Abstracts

Spatial decay of rotating waves in parabolic systems

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Consider a reaction diffusion system

$$(1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

with diffusion matrix $A\in\mathbb{R}^{N,N}$, nonlinearity $f\in C^2(\mathbb{R}^N,\mathbb{R}^N)$, initial data $u_0:\mathbb{R}^d\to\mathbb{R}^N$ and solution $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$.
A rotating wave of (1) is a special solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ of the form

$$u_\star(x,t) = v_\star(e^{-tS}x),$$

where $v_\star:\mathbb{R}^d\to\mathbb{R}^N$ is the profile (pattern) and $0\neq S\in\mathbb{R}^{d,d}$ is a skew-symmetric matrix. Examples of rotating waves are spiral waves, scroll waves, spinning solitons, etc.
If u solves (1) then the function $v(x,t)=u(e^{tS}x,t)$, transformed into a rotating frame, solves

$$(2)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

The linear operator is of Ornstein-Uhlenbeck type with an unbounded drift term containing angular derivatives

$$\langle Sx,\nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j\frac{\partial}{\partial x_i}v(x) = \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\right)v(x).$$

Observe that $v_\star$ is a stationary solution of (2), meaning that $v_\star$ solves

$$(3)\qquad A\triangle v(x) + \langle Sx,\nabla v(x)\rangle + f(v(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

Investigating steady state problems of this type is motivated by the stability theory of rotating patterns in several space dimensions, [1]. Equation (3) determines the shape and the angular speed of a rotating wave.
In this talk, we prove under certain conditions that every classical solution of (3) which falls below a certain threshold at infinity, must decay exponentially in space, meaning that the pattern is exponentially localized. This guarantees an exponentially small cut-off error if we restrict (3) to a bounded domain and justifies the numerical computation of rotating waves from boundary value problems on bounded domains.
We require $f(v_\infty)=0$ and $\mathrm{Re}\,\sigma\left(Df(v_\infty)\right)<0$ for some $v_\infty\in\mathbb{R}^N$. In addition to $\mathrm{Re}\,\sigma(A)>0$ we impose the cone-condition

$$|\mathrm{Im}\,\lambda|\,|p-2| \leqslant 2\sqrt{p-1}\mathrm{Re}\,\lambda \quad \forall\,\lambda\in\sigma(A) \text{ for some } 1<p<\infty$$

and assume that A, $Df(v_\infty) \in \mathbb{R}^{N,N}$ are simultaneously diagonalizable over $\mathbb{C}$. Further, we choose constants a_0, b_0, $a_{\max} > 0$ such that

$$a_0 \leqslant \operatorname{Re}\lambda,\ |\lambda| \leqslant a_{\max}\ \forall\,\lambda \in \sigma(A), \quad \operatorname{Re}\mu \leqslant -b_0 < 0\ \forall\,\mu \in \sigma(Df(v_\infty)).$$

Following [6], we call a positive function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ a weight function of exponential growth rate $\eta \geqslant 0$ provided that

$$\exists\, C_\theta > 0 :\ \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \quad \forall\, x, y \in \mathbb{R}^d.$$

Finally, the exponentially weighted Sobolev spaces for $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$ are defined by

$$L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^N) \mid \|\theta v\|_{L^p} < \infty \right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)\ \forall\,|\beta| \leqslant k \right\}.$$

Under these assumptions the following statement holds:

Theorem 1. *For every $1 < p < \infty$, $0 < \vartheta < 1$ and for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ with*

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

there exists $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ with the following property:
Every classical solution $v_\star$ of equation (3) such that $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and

$$\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N).$$

In this talk we present the main idea of the proof based upon a linearization at infinity, also known as far-field linearization. Our investigations of the associated Ornstein-Uhlenbeck operator generalizes the results of [3], [4]. We determine the maximal domain of the operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, analyze its constant and variable coefficient perturbations and derive resolvent estimates.
We apply the theory to the cubic-quintic complex Ginzburg-Landau equation

$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right), \quad u = u(x,t) \in \mathbb{C},$$

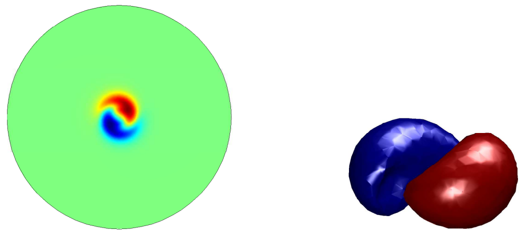
where $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{C}$, $d \in \{2, 3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called spinning soliton solutions, [2], see Figure 1. The solitons are localized in the sense of Theorem 1 with the bound

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{1}{3p^2} < \frac{1}{3p^2} \quad \text{for} \quad 2 \leqslant p \leqslant 6.$$

Details of the results may be found in the preprint [5] which forms the core of the authors' PhD thesis.



(a) $d = 2$ (b) $d = 3$

FIGURE 1. Spinning solitons of the Ginzburg-Landau equation

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