Rotating waves in parabolic systems

Spatial decay and spectral properties[1]

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1 Rotating patterns in $\mathbb{R}^d$

Reaction-diffusion system:

$$u_t(x,t) = A\triangle u(x,t) + f(u(x,t)),\ x\in\mathbb{R}^d,\ t\geq 0,\ d\geq 2, \qquad (1)$$

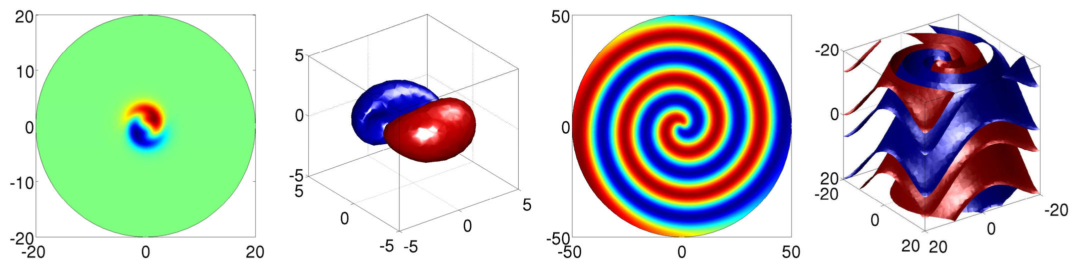
$u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$, $A\in\mathbb{R}^{m,m}$, $f:\mathbb{R}^m\to\mathbb{R}^m$.

Rotating wave: Special solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^m$ of (1) with

$$u_\star(x,t) = v_\star(e^{-tS_\star}(x-x_\star)),\ x\in\mathbb{R}^d,\ t\geq 0,$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^m$ pattern (profile), $S_\star\in\mathbb{R}^{d,d}$, $S_\star^T=-S_\star$ angular velocity matrix, $x_\star\in\mathbb{R}^d$ center of rotation.

Rotating patterns in various examples:



Co-rotating frame: $v(x,t)=u(e^{tS_\star}x+x_\star,t)$, $x\in\mathbb{R}^d$, $t\geq 0$, solves

$$v_t(x,t) = A\triangle v(x,t) + \langle S_\star x,\nabla v(x,t)\rangle + f(v(x,t)). \qquad (2)$$

Steady state equation: $v_\star$ stationary solution of (2), i.e.

$$A\triangle v_\star(x) + \langle S_\star x,\nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d. \qquad (3)$$

Ornstein-Uhlenbeck operator[2]**:**

$$\left[\mathcal{L}_0 v\right](x) = A\triangle v_\star(x) + \langle S_\star x,\nabla v_\star(x)\rangle,\ x\in\mathbb{R}^d,$$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \left\{v\in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^m)\cap L^p(\mathbb{R}^d,\mathbb{R}^m)\mid \mathcal{L}_0 v\in L^p(\mathbb{R}^d,\mathbb{R}^m)\right\},$$

with diffusion term and drift term

$$A\triangle v(x) = A\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}v(x),\quad \langle S_\star x,\nabla v(x)\rangle = \sum_{i=1}^d (S_\star x)_i\frac{\partial}{\partial x_i}v(x).$$

Drift term is rotational by skew-symmetry of $S_\star$

$$\langle S_\star x,\nabla v(x)\rangle = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d}(S_\star)_{ij}\left(x_j\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial x_j}\right)v(x).$$

Ornstein-Uhlenbeck semigroup:

$$\left[T(t)v\right](x) = \int_{\mathbb{R}^d} H(x,\xi,t)v(\xi)d\xi,\ x\in\mathbb{R}^d,\ t>0.$$

with Kolmogorov kernel[3]

$$H(x,\xi,t) = (4\pi tA)^{-\frac{d}{2}}\exp\left(-(4tA)^{-1}\left|e^{tS_\star}x-\xi\right|^2\right), x,\xi\in\mathbb{R}^d, t>0.$$

2 Spatial decay of rotating waves

Theorem 1 **(Exponential decay of $v_\star$).** *For every $0<\vartheta<1$ and every positive, radial, nondecreasing weight function $\theta\in C(\mathbb{R}^d,\mathbb{R})$ of exponential growth rate $\eta\geqslant 0$ with*

$$0\leq\eta^2\leq\vartheta\frac{2}{3}\frac{s(-A)\ s(Df(v_\infty))}{(\rho(A))^2\ p^2},\qquad s(A)\text{ spectral bound, } \rho(A)\text{ spectral radius,}$$

there exists $K_1>0$ such that:
Every classical solution $v_\star$ of (3) with $v_\star - v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^m)$ and

$$\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\leqslant K_1 \text{ for some } R_0>0$$

satisfies

$$v_\star - v_\infty\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{R}^m).$$

Weight function of exponential growth rate[4] **$\eta\geq 0$:** $\theta\in C(\mathbb{R}^d,\mathbb{R})$ with

$$\exists C_\theta>0:\ \theta(x+y)\leqslant C_\theta\theta(x)e^{\eta|y|}\ \forall x,y\in\mathbb{R}^d.$$

Exponentially weighted Sobolev spaces: $1\leq p\leq\infty$, $k\in\mathbb{N}_0$,

$$L^p_\theta(\mathbb{R}^d,\mathbb{R}^m) = \left\{v\in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^m)\mid \|\theta v\|_{L^p}<\infty\right\},$$

$$W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^m) = \left\{v\in L^p_\theta(\mathbb{R}^d,\mathbb{R}^m)\mid D^\beta v\in L^p_\theta(\mathbb{R}^d,\mathbb{R}^m)\ \forall|\beta|\leq k\right\}.$$

General assumptions:

- $A\in\mathbb{R}^{m,m}$ with $A>0$ for $m=1$ and for $m>1$

$$\mu_1(A) = \inf_{\substack{w\neq 0\\ Aw\neq 0}}\frac{\mathrm{Re}\langle w,Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p}\text{ for some } 1<p<\infty$$

  ($\mu_1(A)$ first antieigenvalue of A)

- $f\in C^2(\mathbb{R}^m,\mathbb{R}^m)$
- $v_\infty\in\mathbb{R}^m$, $f(v_\infty)=0$, $\mathrm{Re}\,\sigma(Df(v_\infty))<0$
- A and $Df(v_\infty)$ simultaneously diagonalizable (over $\mathbb{C}$)
- $0\neq S_\star\in\mathbb{R}^{d,d}$, $S_\star^T=-S_\star$

3 Outline of proof (Theorem 1)

1. Far-field linearization: In (3) expand $f(v_\star(x))$ into

$$\underbrace{f(v_\infty)}_{=0} + \Big(\underbrace{Df(v_\infty)}_{\text{stable part}} + \underbrace{\int_0^1 Df(v_\infty+tw_\star(x)) - Df(v_\infty)dt}_{=Q(x),\ Q\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^{m,m})}\Big)w_\star(x).$$

The difference $w_\star(x)=v_\star(x)-v_\infty$ satisfies

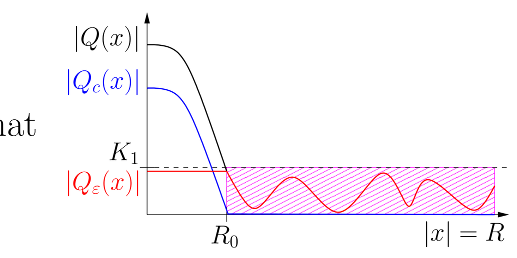
$$\left[\mathcal{L}_0 w_\star\right](x) + (Df(v_\infty)+Q(x))\,w_\star(x) = 0,\quad x\in\mathbb{R}^d.$$

2. Decomposition of variable coefficient Q: Decompose

$$Q(x) = Q_\varepsilon(x) + Q_c(x),\quad x\in\mathbb{R}^d$$

with $Q_\varepsilon, Q_c\in C_{\mathrm{b}}(\mathbb{R}^d,\mathbb{R}^{m,m})$ such that

Q_ε small w.r.t. $\|\cdot\|_{C_{\mathrm{b}}}$,
Q_c compactly supported on $\mathbb{R}^d$.



Result:

$$\left[\mathcal{L}_0 w_\star\right](x) + (Df(v_\infty)+Q_\varepsilon(x)+Q_c(x))\,w_\star(x) = 0,\ x\in\mathbb{R}^d.$$

Perturbed Ornstein-Uhlenbeck operators:

$$\left[\mathcal{L}_Q v\right](x) = \left[\mathcal{L}_0 v\right](x) + Df(v_\infty)v(x) + Q_\varepsilon(x)v(x) + Q_c(x)v(x)$$
$$\left[\mathcal{L}_{Q_\varepsilon} v\right](x) = \left[\mathcal{L}_0 v\right](x) + Df(v_\infty)v(x) + Q_\varepsilon(x)v(x)$$
$$\left[\mathcal{L}_\infty v\right](x) = \left[\mathcal{L}_0 v\right](x) + Df(v_\infty)v(x)$$

Exponential estimates in space

- Characterization of domain for $\mathcal{L}_0$
- Explicit heat kernel estimates for $\mathcal{L}_\infty$
- Small perturbation argument for $\mathcal{L}_{Q_\varepsilon}$
- $Q_c v$ treated as exponentially decaying right hand side of $\mathcal{L}_{Q_\varepsilon}$

4 Spectral properties of rotating waves

Linearized operator:

$$\left[\mathcal{L}v\right](x) = \left[\mathcal{L}_0 v\right](x) + Df(v_\star(x))v(x),\ x\in\mathbb{R}^d,\ d\geq 2.$$

Eigenvalue problem:

$$\left[\mathcal{L}v\right](x) = \lambda v(x),\ x\in\mathbb{R}^d. \qquad (4)$$

Spectrum of $\mathcal{L}$: $\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L})\,\dot\cup\,\sigma_{\mathrm{pt}}(\mathcal{L})$ with

$$\sigma_{\mathrm{pt}}(\mathcal{L}) = \{\lambda\in\sigma(\mathcal{L})\mid\lambda\text{ isolated with finite multiplicity}\},$$
$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \sigma(\mathcal{L})\backslash\sigma_{\mathrm{pt}}(\mathcal{L}),$$

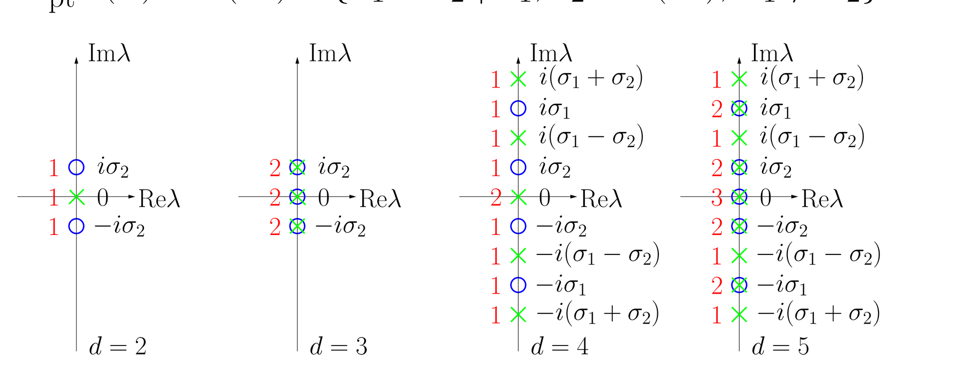
$\sigma_{\mathrm{pt}}(\mathcal{L})$ point spectrum, $\sigma_{\mathrm{ess}}(\mathcal{L})$ essential spectrum.

Theorem 2 **(Exponential decay of eigenfunctions v).** *Classical solutions $v\in L^p(\mathbb{R}^d,\mathbb{C}^m)$ of (4) for $\mathrm{Re}\,\lambda\geqslant -s(Df(v_\infty))+\varepsilon$ satisfy*

$$v\in W^{1,p}_\theta(\mathbb{R}^d,\mathbb{C}^m).$$

Theorem 3 **(Point spectrum in L^p on $i\mathbb{R}$).** $\sigma^{\mathrm{part}}_{\mathrm{pt}}(\mathcal{L})\subseteq\sigma_{\mathrm{pt}}(\mathcal{L})$,

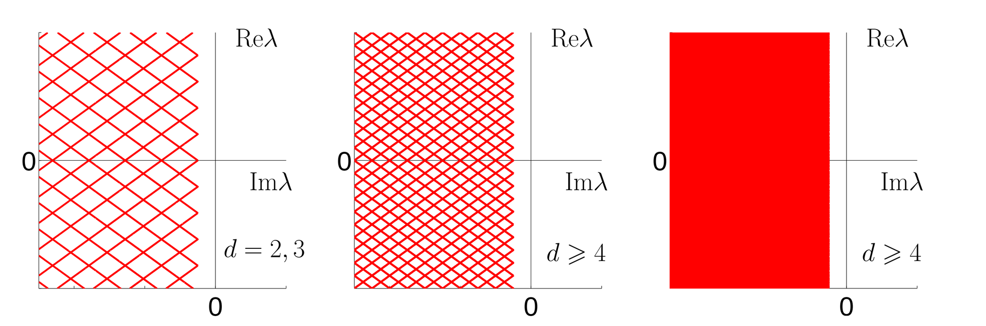
$$\sigma^{\mathrm{part}}_{\mathrm{pt}}(\mathcal{L}) = \sigma(S_\star)\cup\{\lambda_1+\lambda_2\mid\lambda_1,\lambda_2\in\sigma(S_\star),\ \lambda_1\neq\lambda_2\}.$$



Eigenfunctions: $v(x)=\langle Sx+\tau,\nabla v_\star(x)\rangle$ with $S\in\mathbb{C}^{d,d}$, $S^T=-S$, $\tau\in\mathbb{C}^d$. A total of $\frac{d(d+1)}{2}$ eigenvalues and eigenfunctions.

Theorem 4 **(Essential spectrum**[2,5] **in L^p).** $\sigma^{\mathrm{part}}_{\mathrm{ess}}(\mathcal{L})\subseteq\sigma_{\mathrm{ess}}(\mathcal{L})$,

$$\sigma^{\mathrm{part}}_{\mathrm{ess}}(\mathcal{L}) = \left\{-\lambda(\omega)+i\sum_{l=1}^k n_l\sigma_l\;\middle|\;\begin{array}{l}\lambda(\omega)\in\sigma\left(\omega^2A-Df(v_\infty)\right),\\ n_l\in\mathbb{Z},\ \omega\in\mathbb{R}\end{array}\right\}.$$



Density: $0\neq\pm i\sigma_l\in\sigma(S_\star)$, $l=1,\ldots,k\leq\frac{d}{2}$, then

$$\sigma^{\mathrm{part}}_{\mathrm{ess}}(\mathcal{L})\overset{\text{dense}}{\subseteq}\{\lambda\in\mathbb{C}\mid\mathrm{Re}\lambda\leq s(Df(v_\infty))\}\ \Leftrightarrow\ \exists\sigma_n,\sigma_m:\sigma_n\sigma_m^{-1}\notin\mathbb{Q}.$$

Dispersion relation[6]**:** $\lambda\in\sigma_{\mathrm{ess}}(\mathcal{L})$ if for some $\omega\in\mathbb{R}$, $n_l\in\mathbb{Z}$

$$\det\left(\lambda I_N+\omega^2A+i\sum_{l=1}^k n_l\sigma_l I_N - Df(v_\infty)\right) = 0.$$

5 Numerical computations of rotating waves, their spectra and eigenfunctions

Quintic-cubic Ginzburg-Landau equation:

$$u_t = \alpha\triangle u + \delta u + \beta|u|^2u + \gamma|u|^4u,\quad x\in\mathbb{R}^3,\ u(x,t)\in\mathbb{C},$$

with $\alpha,\ \beta,\ \gamma\in\mathbb{C}$, $\mathrm{Re}\,\alpha>0$, $\delta<0$.

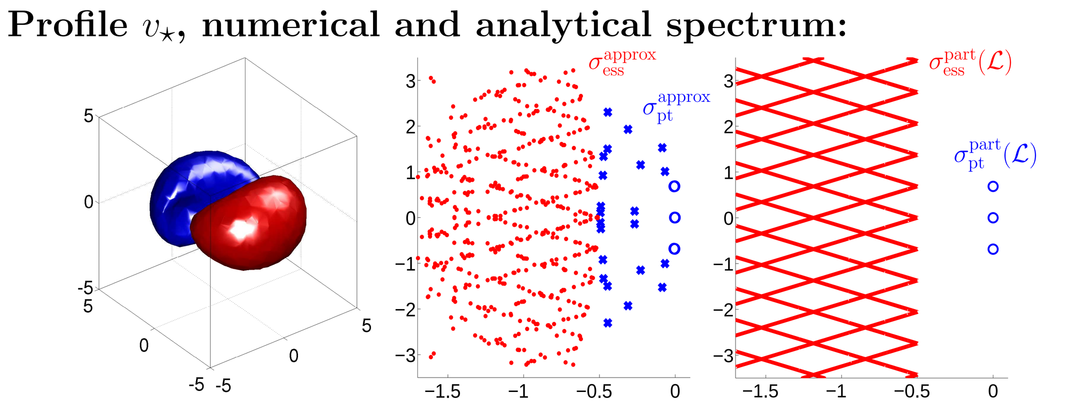
3D Spinning solitons: For parameters[7]

$$\alpha=\frac{1}{2}+\frac{1}{2}i,\quad\beta=\frac{5}{2}+i,\quad\gamma=-1-\frac{1}{10}i,\quad\delta=-\frac{1}{2}$$

solitons are exponentially localized by Theorem 1 with bound

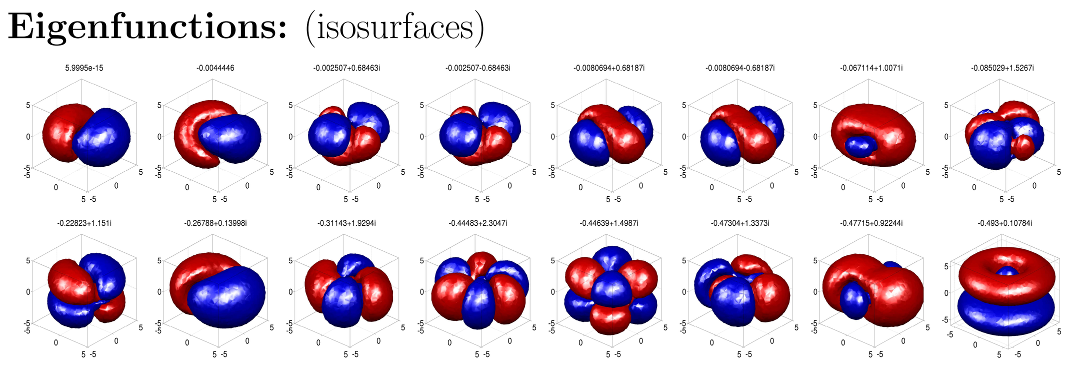
$$0\leq\eta^2\leq\vartheta\frac{1}{3p^2}<\frac{1}{3p^2}\quad\text{for}\quad p\in]4-2\sqrt{2},4+2\sqrt{2}[.$$

Profile $v_\star$, numerical and analytical spectrum:



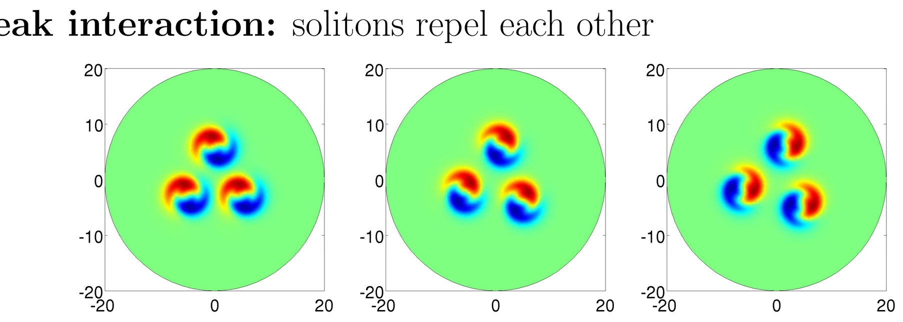
$$S_\star = \begin{pmatrix}0 & 0.6855 & -0.01558\\ -0.6855 & 0 & 0.01086\\ 0.01558 & -0.01086 & 0\end{pmatrix},\ x_\star = \begin{pmatrix}0.179489\\ 0.191649\\ -0.007199\end{pmatrix}$$

Eigenfunctions: (isosurfaces)

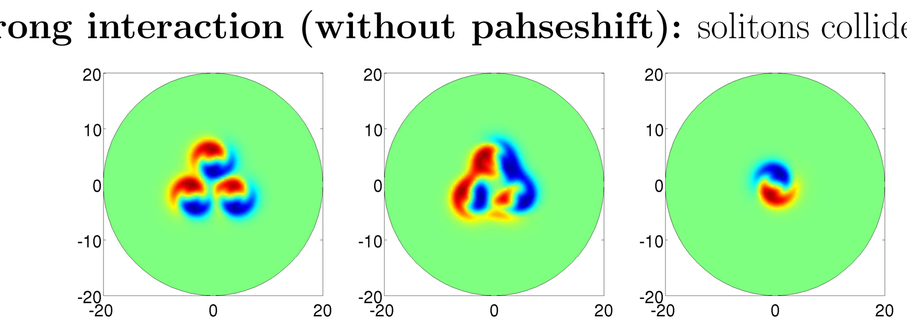


6 Interaction of rotating waves

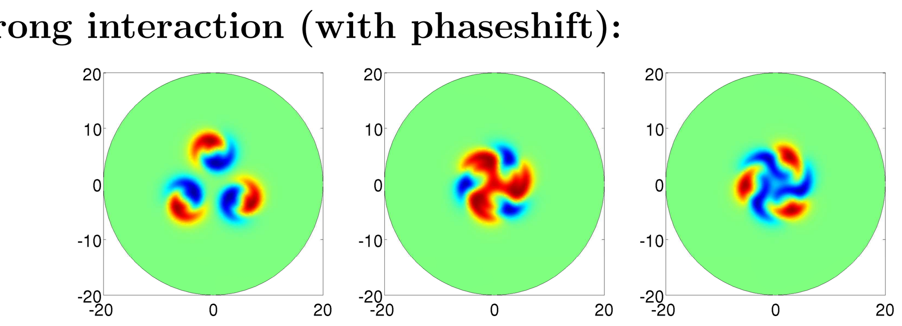
Weak interaction: solitons repel each other



Strong interaction (without pahseshift): solitons collide



Strong interaction (with phaseshift):



Aims

- Nonlinear stability of rotating waves[5] for $d\geq 3$
- Approximation theorem for rotating waves (on bounded domains)
- Discard assumption $v_\star - v_\infty\in L^p(\mathbb{R}^d,\mathbb{R}^m)$ in Theorem 1
- Exponential decay in space of bounded continuous functions
- Stability of freezing method[8] and decompose and freeze method[8]

References

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[2] Characterization and identification of maximal domain generalizes G. Metafune, D. Pallara, V. Vespri (*Houston J. Math.* 2005). For essential spectrum of drift term see G. Metafune (*Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 2001).

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[5] For essential spectrum for $d=p=2$ and nonlinear stability of rotating waves for $d=2$ see W.-J. Beyn, J. Lorenz (*Dyn. Partial Differ. Equ.* 2008).

[6] For spectra and dispersion relation for general spiral waves see B. Sandstede, A. Scheel (*Phys. Rev. E* 2000, *Phys. Rev. Lett.* 2001), B. Fiedler, A. Scheel (*Trends in Nonl. Anal.* Springer 2003).

[7] Parameters from L.-C. Crasovan, B.A. Malomed, D. Mihalache (*Pramana-journal of Physics* 2001)

[8] Freezing method cf. W.-J. Beyn, V. Thümmler (*SIAM J. Appl. Dyn. Syst.* 2004). Decompose and freeze method cf. W.-J. Beyn, S. Selle, V. Thümmler (*SIAM J. Appl. Dyn. Syst.* 2008)