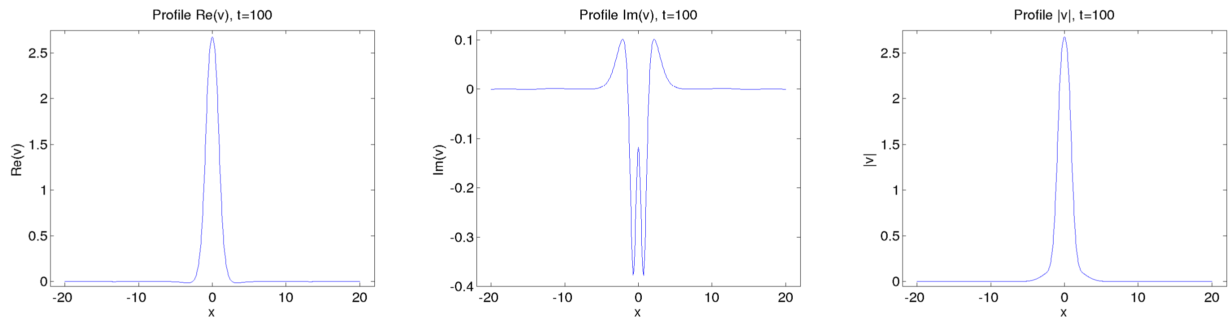
Spatial-temporal pattern:



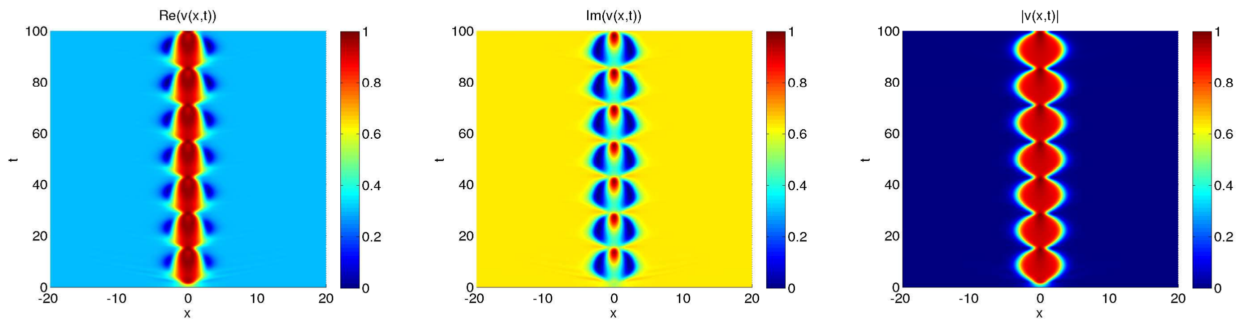
II. Frozen system (freezing method still not developed): Consider the associated frozen system

$$
\begin{array}{lll}
v_t & = & \alpha \triangle v + v\left(\mu + \beta\,|v|^2 + \gamma\,|v|^4\right) + \lambda_1 v_x + i\lambda_2 v \quad , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 \quad , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = & u_0 \quad , x \in B_R(0),\ t = 0 \\
(\gamma_1)_t & = & \lambda_1 \quad , t \in [0,\infty[\\
\gamma_1(0) & = & 0 \quad , t = 0 \\
(\gamma_2)_t & = & \lambda_2 \quad , t \in [0,\infty[\\
\gamma_2(0) & = & 0 \quad , t = 0 \\
0 & = & \langle \hat{v}_x, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} \quad , t \in [0,\infty[\\
0 & = & \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} \quad , t \in [0,\infty[
\end{array}
$$

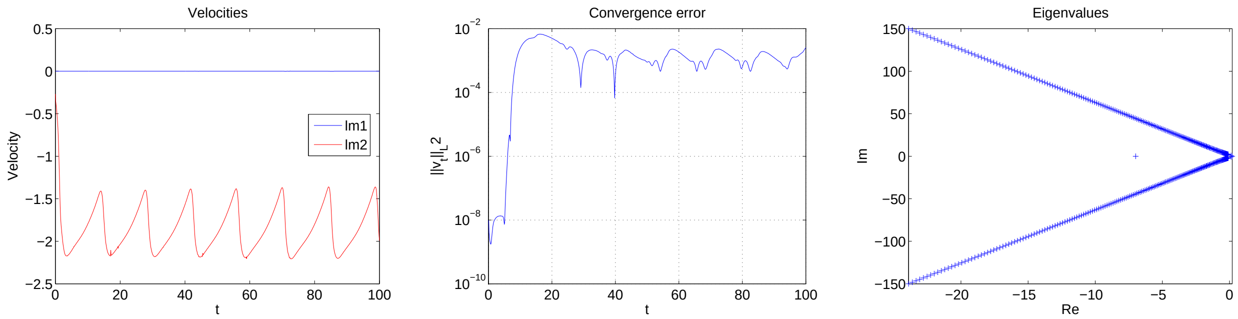
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = \frac{8}{100} + \frac{1}{2}i$, $\beta = \frac{66}{100} + i$, $\gamma = -\frac{1}{10} - \frac{1}{10}i$, $R = 20$ and initial data $\mathrm{Re}\, u_0 = \mathrm{sech}(x)$, $\mathrm{Im}\, u_0 = 0$. As reference functions $\mathrm{Re}\,\hat{v}(x)$ and $\mathrm{Im}\,\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 10$, respectively, with the parameters mentioned above. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-5}$ and intermediate timesteps.
Frozen solutions:



Spatial-temporal pattern:



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:

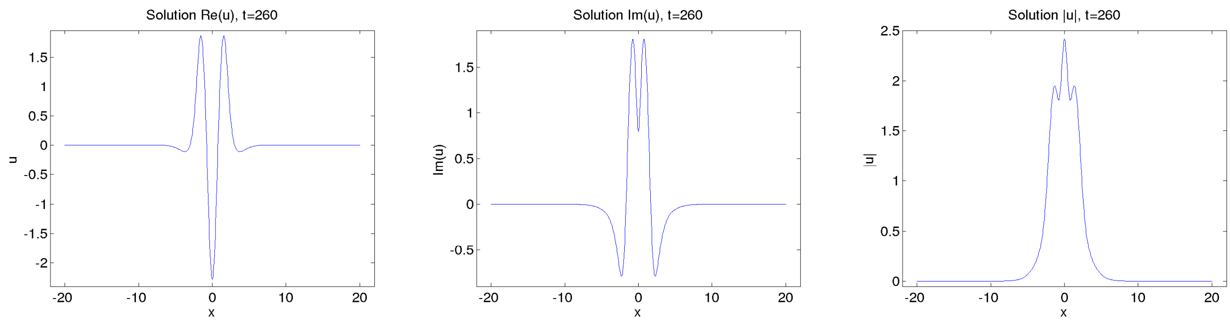


- Creeping soliton:

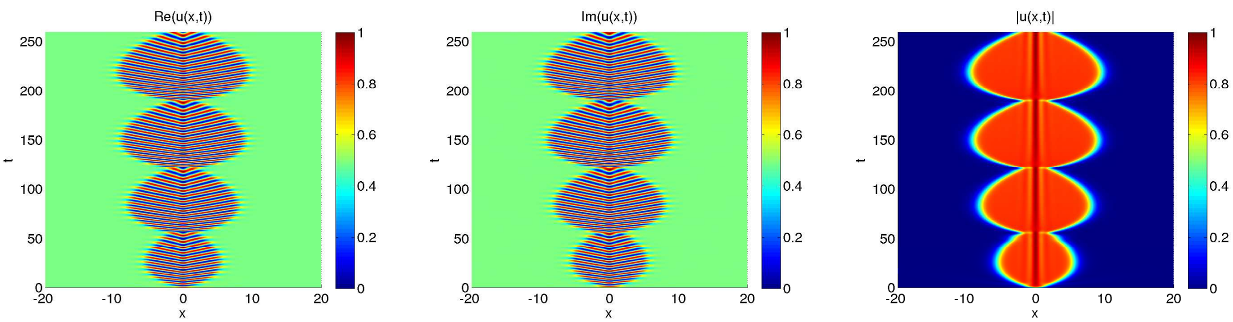
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= \alpha\triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) &&, x \in B_R(0),\ t\in[0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x\in\partial B_R(0),\ t\in[0,\infty[\\ u(0) &= u_0 &&, x\in B_R(0),\ t=0 \end{aligned}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = \frac{101}{1000} + \frac{1}{2}i$, $\beta = \frac{13}{10} + i$, $\gamma = -\frac{3}{10} - \frac{101}{1000}i$, $R = 20$ and initial data $\operatorname{Re} u_0 = \operatorname{sech}(x)$, $\operatorname{Im} u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.3$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps ([5]).
Nonfrozen solution:



Spatial-temporal pattern:



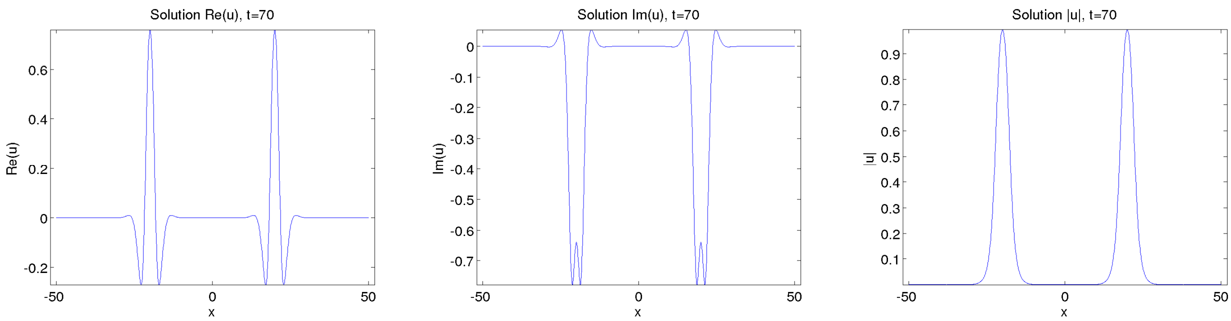
II. Frozen system (freezing method still not developed).

- Rotating 2-pulse (rotating multipulse):

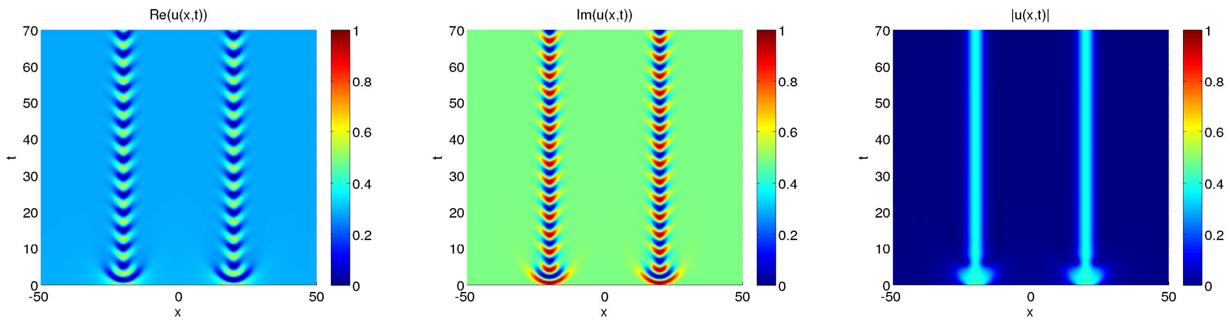
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{lcll} u_t & = & \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right) & ,\ x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} & = & 0 & ,\ x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) & = & u_0 & ,\ x \in B_R(0),\ t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3+i$, $\gamma = -\frac{11}{4}+i$, $R = 50$ and initial data $\operatorname{Re} u_0 = \frac{5}{2\left(1+\left(\frac{x+20}{2}\right)^2\right)} + \frac{5}{2\left(1+\left(\frac{x-20}{2}\right)^2\right)}$, $\operatorname{Im} u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:

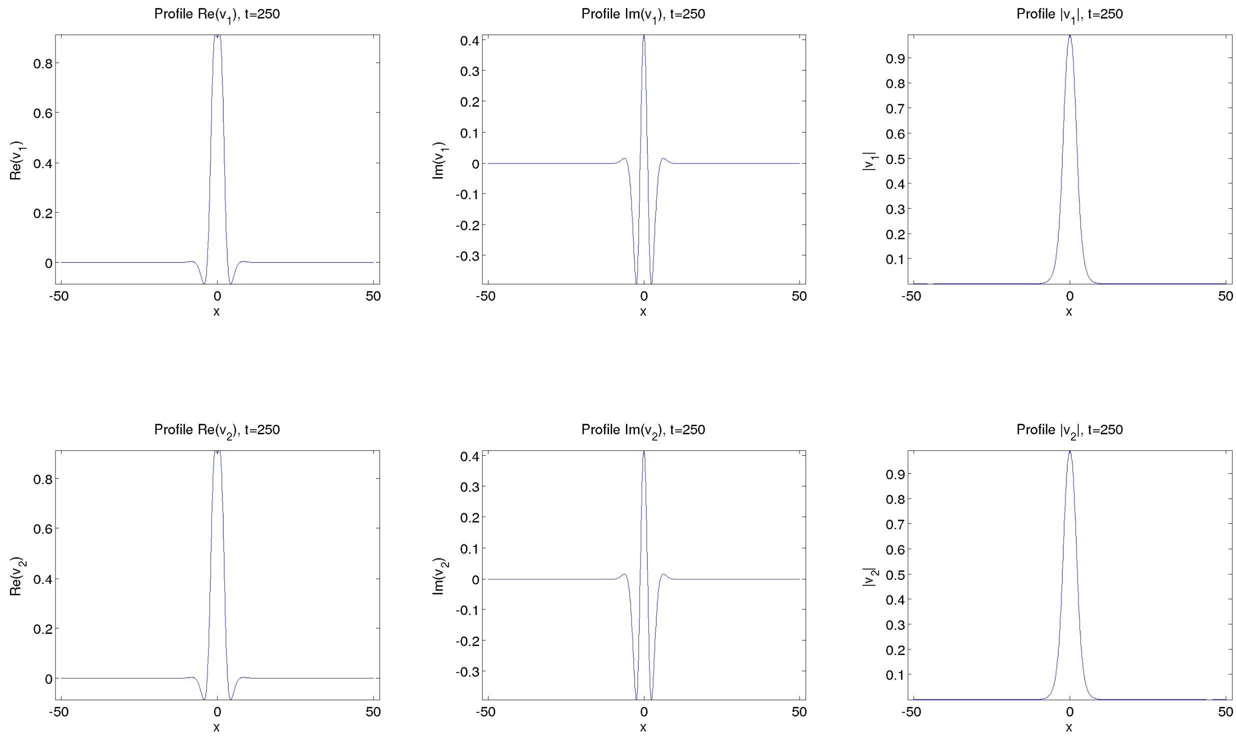


Spatial-temporal pattern:

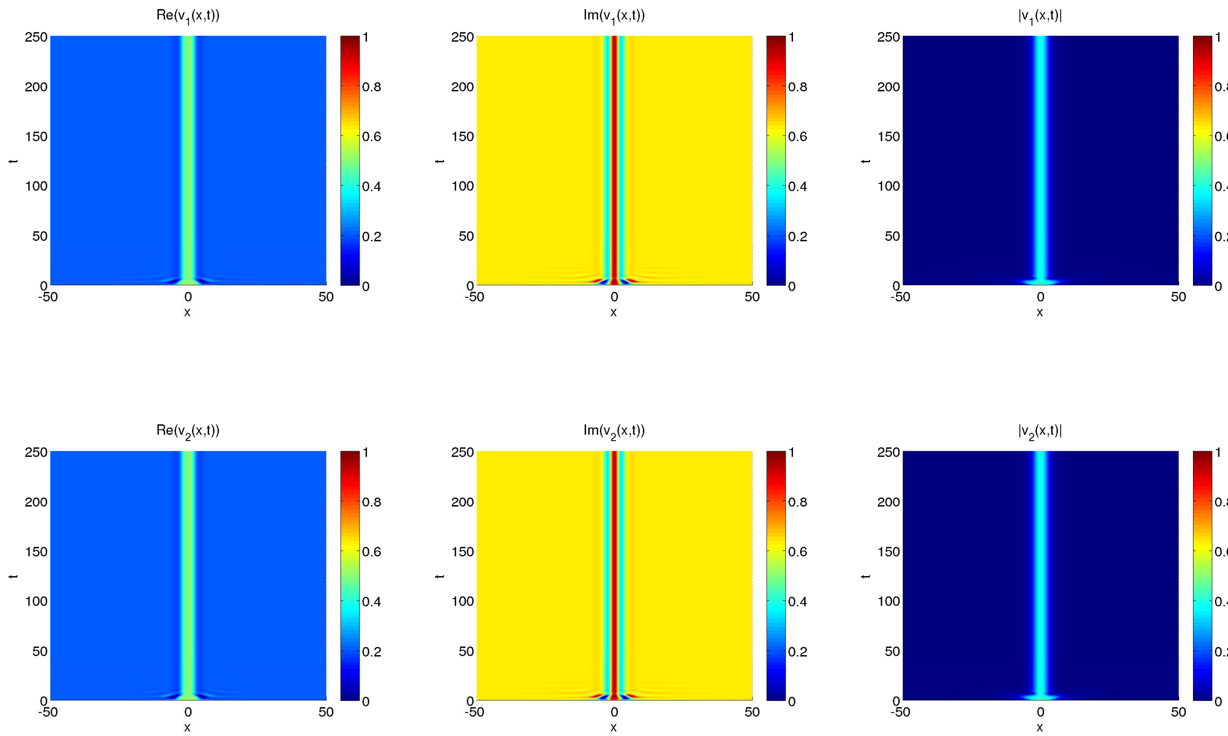


II. Frozen system (with freezing method ([73])): Consider the associated frozen system

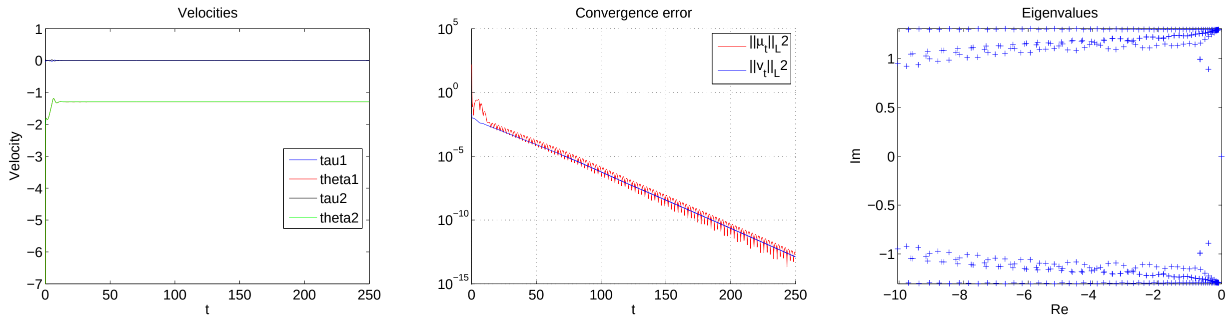
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3+i$, $\gamma = -\frac{11}{4}+i$, $R = 50$ and initial data $\operatorname{Re} v_{10} = \frac{5}{2\left(1+\left(\frac{x}{2}\right)^2\right)}$, $\operatorname{Im} v_{10} = 0$, $\operatorname{Re} v_{20} = \operatorname{Re} v_{10}$, $\operatorname{Im} v_{20} = 0$. As reference functions and bump function we use $\operatorname{Re} \hat{v}_1(x) = \operatorname{Re} v_{10}$, $\operatorname{Im} \hat{v}_1(x) = \operatorname{Im} v_{10}$, $\operatorname{Re} \hat{v}_2(x) = \operatorname{Re} v_{20}$, $\operatorname{Im} \hat{v}_2(x) = \operatorname{Im} v_{20}$ and $\varphi(x) = \operatorname{sech}(0.5 \cdot x)$, respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.2$, $rtol = 10^{-2}$, $atol = 10^{-6}$ and intermediate timesteps.
Frozen solutions:



Spatial-temporal pattern:



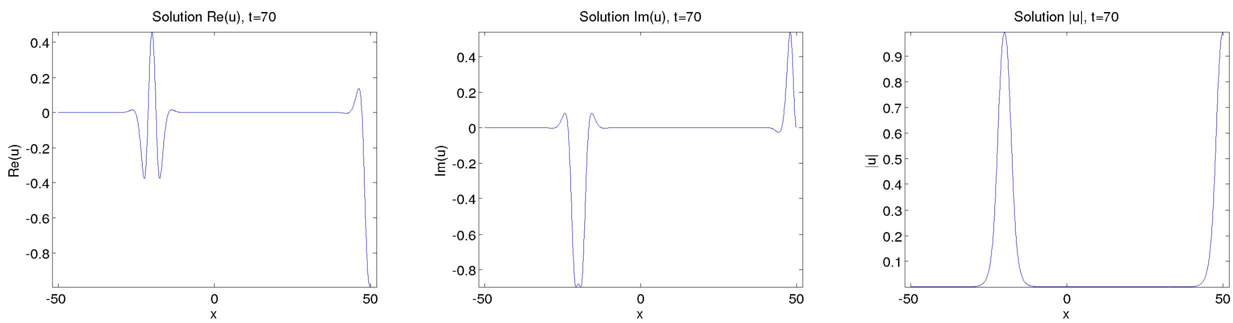
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



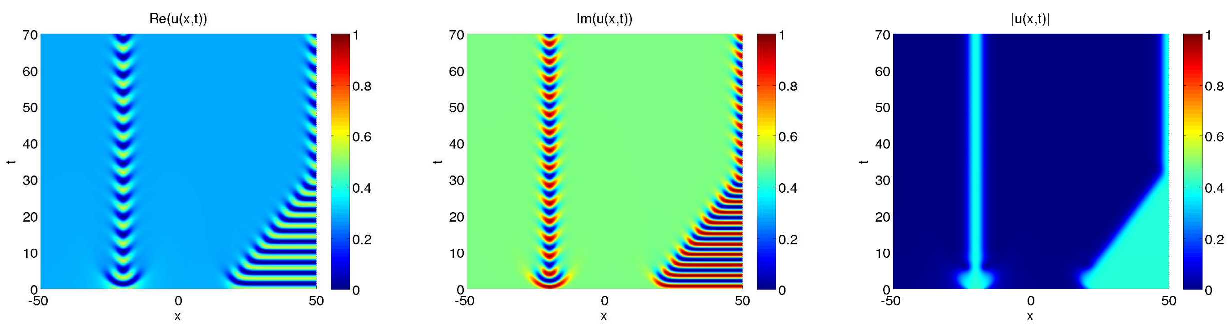
• Rotating pulse combined with a traveling rotating front (traveling rotating multistructure):
I. Nonfrozen solution: Consider the nonfrozen system

$$
\begin{aligned}
u_t &= \alpha\triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) &&, x\in B_R(0),\, t\in[0,\infty[\\
\frac{\partial u}{\partial n} &= 0 &&, x\in\partial B_R(0),\, t\in[0,\infty[\\
u(0) &= u_0 &&, x\in B_R(0),\, t=0
\end{aligned}
$$

where $d=1$ (i. e. $B_R(0)=[-R,R]$). For the numerical computations we use parameters $\mu=-\frac{1}{10}$, $\alpha=1$, $\beta=3+i$, $\gamma=-\frac{11}{4}+i$, $R=50$ and initial data $\operatorname{Re}u_0 = \frac{5}{2\left(1+\left(\frac{x+20}{2}\right)^2\right)} + \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x-20}{\sqrt{2}}\right)}$, $\operatorname{Im}u_0=0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x=0.1$. For the temporal discretization we use BDF(2) with $\triangle t=0.1$, $rtol=10^{-3}$, $atol=10^{-4}$ and intermediate timesteps.
Nonfrozen solution:

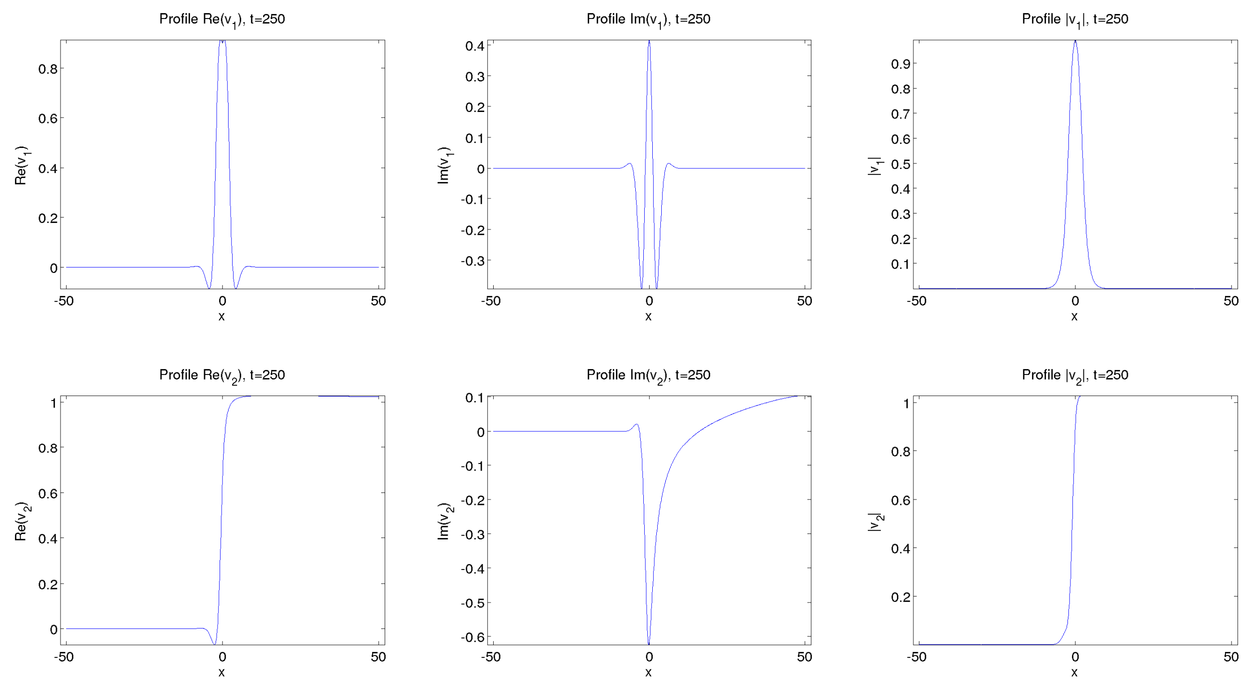


Spatial-temporal pattern:

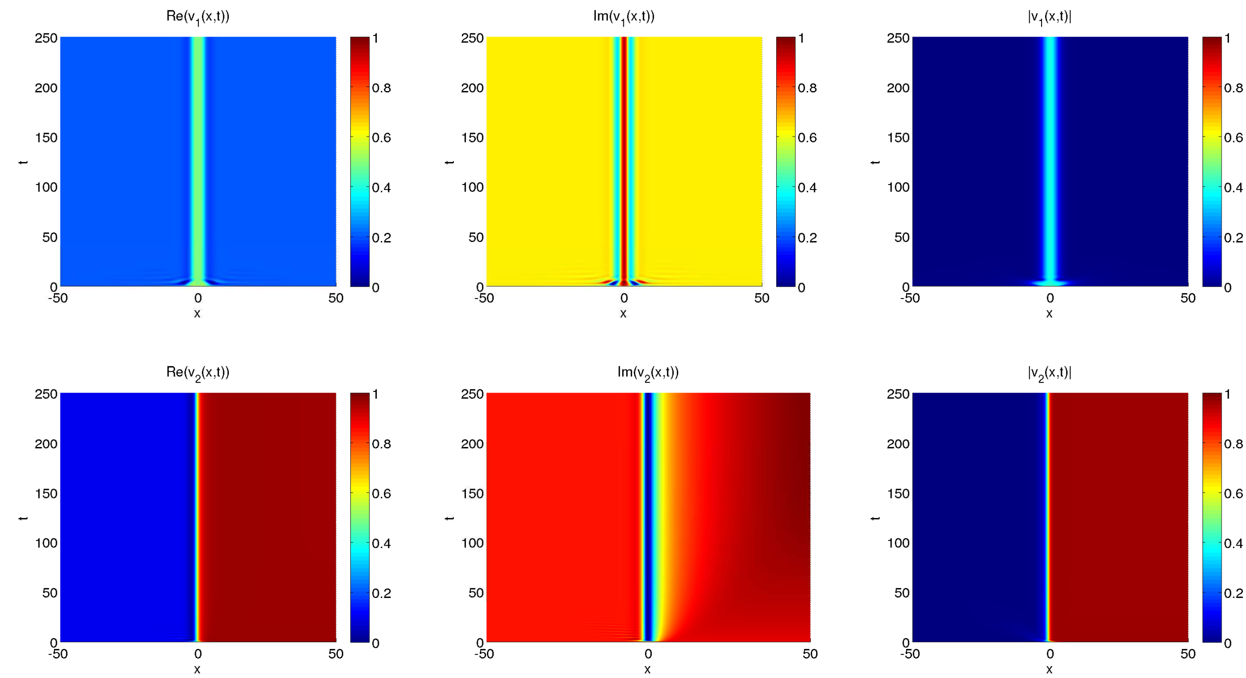


II. Frozen system (with freezing method ([73])): Consider the associated frozen system

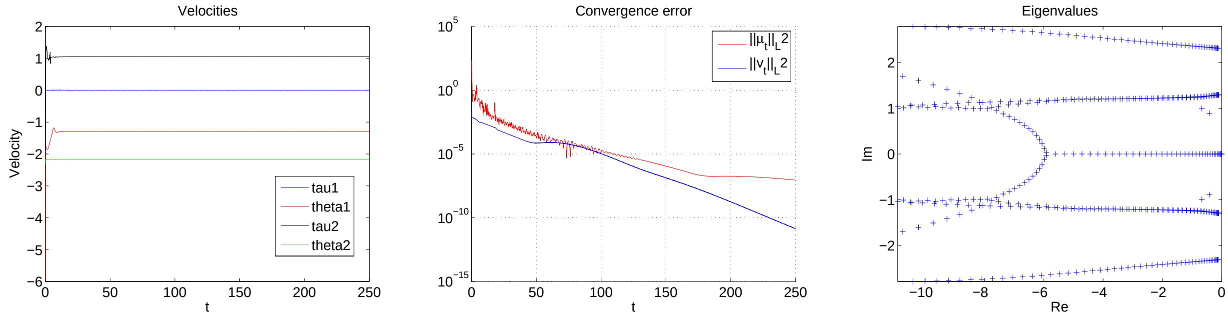
where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 50$ and initial data $\operatorname{Re} v_{10} = \frac{5}{2\left(1+\left(\frac{x}{2}\right)^2\right)}$, $\operatorname{Im} v_{10} = 0$, $\operatorname{Re} v_{20} = \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x}{\sqrt{2}}\right)}$, $\operatorname{Im} v_{20} = 0$. As reference functions and bump function we use $\operatorname{Re} \hat{v}_1(x) = \operatorname{Re} v_{10}$, $\operatorname{Im} \hat{v}_1(x) = \operatorname{Im} v_{10}$, $\operatorname{Re} \hat{v}_2(x) = \operatorname{Re} v_{20}$, $\operatorname{Im} \hat{v}_2(x) = \operatorname{Im} v_{20}$ and $\varphi(x) = \operatorname{sech}(0.5 \cdot x)$, respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-2}$, $atol = 10^{-3}$ and intermediate timesteps.
Frozen solutions:



Spatial-temporal pattern:



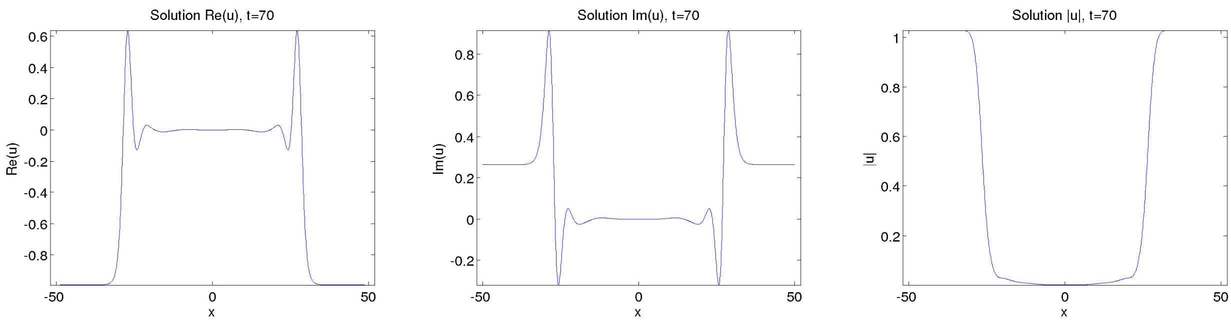
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



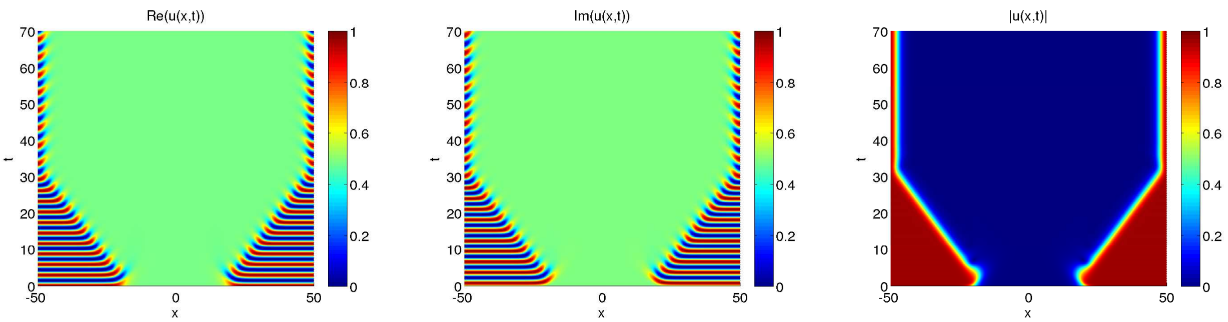
• Traveling rotating 2-front (traveling rotating multifront):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{rcll} u_t &=& \alpha\triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) &, x \in B_R(0),\ t \in [0,\infty[\\ \frac{\partial u}{\partial n} &=& 0 &, x \in \partial B_R(0),\ t \in [0,\infty[\\ u(0) &=& u_0 &, x \in B_R(0),\ t = 0 \end{array}$$

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 50$ and initial data $\operatorname{Re} u_0 = \frac{2}{11}\left(3 + \sqrt{9 + 11\mu}\right) - \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x+20}{\sqrt{2}}\right)} + \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x-20}{\sqrt{2}}\right)}$, $\operatorname{Im} u_0 = 0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Nonfrozen solution:

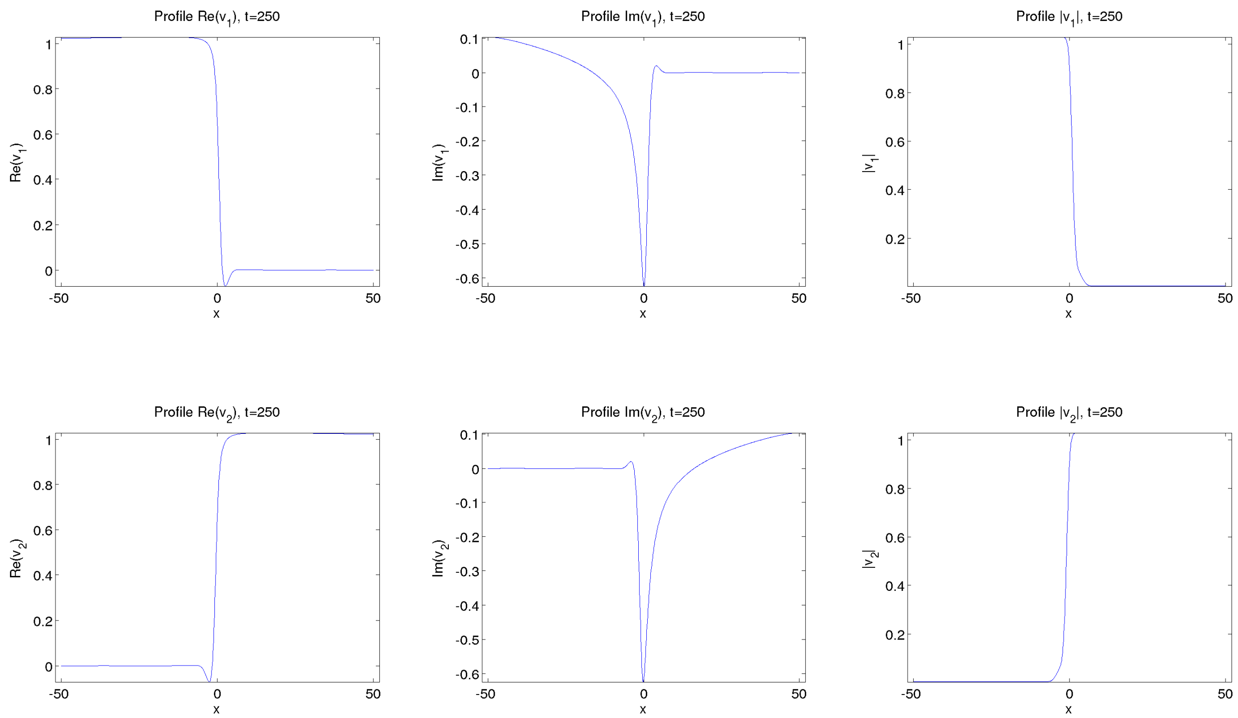


Spatial-temporal pattern:

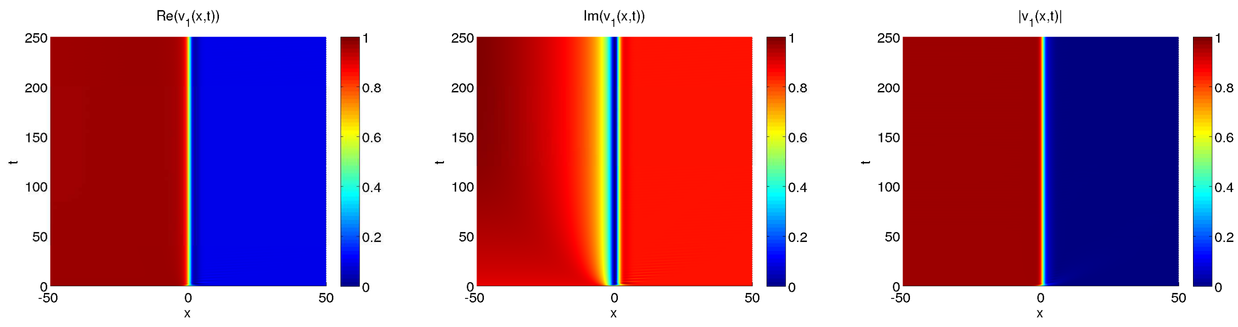


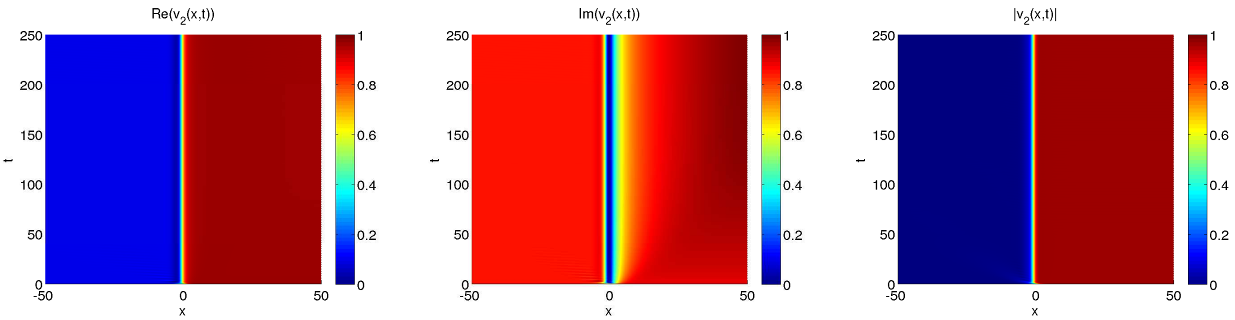
II. Frozen system (with freezing method ([73])): Consider the associated frozen system

where $d = 1$ (i. e. $B_R(0) = [-R, R]$). For the numerical computations we use parameters $\mu = -\frac{1}{10}$, $\alpha = 1$, $\beta = 3 + i$, $\gamma = -\frac{11}{4} + i$, $R = 50$ and initial data $\operatorname{Re} v_{10} = \frac{2}{11}\left(3 + \sqrt{9 + 11\mu}\right) - \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x}{\sqrt{2}}\right)}$, $\operatorname{Im} v_{10} = 0$, $\operatorname{Re} v_{20} = \frac{\frac{2}{11}\left(3+\sqrt{9+11\mu}\right)}{1+\exp\left(-\frac{x}{\sqrt{2}}\right)}$, $\operatorname{Im} v_{20} = 0$. As reference functions and bump function we use $\operatorname{Re} \hat{v}_1(x) = \operatorname{Re} v_{10}$, $\operatorname{Im} \hat{v}_1(x) = \operatorname{Im} v_{10}$, $\operatorname{Re} \hat{v}_2(x) = \operatorname{Re} v_{20}$, $\operatorname{Im} \hat{v}_2(x) = \operatorname{Im} v_{20}$ and $\varphi(x) = \operatorname{sech}(0.5 \cdot x)$, respectively. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.1$. For the temporal discretization we use BDF(2) with $\triangle t = 0.2$, $rtol = 10^{-2}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:

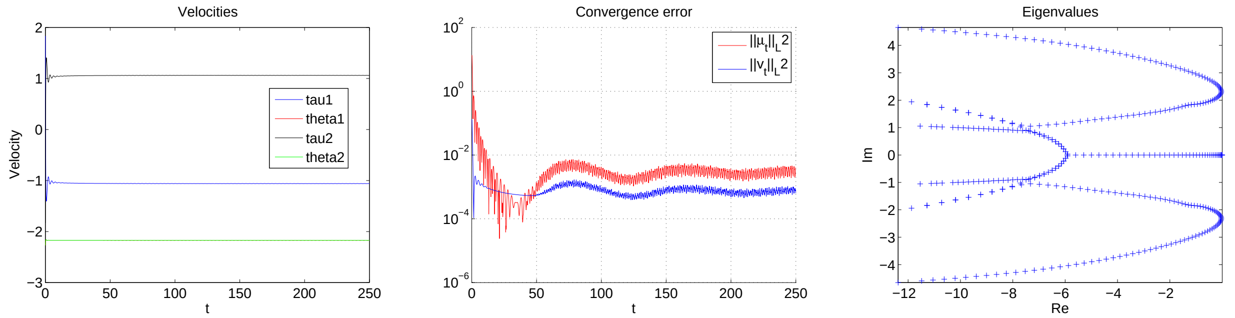


Spatial-temporal pattern:





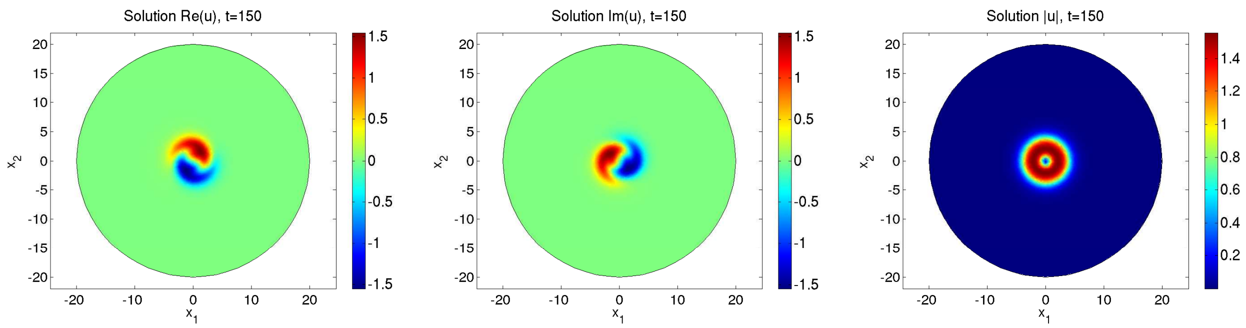
Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



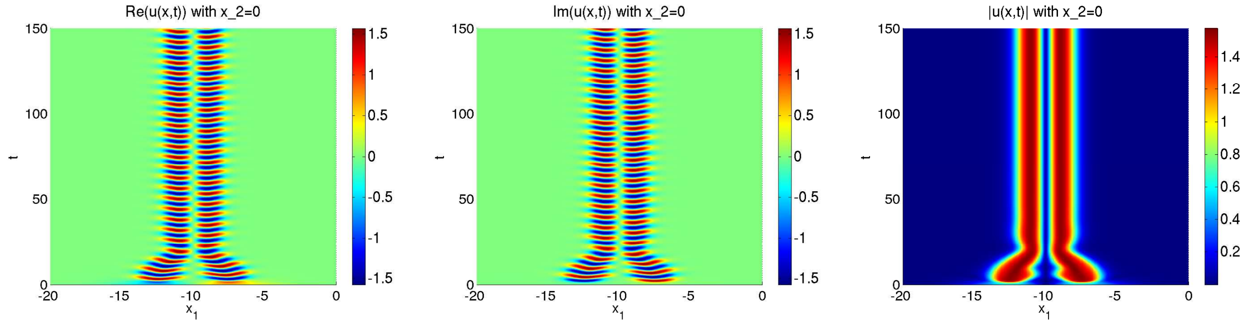
• Spinning soliton (spinning solitary wave, localized vortex solution):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{array}{rcll} u_t &=& \alpha\triangle u + u\left(\mu+\beta\,|u|^2+\gamma\,|u|^4\right) & , x\in B_R(0),\, t\in[0,\infty[\\ \frac{\partial u}{\partial n} &=& 0 & , x\in \partial B_R(0),\, t\in[0,\infty[\\ u(0) &=& u_0 & , x\in B_R(0),\, t=0 \end{array}$$

where $d=2$ (i. e. $B_R(0)=\{x\in\mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2}\leqslant R\}$). For the numerical computations we use parameters $\mu=-\frac{1}{2}$, $\alpha=\frac{1}{2}+\frac{1}{2}i$, $\beta=\frac{5}{2}+i$, $\gamma=-1-\frac{1}{10}i$, $R=20$ and initial data $\mathrm{Re}\,u_0=\frac{x_1}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$, $\mathrm{Im}\,u_0=\frac{x_2}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$ (or using polar coordinates $u_0(r,\phi)=\frac{r}{5}\exp(i\phi)\exp\left(-\frac{r^2}{7^2}\right)$). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x=0.5$. For the temporal discretization we use BDF(2) with $\triangle t=0.1$, $rtol=10^{-4}$, $atol=10^{-5}$ and intermediate timesteps ([25],[24],[78],[14],[12]).
Nonfrozen solution:



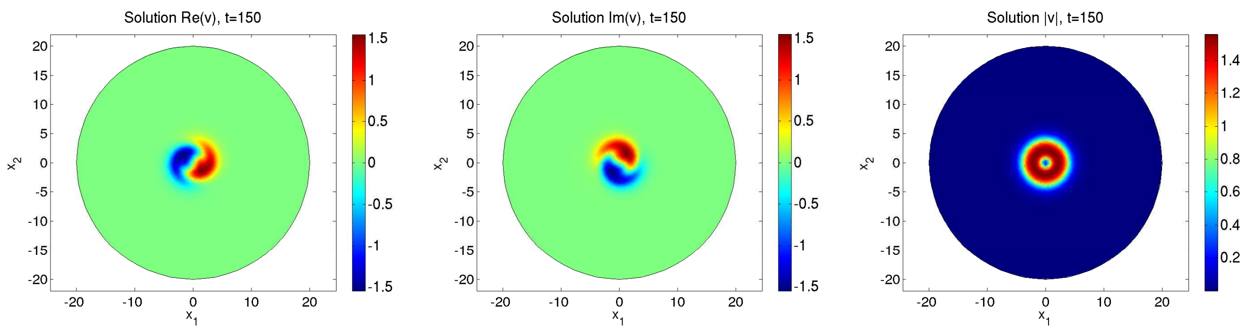
Spatial-temporal pattern: (with $x_1\in[-20,20]$ and $x_2=0$)



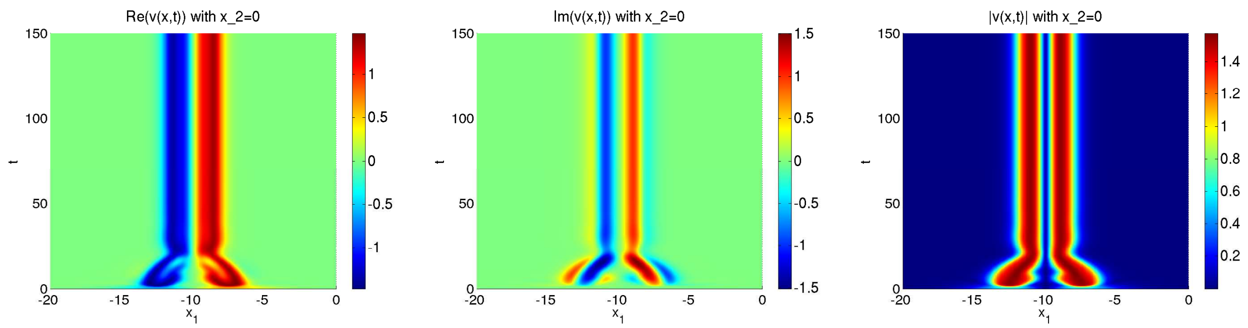
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lll}
v_t & = \alpha \triangle v + v\left(\mu + \beta |v|^2 + \gamma |v|^4\right) + \lambda_1 v_{x_1} + \lambda_2 v_{x_2} + i\lambda_3 v & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = u_0 & , x \in B_R(0),\ t = 0 \\
(\gamma_1)_t & = \lambda_1 & , t \in [0,\infty[\\
\gamma_1(0) & = 0 & , t = 0 \\
(\gamma_2)_t & = \lambda_2 & , t \in [0,\infty[\\
\gamma_2(0) & = 0 & , t = 0 \\
(\gamma_3)_t & = \lambda_3 & , t \in [0,\infty[\\
\gamma_3(0) & = 0 & , t = 0 \\
0 & = \langle \hat{v}_{x_1}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[\\
0 & = \langle \hat{v}_{x_2}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[\\
0 & = \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[
\end{array}
$$

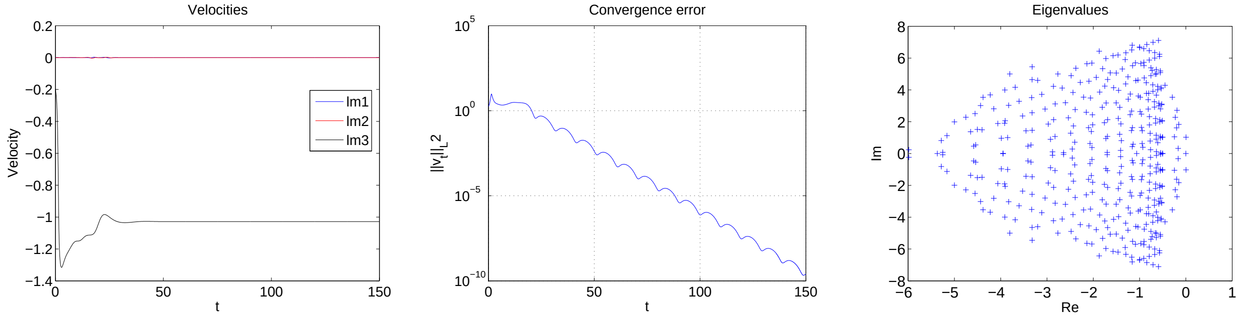
where $d = 2$ (i. e. $B_R(0) = \{x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leqslant R\}$). For the numerical computations we use parameters $\mu = -\frac{1}{2}$, $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{1}{10}i$, $R = 20$ and initial data $\operatorname{Re} u_0 = \frac{x_1}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$, $\operatorname{Im} u_0 = \frac{x_2}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$ (or using polar coordinates $u_0(r,\phi) = \frac{r}{5}\exp(i\phi)\exp\left(-\frac{r^2}{7^2}\right)$). As reference functions $\operatorname{Re}\hat{v}(x)$ and $\operatorname{Im}\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 150$, respectively, with the parameters mentioned above replacing $rtol = 10^{-2}$ and $atol = 10^{-4}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.5$. For the temporal discretization we use BDF(2) with $\triangle t = 0.05$, $rtol = 10^{-2}$, $atol = 10^{-7}$ and intermediate timesteps.
Frozen solutions:



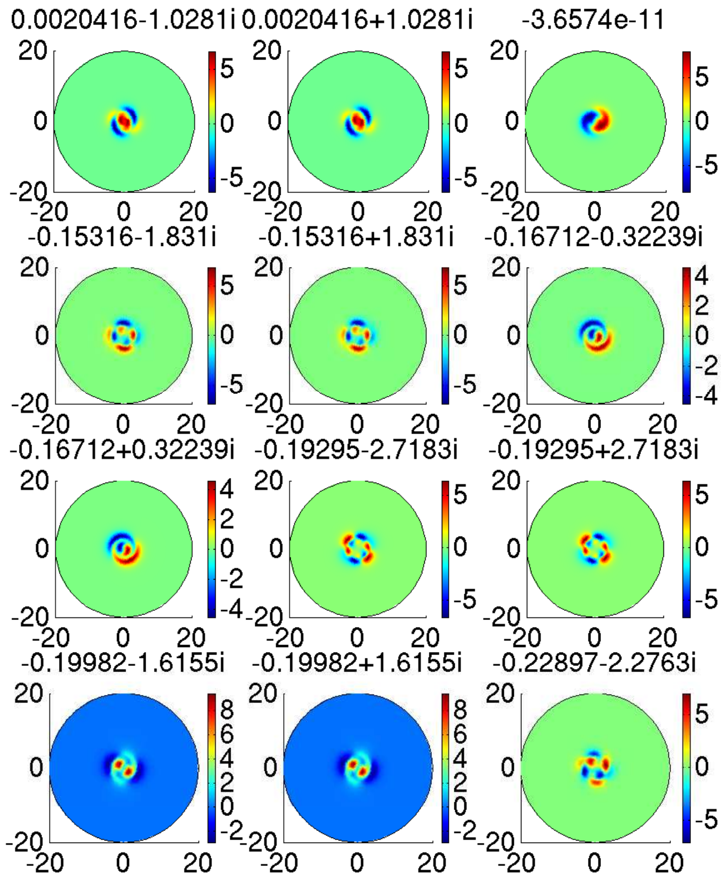
Spatial-temporal pattern: (with $x_1 \in [-20, 20]$ and $x_2 = 0$)



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



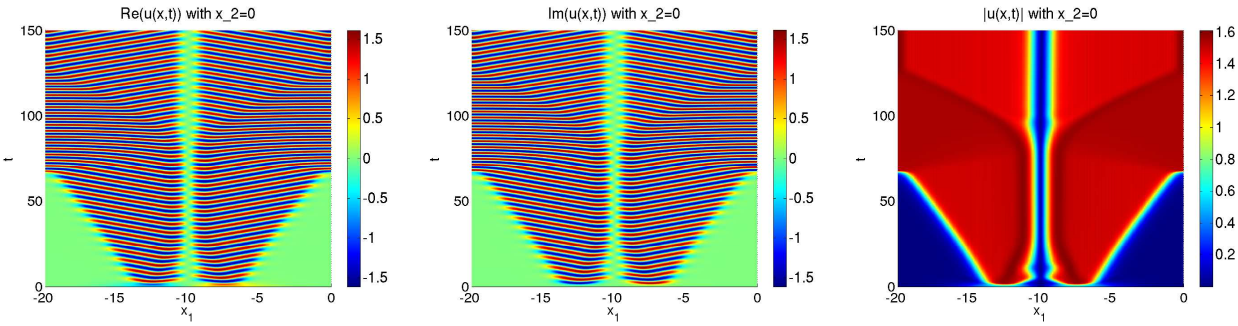
Real parts of the eigenfunctions:



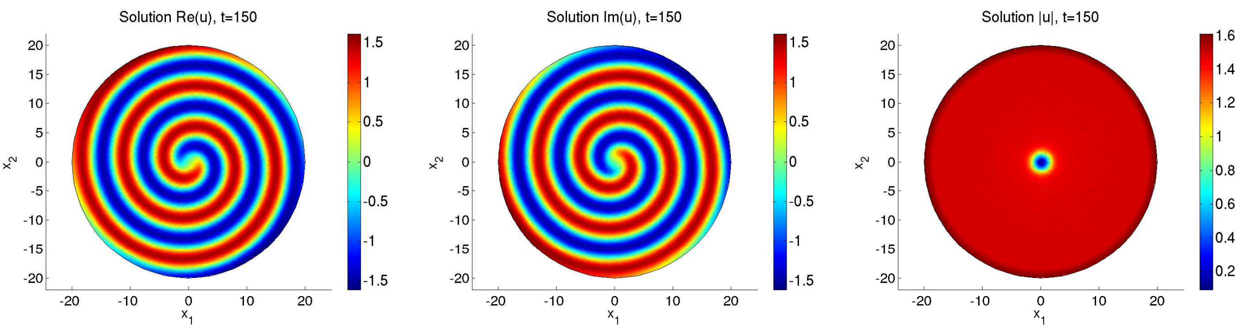
• Rotating spiral wave (rigidly rotating spiral):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= \alpha \triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) &&, x \in B_R(0),\, t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\, t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\, t = 0 \end{aligned}$$

where $d = 2$ (i. e. $B_R(0) = \{x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leqslant R\}$). For the numerical computations we use parameters $\mu = -\frac{1}{2}$, $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{13}{5} + i$, $\gamma = -1 - \frac{1}{10}i$, $R = 20$ and initial data $\mathrm{Re}\, u_0 = \frac{x_1}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$, $\mathrm{Im}\, u_0 = \frac{x_2}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$ (or using polar coordinates $u_0(r,\phi) = \frac{r}{5}\exp(i\phi)\exp\left(-\frac{r^2}{7^2}\right)$). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.5$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-4}$, $atol = 10^{-5}$ and intermediate timesteps ([78]).
Nonfrozen solution:



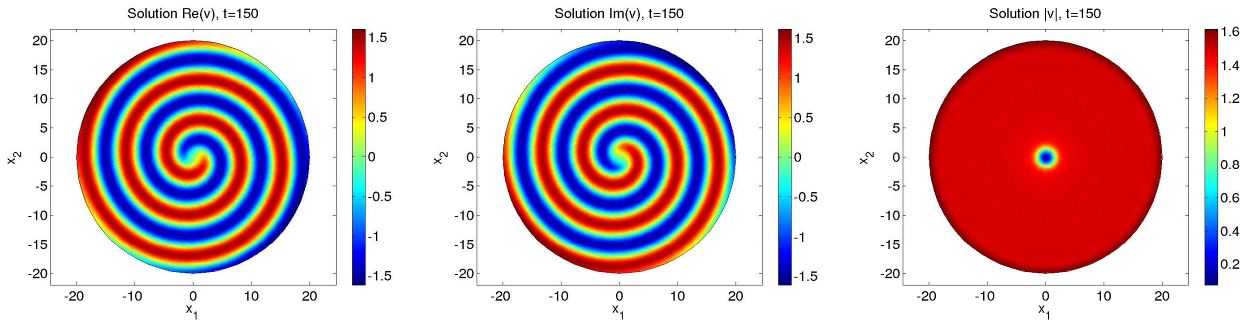
Spatial-temporal pattern: (with $x_1 \in [-20, 20]$ and $x_2 = 0$)



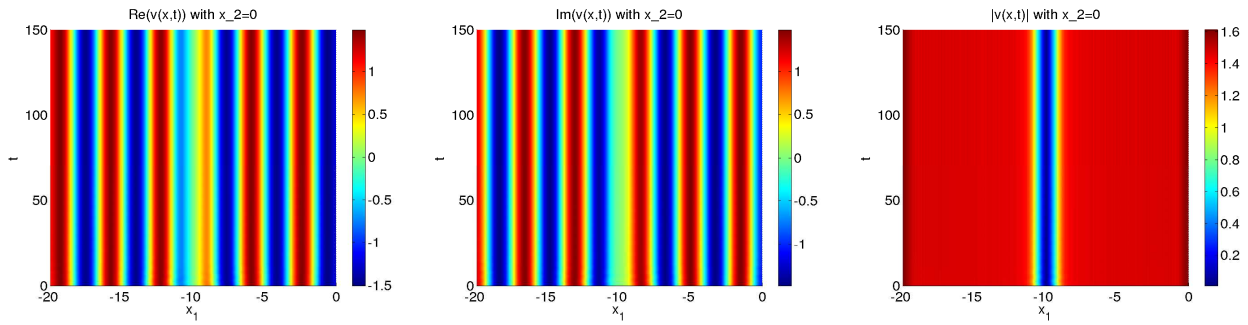
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$\begin{array}{lcll}
v_t & = & \alpha\triangle v + v\left(\mu + \beta\,|v|^2 + \gamma\,|v|^4\right) + \lambda_1 v_{x_1} + \lambda_2 v_{x_2} + i\lambda_3 v & ,\ x\in B_R(0),\ t\in[0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & ,\ x\in\partial B_R(0),\ t\in[0,\infty[\\
v(0) & = & u_0 & ,\ x\in B_R(0),\ t=0 \\
(\gamma_1)_t & = & \lambda_1 & ,\ t\in[0,\infty[\\
\gamma_1(0) & = & 0 & ,\ t=0 \\
(\gamma_2)_t & = & \lambda_2 & ,\ t\in[0,\infty[\\
\gamma_2(0) & = & 0 & ,\ t=0 \\
(\gamma_3)_t & = & \lambda_3 & ,\ t\in[0,\infty[\\
\gamma_3(0) & = & 0 & ,\ t=0 \\
0 & = & \langle \hat{v}_{x_1}, v-\hat{v}\rangle_{L^2(B_R(0),\mathbb{C})} & ,\ t\in[0,\infty[\\
0 & = & \langle \hat{v}_{x_2}, v-\hat{v}\rangle_{L^2(B_R(0),\mathbb{C})} & ,\ t\in[0,\infty[\\
0 & = & \langle i\hat{v}, v-\hat{v}\rangle_{L^2(B_R(0),\mathbb{C})} & ,\ t\in[0,\infty[
\end{array}$$

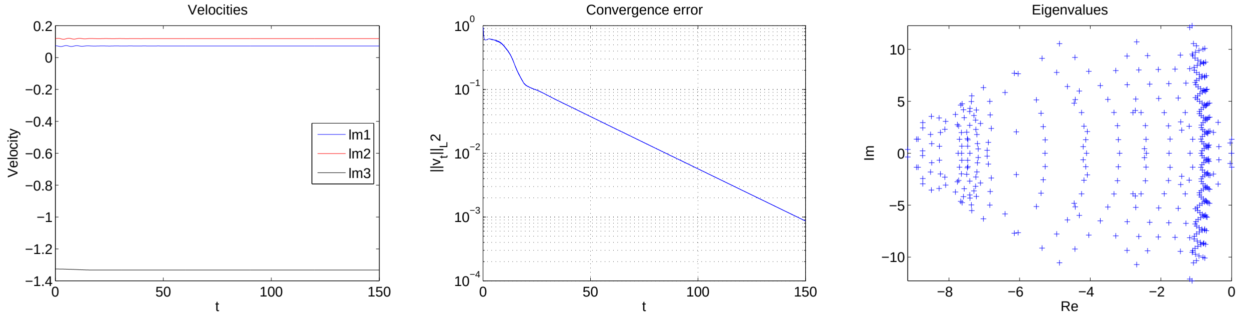
where $d=2$ (i. e. $B_R(0)=\{x\in\mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2}\leqslant R\}$). For the numerical computations we use parameters $\mu=-\frac{1}{2}$, $\alpha=\frac{1}{2}+\frac{1}{2}i$, $\beta=\frac{13}{5}+i$, $\gamma=-1-\frac{1}{10}i$, $R=20$ and initial data $\operatorname{Re}u_0=\frac{x_1}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$, $\operatorname{Im}u_0=\frac{x_2}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$ (or using polar coordinates $u_0(r,\phi)=\frac{r}{5}\exp\left(i\phi\right)\exp\left(-\frac{r^2}{7^2}\right)$). As reference functions $\operatorname{Re}\hat{v}(x)$ and $\operatorname{Im}\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t=150$, respectively, with the parameters mentioned with $rtol=10^{-2}$ and $atol=10^{-4}$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x=0.5$. For the temporal discretization we use BDF(2) with $\triangle t=0.1$, $rtol=10^{-2}$, $atol=10^{-5}$ and intermediate timesteps.
Frozen solutions:



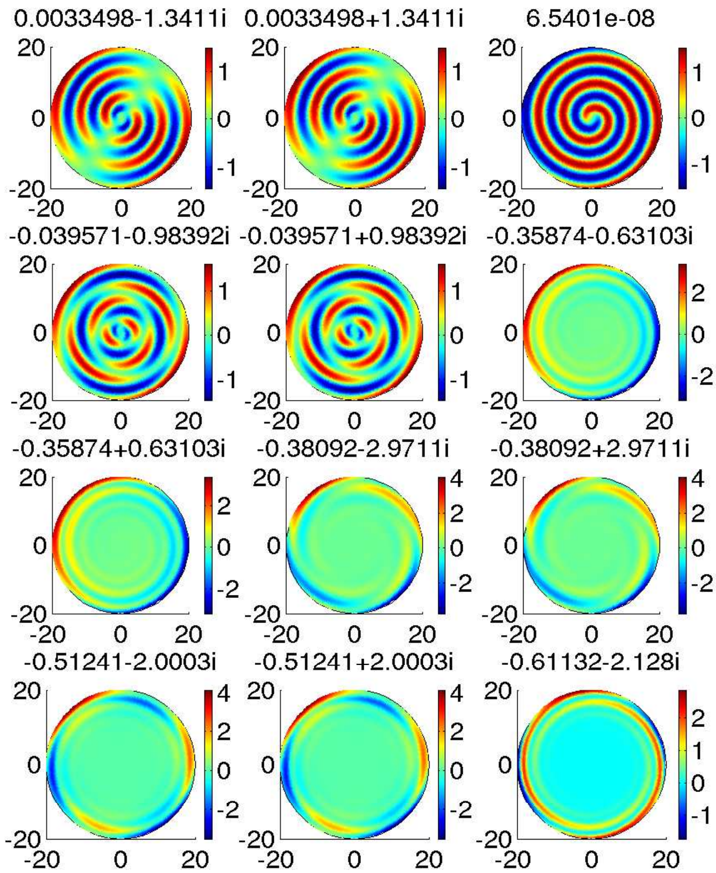
Spatial-temporal pattern: (with $x_1\in[-20,20]$ and $x_2=0$)



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



Real parts of the eigenfunctions:



• Spinning 2-soliton (spinning multisoliton, localized 2-vortex solution):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= \alpha\triangle u + u\left(\mu + \beta\left|u\right|^2 + \gamma\left|u\right|^4\right) &&, x\in B_R(0),\ t\in[0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x\in\partial B_R(0),\ t\in[0,\infty[\\ u(0) &= u_0 &&, x\in B_R(0),\ t=0 \end{aligned}$$

where $d=2$ (i. e. $B_R(0)=\{x\in\mathbb{R}^2\mid \|x\|_{\mathbb{R}^2}\leqslant R\}$). For the numerical computations we use parameters $\mu=-\frac{1}{2}$, $\alpha=\frac{1}{2}+\frac{1}{2}i$, $\beta=\frac{5}{2}+i$, $\gamma=-1-\frac{1}{10}i$, $R=20$ and initial data $\operatorname{Re}u_0=\frac{x_1}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$,

$\operatorname{Im} u_0 = \frac{x_2}{5}\exp\left(-\frac{x_1^2+x_2^2}{49}\right)$ (or using polar coordinates $u_0(r,\phi) = \frac{r}{5}\exp(i\phi)\exp\left(-\frac{r^2}{7^2}\right)$). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 0.5$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-4}$, $atol = 10^{-5}$ and intermediate timesteps.
Nonfrozen solution:

Spatial-temporal pattern: (with $x_1 \in [-20, 20]$ and $x_2 = 0$)

Explicit solutions: ([80],[56],[57])

Additional informations: In case of $\operatorname{Im}(\alpha) = \operatorname{Im}(\beta) = \operatorname{Im}(\gamma) = \operatorname{Im}(\mu) = 0$ (i.e. $\alpha, \beta, \gamma, \mu \in \mathbb{R}$) the equation is called QUINTIC REAL GINZBURG-LANDAU EQUATION. If $\gamma = 0$ then we obtain the CUBIC GINZBURG-LANDAU EQUATION. In the following cases there exists soliton solutions ([80])

1. $\operatorname{Re}(\alpha) < 2 \cdot \operatorname{Re}(\beta)$, $\mu \in \mathbb{R}$ with $\mu < 0$, $\operatorname{Re}(\gamma) < 0$
2. $\operatorname{Re}(\alpha) > 2 \cdot \operatorname{Re}(\beta)$, $\mu \in \mathbb{R}$ with $\mu > 0$, $\operatorname{Re}(\gamma) < 0$
3. $\operatorname{Re}(\alpha) = \operatorname{Re}(\beta) = \operatorname{Re}(\gamma) = \mu = 0$

In case of $\operatorname{Re}(\beta) > 0$ there exists soliton-like solutions ([80]). Multi-armed spirals are everywhere unstable, moreover the only stable rotating spirals are the one-armed. The QCGL equation describes a subcritical Hopf bifurcation at $\operatorname{Re}(\mu) = 0$ with three branches:

$$|u| = 0 \text{ and } |u|_\pm = \sqrt{\frac{-\operatorname{Re}(\beta) \pm \sqrt{(\operatorname{Re}(\beta))^2 - 4\operatorname{Re}(\gamma)\operatorname{Re}(\mu)}}{2\operatorname{Re}(\gamma)}}$$

The first branch is stable if $\operatorname{Re}(\mu) < 0$ and unstable if $\operatorname{Re}(\mu) > 0$. The second exists only for $\frac{(\operatorname{Re}(\beta))^2}{4\operatorname{Re}(\gamma)} < \operatorname{Re}(\mu) < 0$ and is always unstable. The third exists only for $\frac{(\operatorname{Re}(\beta))^2}{4\operatorname{Re}(\gamma)} < \operatorname{Re}(\mu)$ and is always stable ([85]).

Literature: [77], [54], [82], [74], [84], [58], [47], [69], [2], [24], [75], [25], [26], [79], [14], [87], [56], [57], [40], [81], [80], [5], [76]

1.11 λ-ω system

Name: λ-ω SYSTEM

Equations:

$$u_t = D\triangle u + (\lambda(|u|) + i\omega(|u|)) \cdot u$$

$u = u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $D \in \mathbb{C}$, $\lambda : \mathbb{R} \to \mathbb{R}$ with $\lambda = \lambda(|u|)$ (i.e. $\lambda(|u|) = 1 - |u|^2$), $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega = \omega(|u|)$ (i.e. $\omega(|u|) = -\alpha|u|^2$ with $\alpha \in \mathbb{R}$).

Notations:

$\mathrm{Re}\, u(x,t)$	:	concentration at position x and time t of the first reactant
$\mathrm{Im}\, u(x,t)$	:	concentration at position x and time t of the second reactant
D	:	diffusion coefficient
$\mathrm{Re}\, f(u), \mathrm{Im}\, f(u)$	:	$f(u) = (\lambda(|u|) + i\omega(|u|)) \cdot u$, reaction kinetics
α	:	reaction parameter (positive constant)

Short description: The λ-ω system ([53],[61]) describes chemical reaction processes ([53],[52]), physiological processes in the study of cardiac arrhythmias ([66]), time evolution of biological systems ([61]) and is often used to analyse the mechanism of pattern formation ([46]) as well as to study the oneset of turbulent behavior ([55]). An example of an emerging technological application based on pattern forming systems is given by memory devices using magnetic domain patterns ([27]). This model exhibits rotating spirals as well as scroll wave and scroll ring solutions ([10],[30]).

Phenomena:

- Rotating spiral wave (rigidly rotating spiral)

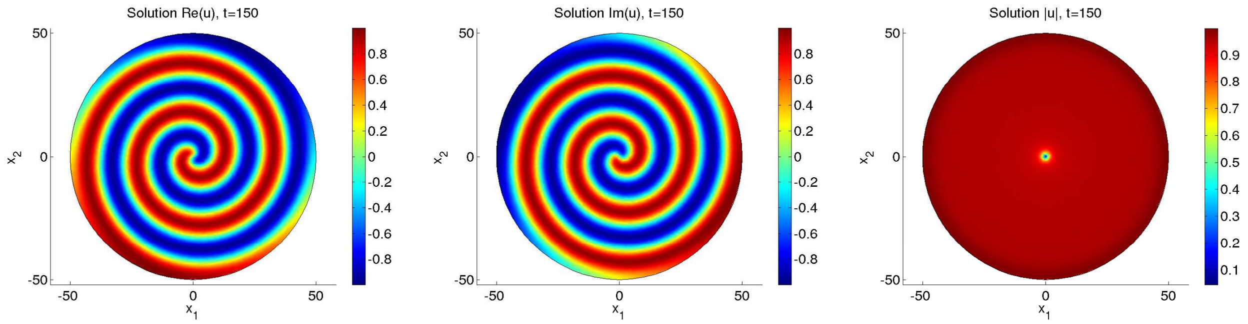
Set of parameter values:

d	D	$\lambda(\|u\|)$	$\omega(\|u\|)$	R	$\triangle x$	$\triangle t$	Boundary	Phenomena
2	1	$1 - \|u\|^2$	$-\|u\|^2$	50	1.0	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$	rotating spiral wave

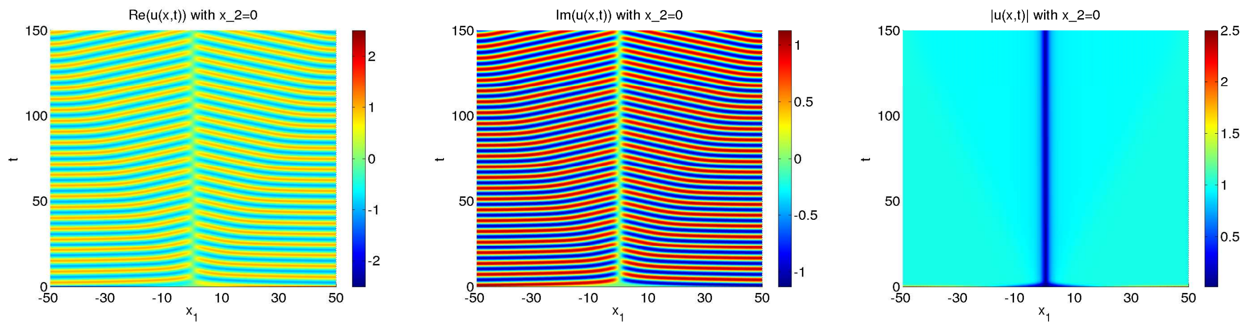
Numerical results:
• Rotating spiral wave (rigidly rotating spiral):
I. Nonfrozen solution: Consider the nonfrozen system

$$\begin{aligned} u_t &= D\triangle u + (\lambda(|u|) + i\omega(|u|)) \cdot u &&, x \in B_R(0),\, t \in [0,\infty[\\ \frac{\partial u}{\partial n} &= 0 &&, x \in \partial B_R(0),\, t \in [0,\infty[\\ u(0) &= u_0 &&, x \in B_R(0),\, t = 0 \end{aligned}$$

where $d = 2$ (i. e. $B_R(0) = \{x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leqslant R\}$). For the numerical computations we use parameters $D = 1$, $\lambda(|u|) = 1 - |u|^2$, $\omega(|u|) = -|u|^2$, $R = 50$ and initial data $u_{10} = \frac{x}{20}$, $u_{20} = \frac{y}{20}$ (or using polar coordinates $u_0(r,\phi) = \frac{r}{R}(\cos(\varphi) + i\sin(\varphi))$). Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 1.0$. For the temporal discretization we use BDF(2) with $\triangle t = 0.1$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps ([13],[53],[12]).
Nonfrozen solution:



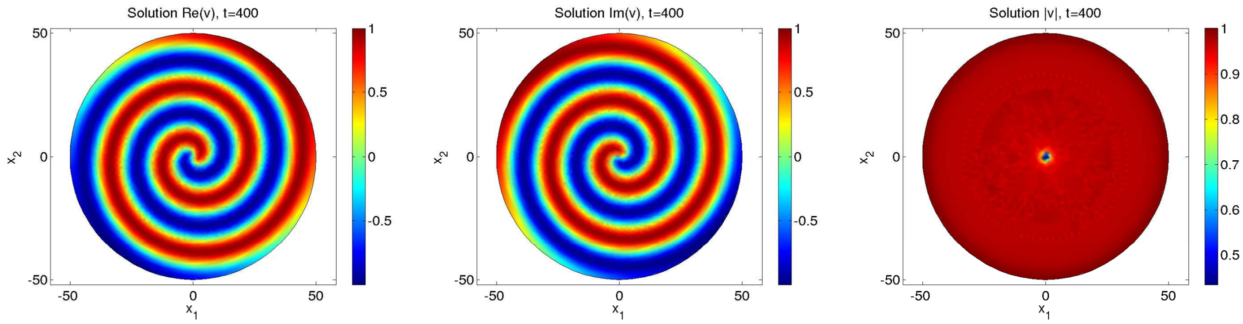
Spatial-temporal pattern: (with $x_1 \in [-50, 50]$ and $x_2 = 0$)



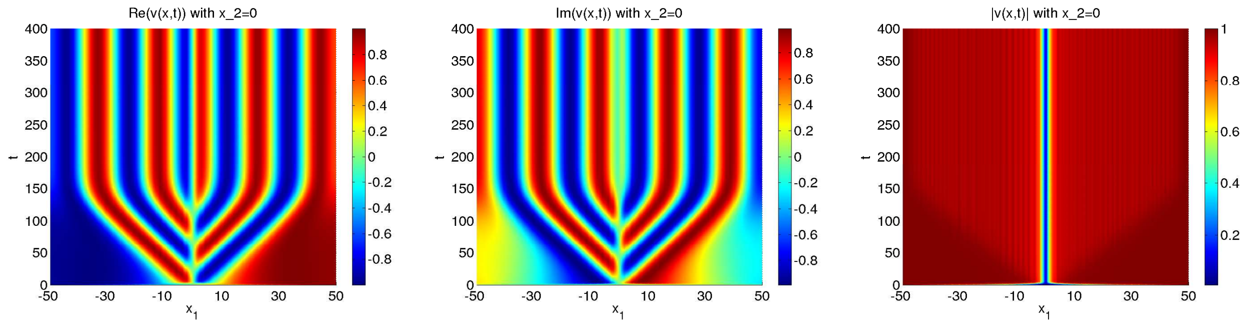
II. Frozen system (with freezing method ([12])): Consider the associated frozen system

$$
\begin{array}{lcll}
v_t & = & D\triangle v + (\lambda(|v|) + i\omega(|v|)) \cdot v + \lambda_1 v_{x_1} + \lambda_2 v_{x_2} + i\lambda_3 v & , x \in B_R(0),\ t \in [0,\infty[\\
\frac{\partial v}{\partial n} & = & 0 & , x \in \partial B_R(0),\ t \in [0,\infty[\\
v(0) & = & u_0 & , x \in B_R(0),\ t = 0 \\
(\gamma_1)_t & = & \lambda_1 & , t \in [0,\infty[\\
\gamma_1(0) & = & 0 & , t = 0 \\
(\gamma_2)_t & = & \lambda_2 & , t \in [0,\infty[\\
\gamma_2(0) & = & 0 & , t = 0 \\
(\gamma_3)_t & = & \lambda_3 & , t \in [0,\infty[\\
\gamma_3(0) & = & 0 & , t = 0 \\
0 & = & \langle \hat{v}_{x_1}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[\\
0 & = & \langle \hat{v}_{x_2}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[\\
0 & = & \langle i\hat{v}, v - \hat{v} \rangle_{L^2(B_R(0),\mathbb{C})} & , t \in [0,\infty[
\end{array}
$$

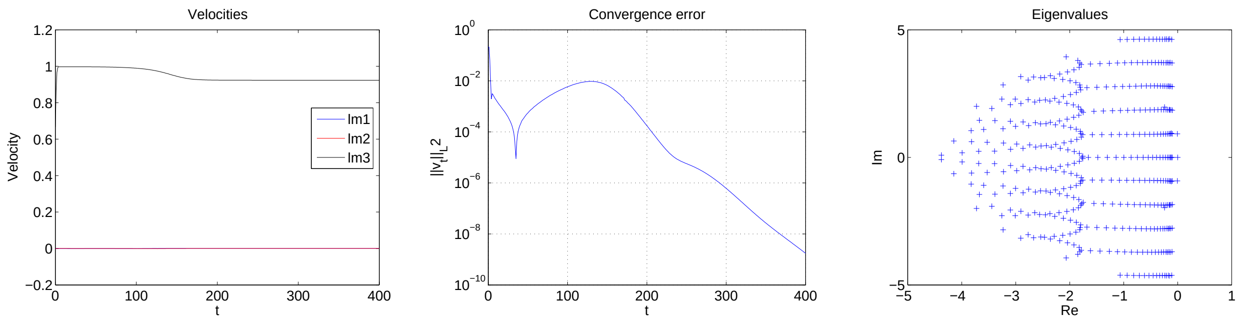
where $d = 2$ (i. e. $B_R(0) = \{x \in \mathbb{R}^2 \mid \|x\|_{\mathbb{R}^2} \leqslant R\}$). For the numerical computations we use parameters $D = 1$, $\lambda(|v|) = 1 - |v|^2$, $\omega(|v|) = -|v|^2$, $R = 50$, and initial data $u_{10} = \frac{x}{20}$, $u_{20} = \frac{y}{20}$ (or using polar coordinates $u_0(r,\phi) = \frac{r}{R}(\cos(\varphi) + i\sin(\varphi))$). As reference functions $\operatorname{Re}\hat{v}(x)$ and $\operatorname{Im}\hat{v}(x)$ we choose the real and imaginary part of the nonfrozen solution at time $t = 150$, respectively, with the parameters mentioned above and $\triangle x = 2.0$. Moreover, for the spatial discretization we use FEM (continuous piecewise linear finite elements) with $\triangle x = 2.0$. For the temporal discretization we use BDF(2) with $\triangle t = 1.0$, $rtol = 10^{-3}$, $atol = 10^{-4}$ and intermediate timesteps.
Frozen solutions:



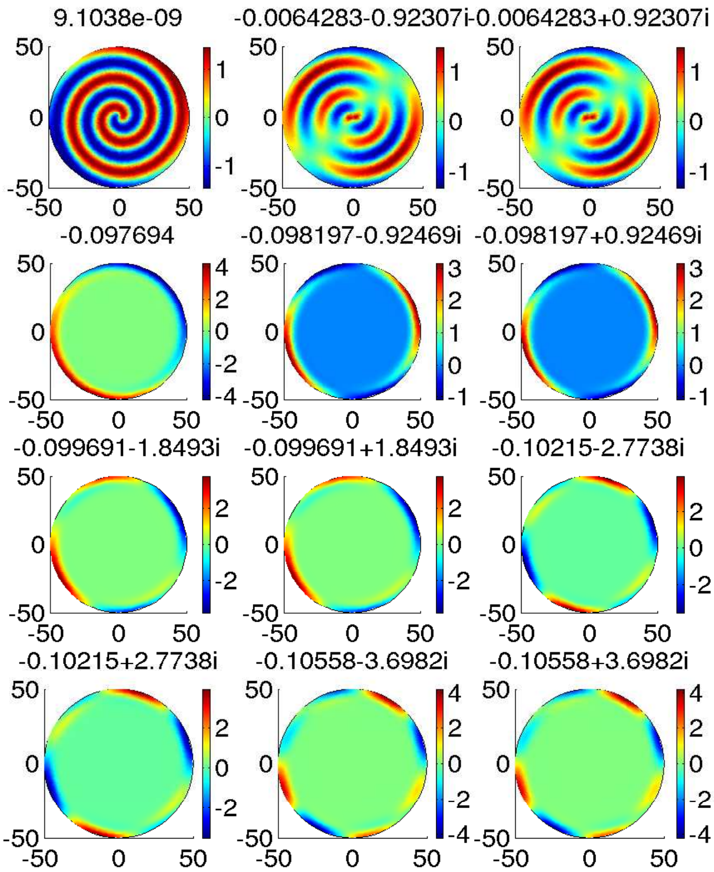
Spatial-temporal pattern: (with $x_1 \in [-50, 50]$ and $x_2 = 0$)



Velocities, convergence error of freezing method and eigenvalues of the linearization about the frozen solution:



Real parts of the eigenfunctions:



Explicit solutions: not available

Literature: [12], [61], [18], [52], [46], [55], [27], [66], [53], [10], [13], [30]

1.12 Autocatalysis model

Name: AUTOCATALYSIS MODEL

Equations:

$$u_t = D_u \triangle u - uv^m$$
$$v_t = D_v \triangle v + uv^m$$

$u = u(x,t) \in \mathbb{R}$, $v = v(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^d$, $d \in \{1,2,3\}$, $t \in [0,\infty[$, $D_u, D_v \in \mathbb{R}$, $m \in \mathbb{N}$.

Notations:

$u(x,t)$	:	concentration of the reactant at position x and time t
$v(x,t)$	:	concentration of the autocatalyst at position x and time t
D_u, D_v	:	diffusion coefficients (positive constant)
m	:	other system parameter (positve integer)

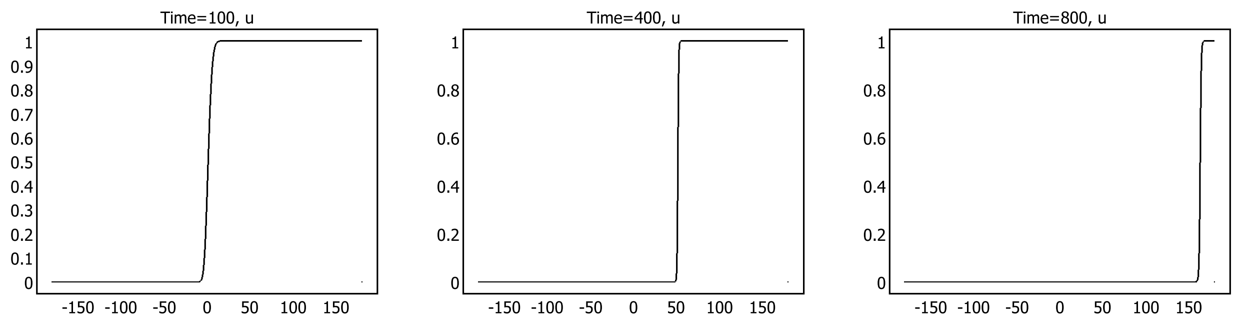
Short description: not available

Set of parameters:

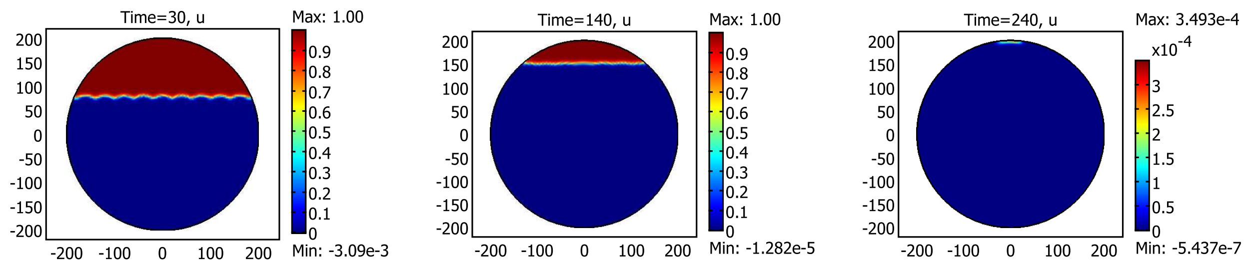
d	D_u	D_v	m	R	$\triangle x$	$\triangle t$	Boundary
1	0.1	1	9	180	0.1	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial[-180,180]$
2	2	1	2	200	5.0	0.1	$\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$

Phenomena:

- Traveling wave front (traveling wave, traveling front): $d = 1$, $D_u = 0.1$, $D_v = 1$, $m = 9$, $R = 180$, $\triangle x = 0.1$, $\triangle t = 0.1$, $\frac{\partial u}{\partial n} = 0$ on $\partial[-180,180]$ (Neumann boundary), $u_0 = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$, $v_0 = 1 - u_0$.



- Traveling wave front (traveling wave, traveling front): $d = 2$, $D_u = 2$, $D_v = 1$, $m = 2$, $R = 200$, $\triangle x = 5.0$, $\triangle t = 0.1$, $\frac{\partial u}{\partial n} = 0$ on $\partial B_R(0)$ (Neumann boundary), $u_0 = \frac{1}{1+\exp\left(-\frac{y-50-5\cos\left(\frac{\pi x}{20}\right)}{4}\right)}$, $v_0 = 1 - u_0$.



Explicit solutions: not available

Literature: not available

Bibliography

[1] Ablowitz, Zeppetella: *Explicit solutions of Fisher's equation for a special wave speed.* Bull. Math. Biology, Vol. 41, pp. 835–840, 1979.

[2] Afanasjev, Akhmediev, Soto-Crespo: *Three forms of localized solutions of the quintic complex Ginzburg-Landau equation.* Phys. Rev. E, volume 53, no. 2, pp. 1931–1939, 1996.

[3] Agrawal: *Nonlinear Fiber Optics.* Academic Press, San Diego, 1995.

[4] Airault: *Équations asymptotiques pour des cas spéciaux de l'équation de Nagumo.* C. R. Acad. Sci. Paris Sér. I Math., volume 301, no. 6, pp. 295–298, 1985.

[5] Akhmediev, Ankiewicz: *Dissipative Solitons: From Optics to Biology and Medicine.* Lecture Notes in Physics, vol. 751, Springer, 2008.

[6] Aranson, Kramer: *The world of the complex Ginzburg-Landau equation.* Rev. Mod. Phys., volume 74, no. 1, pp. 99–143, American Physical Society, 2002.

[7] Barkley: *A model for fast computer simulation of waves in excitable media.* Phys. D, volume 49, pp. 61–70, Elsevier Science Publishers B. V., Amsterdam, The Netherlands, 1991.

[8] Barkley: *Euclidean symmetry and the dynamics of rotating spiral waves.* Phys. Rev. Lett., volume 72, pp. 164–167, 1994.

[9] Barkley: *Barkley model.* http://www.scholarpedia.org/article/Barkley_model, Scholarpedia, 2008.

[10] Barkley, Dowle, Mantel: *Fast simulation of waves in three-dimensional excitable media.* Internat. J. Bifur. Chaos Appl. Sci. Engrg., volume 7, no. 11, pp. 2529–2545, 1997.

[11] Beyn, Thümmler, Selle: *Freezing multipulse and multifronts.* SIAM Journal on Applied Dynamical Systems, volume 7, no. 2, pp. 577–608, 2008.

[12] Beyn, Thümmler: *Freezing solutions of equivariant evolution equations.* SIAM J. Appl. Dyn. Syst. Volume 3, Issue 2, pp. 85–116, 2004.

[13] Beyn, Thümmler: *Phase conditions, symmetries, and PDE continuation.* Numerical Continuation Methods for Dynamical Systems (B. Krauskopf, H. Osinga, J. Galan-Vioque Eds.), pp. 301–330, 2007.

[14] Beyn, Thümmler: *Dynamics of pattern in nonlinear equivariant equations.* GAMM-Mitt., volume 32, no. 1, pp. 7–25, 2009.

[15] Biktashev: *Dissipation of the excitation wave fronts.* Phys. Rev. Lett., volume 89, no. 16, pp. 168102, 2002.

[16] Biktashev, Foulkes: *Riding a Spiral Wave: Numerical Simulation of Spiral Waves in a Co-Moving Frame of Reference.* 2010.

[17] Bini, Cherubini, Filippi: *Viscoelastic FitzHugh-Nagumo models.* Physical Review E, volume 72, 2005.

[18] Borzi, Griesse: *Distributed optimal control of lambda-omega systems.* Journal of Numerical Mathematics 14 1, pp. 17–40, 2006.

[19] Bär, Brusch: *Breakup of spiral waves caused by radial dynamics: Eckhaus and finite wavenumber instabilities.* New J. Phys. 6 5, 2004.

[20] Bär, Eiswirth: *Turbulence due to spiral breakup in a continuous excitable medium.* Phys. Rev. E, volume 48, no. 3, R1635–R1637, 1993.

[21] Bär, Or-Guil. Phys. Rev. Lett., volume 82, pp. 1160, 1999.

[22] Brazhnik, Tyson: *On Traveling Wave Solutions of Fisher's Equation in Two Spatial Dimensions.* SIAM Journal on Applied Mathematics, Vol. 60, No. 2, pp. 371–391, 1999.

[23] Chen, Guo: *Analytic solutions of the nagumo equation.* IMA Journal of Applied Mathematics, volume 48, no. 2, pp. 107–115, 1992.

[24] Crasovan, Malomed, Mihalache: *Stable vortex solitons in the two-dimensional Ginzburg-Landau equation.* Phys. Rev. E, volume 63, no. 1, 016605, 2000.

[25] Crasovan, Malomed, Mihalache: *Spinning solitons in cubic-quintic nonlinear media.* Pramana, volume 57, no. 5–6, pp. 1041–1059, 2001.

[26] Crasovan, Malomed, Mihalache-Mazilu Lederer: *Three-dimensional spinning solitons in the cubic-quintic nonlinear medium.* Phys. Rev. E, volume 61, no. 6, pp. 7142–7145, 2000.

[27] Dahlberg, Zhu: *Micromagnetic microscopy and modeling.* Physics Today 48, p. 34, 1995.

[28] Danilov, Maslov, Volosov: *Mathematical Modelling of Heat and Mass Transfer Processes.* Kluwer, Dordrecht, 1995.

[29] EqWorld: *The World of Mathematical Equations.* http://eqworld.ipmnet.ru/en/solutions/npde/npde-

[30] Fiedler, Mantel: *Crossover collision of scroll wave filaments.* Doc. Math., 5:695–731 (electronic), 2000.

[31] Fisher: *The wave of advance of advantageous genes.* Ann. Eug. 7, pp. 355–369, 1937.

[32] FitzHugh: *Mathematical models of threshold phenomena in the nerve membrane.* Bull. Math. Biophysics 17, pp. 257–278, 1955.

[33] FitzHugh: *Impulses and physiological states in theoretical models of nerve membrane.* Biophysical J. 1, pp. 445–466, 1961.

[34] FitzHugh: *Motion picture of nerve impulse propagation using computer animation.* Journal of Applied Physiology 25, pp. 628–630, 1968.

[35] FitzHugh: *Mathematical models of excitation and propagation in nerve.* Chapter 1 (pp. 1–85 in H.P. Schwan, ed. Biological Engineering, McGraw-Hill Book Co., N.Y.), 1969.

[36] Ginzburg, Landau: *On the theory of superconductivity.* Zh. Eksp. Teor. Fiz. (USSR), volume 20, pp. 1064, English transl. in: Ter Haar, D. (ed.): Men of Physics: Landau, volume I, pp. 546–568, New York (1965), 1950.

[37] Gross: *Structure of a quantized vortex in boson systems.* Il Nuovo Cimento (Italian Physical Society), Volume 20, Number 3, pp. 454–457, 1961.

[38] Hagberg: *Mathematical Modeling and Analysis.* http://math.lanl.gov/~hagberg/Movies/.

[39] Hagberg, Meron: *Complex Patterns in Reaction-Diffusion Systems: A Tale of Two front*

instabilities. 1994.

[40] Hagberg, Meron: *From Labyrinthine Patterns to Spiral Turbulence.* Phys. Rev. Lett., volume 72, no. 15, 1994.

[41] Hagberg, Marts, Meron-Lin: *Resonant and nonresonant pattern in forced oscillators.*

[42] Hagberg, Yochelis, Meron-Lin Swinney: *Development of Standing-Wave Labyrinthine Patterns.* SIAM J. Appl. Dyn. Syst. Volume 1, Issue 2, pp. 236–247, 2002.

[43] Henry: *Geometric theory of semilinear parabolic equations.* Lecture Notes in Mathematics, volume 840, Springer-Verlag Berlin Heidelberg New York, 1981.

[44] Hermann, Gottwald: *The large core limit of spiral waves in excitable media: A numerical approach.* 2010.

[45] Hodgkin, Huxley: *A quantitative description of membrane current and its application to conduction and excitation in nerve.* J. Physical., volume 117, pp. 550, 1952.

[46] Hohenberg, Cross: *Pattern formation outside of equilibrium.* Rev. Mod. Phys. 65, pp. 8511112, 1993.

[47] Hohenberg, van Saarloos: *Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations.* Phys. D, volume 56, no. 4, pp. 303–367, 1992.

[48] Kaliappan: *An exact solution for travelling waves of* $u_t = Du_{xx} + u - u^k$. Physica D, Vol. 11, No. 3, pp. 368–374, 1984.

[49] Kawahara, Tanaka: *Interactions of traveling fronts: an exact solution of a nonlinear diffusion equations.* Phys. Lett., Vol. 97, p. 311, 1983.

[50] Keener, Tyson: *The dynamics of scroll waves in excitable media.* SIAM Rev., volume 34, no. 1, pp. 1–39, 1992.

[51] Kudryashov: *On exact solutions of families of Fisher equations.* Theor. and Math. Phys., Vol. 94, No. 2, pp. 211–218, 1993.

[52] Kuramoto: *Chemical Oscillations, Waves, and Turbulence.* Springer-Verlag, 1984.

[53] Kuramoto, Koga: *Turbulized rotating chemical wavies.* Progress of Theoretical Physics, volume 66, no. 3, pp. 1081–1085, 1981.

[54] Mancas, Choudhury: *The complex cubic-quintic Ginzburg-Landau equation: Hopf bifurcations yielding traveling waves.* Math. Comput. Simul. 74 4-5, pp. 281–291, Elsevier Science Publishers B. V., Amsterdam, The Netherlands, 2007.

[55] Manneville: *Dissipative structures and weak turbulence.* Academic Press, 1990.

[56] Marcq, Chaté, Conte: *Exact solutions of the one-dimensional quintic complex Ginzburg-Landau equation.* Phys. D, volume 73, no. 4, pp. 305–317, 1994.

[57] Marcq, Chaté, Conte: *Explicit and implicit solutions for the one-dimensional cubic and quintic complex Ginzburg-Landau equations.* Appl. Math. Lett., volume 10, no. 10, pp. 1007–1012, 2006.

[58] Mielke: *The Ginzburg-Landau equation in its role as a modulation equation.* In B. Fiedler editor, Handbook of dynamical systems, volume 2, pp. 759–834, North Holland, Amsterdam, 2002.

[59] Miura: *Accurate computation of the stable solitary waves for the FitzHugh-Nagumo equations.*

Journal of Mathematical Biology, volume 13, pp. 247–269, 1982.

[60] Moores. Opt. Commun., volume 96, 65, 1993.

[61] Murray: *Mathematical biology, II: Spatil models and biomedical applications.* Springer, 2004.

[62] Nagumo, Arimoto, Yoshizawa: *An active pulse transmission line simulating nerve axon.* In Proceedings of the IRE, volume 50, pp. 2061–2070, 1962.

[63] Newell, Whitehead: *Finite bandwidth, finite amplitude convection.* J. Fluid Mech., Volume 38, pp. 279–303, 1969.

[64] Nii: *A topological proof of stability of N-front solutions of the FitzHugh-Nagumo equations.* Journal of Dynamics and Differential Equations , volume 11, no. 3, 1999.

[65] Otten: *Attraktoren für Finite-Elemente Diskretisierungen parabolischer Differentialgleichungen.* Diplomarbeit, Fakultät für Mathematik, Universität Bielefeld, 2009.

[66] Panfilov: *Electrophysiological model of the heart and its application to studying of cardiac arrhythmias.* Workshop on Issues in Cardiovascular - Respiratory Control Modeling, Graz, Austria, June 14–16, 2001.

[67] Pitaevskii: *Vortex Lines in an Imperfect Bose Gas.* Soviet Physics JETP-USSR (Woodbury, New York: American Institute of Physics), Volume 13, Issue 2, pp. 451–454, 1961.

[68] Rocsoreanu, Giurgiteanu, Georgescu: *The FitzHugh-Nagumo Model: Bifurcation and Dynamics.* Springer Netherlands, 2000.

[69] Rosanov, Fedorov, Shatsev: *Motion of clusters of weakly coupled two-dimensional cavity solitons.* Journal of Experimental and Theoretical Physics, volume 102, no. 4, pp. 547–555, 2006.

[70] Sandstede, Scheel: *Absolute versus convective instability of spiral waves.* Phys. Rev. E, pp. 7708–7714, 2000.

[71] Schrödinger: *An Undulatory Theory of the Mechanics of Atoms and Molecules.* Phys. Rev., 28 (6), pp. 1049–1070, 1926.

[72] Segel: *Distant side-walls cause slow amplitude modulation of cellular convection.* J. Fluid Mech., Volume 38, pp. 203–224, 1969.

[73] Selle: *Decomposition and stability of multifronts and multipulses.* PhD thesis, Bielefeld University, 2009.

[74] Shi, Dai, Han: *Exact solutions for 2D cubic-quintic Ginzburg-Landau equation.* J. Phys.: Conf. Ser. 96 012148, 2008.

[75] Soto-Crespo, Akhmediev, Ankiewicz: *Pulsating, Creeping, and Erupting Solitons in Dissipative Systems.* Phys. Rev. Lett., volume 85, no. 14, pp. 2937–2940, 2000.

[76] Thümmler: *The Effect of Freezing and Discretization to the Asymptotic Stability of Relative Equilibria.* Journal of Dynamics and Differential Equations, Volume 20, Number 2, pp. 425–477, 2006.

[77] Thümmler: *Numerical bifurcation analysis of relative equilibria with Femlab.* COMSOL Anwenderkonferenz 2006, Femlab GmbH, 2006.

[78] Thümmler: *Numerical computation of relative equilibria with Femlab.* http://www.math.uni-bielefeld.de/~beyn/AG_Numerik/downloads/preprints/misc/femlab.pdf, 2006.

[79] Thual, Fauve: *Localized structures generated by subcritical instabilities.* J. Phys. France, volume 49, no. 11, pp. 1829–1833, 1988.

[80] Trillo, Torruellas: *Spatial Solitons.* Optical Sciences, Springer, 2001.

[81] Wainblat, Malomed: *Interactions between two-dimensional solitons in the diffractive Ginzburg-Landau equation with the cubic-quintic nonlinearity.* 2009.

[82] Wei, Winter: *On a cubic-quintic Ginzburg-Landau equation with global coupling.* Proc. Amer. Math. Soc. 133, pp. 1787–1796, 2005.

[83] Winterbottom: *The Complex Ginzburg-Landau equation.* http://codeinthehole.com/tutorials/cgl/index.html, School of Mathematical Sciences, University of Nottingham, 2002.

[84] Xu, Chang: *Homoclinic orbit for the cubic-quintic Ginzburg-Landau equation.* Chaos, Solitons and Fractals, Volume 10, Number 7, pp. 1161–1170(10), 1999.

[85] Yang, Zhabotinsky, Epstein: *Jumping solitary waves in an autonomous reaction-diffusion system with subcritical wave instability.* Phys. Chem. Chem. Phys., volume 8, pp. 4647–4651, 2006.

[86] Zeldovich, Frank-Kamenetsky: *A theory of thermal propagation of flame.* Acta Physicochim., Volume 9, pp. 341–350, 1938.

[87] Zykov: *Simulation of wave processes in excitable media.* Manchester University Press, 1987.

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