Spatial Decay and Spectral Properties of Rotating Waves in Evolution Equations

Patterns of Dynamics
Conference in Honor of Bernold Fiedler
Free University of Berlin, July 25-29, 2016

Denny Otten
Department of Mathematics
Bielefeld University
Germany

July 28, 2016


Denny Otten

Spatial decay and spectral properties of rotating waves

Berlin 2016
Congratulations to Bernold Fiedler

Dynamics of Patterns
MFO (Oberwolfach)
December 16-22, 2012

Organisers:
Wolf-Jürgen Beyn
Björn Sandstede
Bernold Fiedler

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?

---

You ask me: Did you ever investigated soliton interactions for shifted phases?
Congratulations to Bernold Fiedler

Dynamics of Patterns
MFO (Oberwohlfach)
December 16-22, 2012
Organisers:
Wolf-Jürgen Beyn
Björn Sandstede
Bernold Fiedler

Good news: Yes, I did it!

Congratulations to Bernold Fiedler

Dynamics of Patterns
MFO (Oberwohlfach)
December 16-22, 2012
Organisers:
Wolf-Jürgen Beyn
Björn Sandstede
Bernold Fiedler

Better news: I never published these results!

Congratulations to Bernold Fiedler

Best news: You now get a look into these results!

Congratulations to Bernold Fiedler

Dynamics of Patterns
MFO (Oberwohlfach)
December 16-22, 2012

Organisers:
Wolf-Jürgen Beyn
Björn Sandstede
Bernold Fiedler

... I even applied the decompose and freeze method to it!

Happy Birthday, Bernold!
Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. Spectral properties of linearization at rotating waves

4. Cubic-quintic complex Ginzburg-Landau equation
Outline

1 Rotating patterns in $\mathbb{R}^d$

2 Spatial decay of rotating waves

3 Spectral properties of linearization at rotating waves

4 Cubic-quintic complex Ginzburg-Landau equation
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

$$u_t(x, t) = A\triangle u(x, t) + f(u(x, t)), \quad t > 0, \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$u(x, 0) = u_0(x), \quad t = 0, \quad x \in \mathbb{R}^d.$$  \hfill (1)

where $u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$, $A \in \mathbb{R}^{N,N}$, $f : \mathbb{R}^N \to \mathbb{R}^N$, $u_0 : \mathbb{R}^d \to \mathbb{R}^N$.

Assume a rotating wave solution $u_* : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$ of (1)

$$u_*(x, t) = v_*(e^{-tS} x)$$

$v_* : \mathbb{R}^d \to \mathbb{R}^N$ profile (pattern), $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

$$v_t(x, t) = A\triangle v(x, t) + (Sx, \nabla v(x, t)) + f(v(x, t)), \quad t > 0, \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$v(x, 0) = u_0(x)$$ \hfill (2)

$$\langle Sx, \nabla v(x) \rangle = Dv(x)Sx = \sum_{i=1}^{d} \sum_{j=1}^{d} S_{ij} x_j D_i v(x) \overset{s = S^T}{=} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij} (x_j D_i - x_i D_j) v(x)$$

(drift term)

(rotational term)
Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

\[
\begin{align*}
    u_t(x, t) &= A \triangle u(x, t) + f(u(x, t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
    u(x, 0) &= u_0(x), \ t = 0, \ x \in \mathbb{R}^d. 
\end{align*}
\]

(1)

where $u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$, $A \in \mathbb{R}^{N,N}$, $f : \mathbb{R}^N \to \mathbb{R}^N$, $u_0 : \mathbb{R}^d \to \mathbb{R}^N$.

Assume a **rotating wave solution** $u^* : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$ of (1)

\[ u^*(x, t) = v^*(e^{-tS} x) \]

$v^* : \mathbb{R}^d \to \mathbb{R}^N$ profile (pattern), $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS} x, t)$ solves

\[
\begin{align*}
    v_t(x, t) &= A \triangle v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \ t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
    v(x, 0) &= u_0(x), \ t = 0, \ x \in \mathbb{R}^d. 
\end{align*}
\]

(2)
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

\[ u_t(x, t) = A \Delta u(x, t) + f(u(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \]
\[ u(x, 0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \]

(1)

where $u : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^N$, $A \in \mathbb{R}^{N \times N}$, $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^N$.

Assume a rotating wave solution $u_* : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^N$ of (1)

\[ u_*(x, t) = v_*(e^{-tS}x) \]

$v_* : \mathbb{R}^d \rightarrow \mathbb{R}^N$ profile (pattern), $0 \neq S \in \mathbb{R}^{d \times d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

\[ v_t(x, t) = A \Delta v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \]
\[ v(x, 0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \]

(2)

\[
\langle Sx, \nabla v(x) \rangle = Dv(x)Sx = \sum_{i=1}^{d} \sum_{j=1}^{d} S_{ij}x_jD_i v(x) \quad -s\equiv s^\top \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} S_{ij} (x_jD_i - x_iD_j) v(x)
\]

(drift term) (rotational term)
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

\[ u_t(x, t) = A \triangle u(x, t) + f(u(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \]

\[ u(x, 0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \]

(1)

where $u : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^N$, $A \in \mathbb{R}^{N \times N}$, $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^N$.

Assume a rotating wave solution $u_* : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{R}^N$ of (1)

\[ u_*(x, t) = v_*(e^{-tS}x) \]

$v_* : \mathbb{R}^d \rightarrow \mathbb{R}^N$ profile (pattern), $0 \neq S \in \mathbb{R}^{d \times d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

\[ v_t(x, t) = A \triangle v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \]

\[ v(x, 0) = u_0(x), \quad t = 0, \ x \in \mathbb{R}^d. \]

(2)

Note: $v_*$ is a stationary solution of (2), i.e. $v_*$ solves the rotating wave equation

\[ A \triangle v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2. \]

$A \triangle v_*(x) + \langle Sx, \nabla v_*(x) \rangle$: Ornstein-Uhlenbeck operator.
Rotating Patterns in $\mathbb{R}^d$

Consider a reaction diffusion system

$$
\begin{align*}
    u_t(x, t) &= A \triangle u(x, t) + f(u(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
    u(x, 0) &= u_0(x), \quad t = 0, \ x \in \mathbb{R}^d.
\end{align*}
$$

(1)

where $u : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$, $A \in \mathbb{R}^{N,N}$, $f : \mathbb{R}^N \to \mathbb{R}^N$, $u_0 : \mathbb{R}^d \to \mathbb{R}^N$.

Assume a rotating wave solution $u_* : \mathbb{R}^d \times [0, \infty[ \to \mathbb{R}^N$ of (1)

$$
u_*(x, t) = \nu_*(e^{-tS} x)
$$

$v_* : \mathbb{R}^d \to \mathbb{R}^N$ profile (pattern), $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS} x, t)$ solves

$$
\begin{align*}
    v_t(x, t) &= A \triangle v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \quad t > 0, \ x \in \mathbb{R}^d, \ d \geq 2, \\
    v(x, 0) &= u_0(x), \quad t = 0, \ x \in \mathbb{R}^d.
\end{align*}
$$

(2)

Question: How to show exponential decay of $v_*$ at $|x| = \infty$?

Consequence: Exponentially small error by truncation to bounded domain.
Examples for rotating waves

**Cubic-quintic complex Ginzburg-Landau equation:** (spinning solitons)

\[ u_t = \alpha \Delta u + u \left( \delta + \beta |u|^2 + \gamma |u|^4 \right) \]

\[ u(x, t) \in \mathbb{C}, \ x \in \mathbb{R}^d, \ t \geq 0, \ \alpha, \beta, \gamma \in \mathbb{C}, \ \text{Re} \alpha > 0, \ 
\delta \in \mathbb{R}, \ d \in \{2, 3\}. \]

**\(\lambda\)-\(\omega\) system:** (spiral waves, scroll waves)

\[ u_t = \alpha \Delta u + \left( \lambda(|u|^2) + i \omega(|u|^2) \right) u \]

\[ u(x, t) \in \mathbb{C}, \ x \in \mathbb{R}^d, \ t \geq 0, \ \lambda, \omega : [0, \infty[ \rightarrow \mathbb{R}, \ 
\alpha \in \mathbb{C}, \ \text{Re} \alpha > 0, \ d \in \{2, 3\}. \]

**Barkley model:** (spiral waves, also scroll waves)

\[ u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \Delta u + \left( \frac{1}{\varepsilon} u_1(1 - u_1)(u_1 - \frac{u_2 + b}{a}) \right) \]

\[ u_1 - u_2 \]

with \( u(x, t) \in \mathbb{R}^2, \ x \in \mathbb{R}^d, \ t \geq 0, \ 0 \leq D \ll 1, \ 
\varepsilon, a, b > 0. \)
Nonlinear stability of rotating waves for $d = 2$:

W.-J. Beyn, J. Lorenz.

Ginzburg-Landau equation:

L.D. Landau, V.L. Ginzburg.

L.-C. Crasovan, B.A. Malomed, D. Mihalache.

A. Mielke.
The Ginzburg-Landau equation in its role as a modulation equation, 2002.

$\lambda$-$\omega$ system:

Y. Kuramoto, S. Koga.
Turbulized rotating chemical waves, 1981.

J. D. Murray.

Barkley model:

D. Barkley.
Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. Spectral properties of linearization at rotating waves

4. Cubic-quintic complex Ginzburg-Landau equation
Spatial decay of rotating waves

Theorem 1: (Exponential decay of $v_*$)

Let $f \in C^2 \left( \mathbb{R}^N, \mathbb{R}^N \right)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp \left( \mu \sqrt{|x|^2 + 1} \right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_* \in C^2 \left( \mathbb{R}^d, \mathbb{R}^N \right)$ of

(RWE) $A \Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \; x \in \mathbb{R}^d,$

such that

(TC) $\sup_{|x| \geq R_0} |v_*(x) - v_\infty| \leq K_1$ for some $R_0 > 0$

satisfies

$v_* - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$

for every exponential decay rate

$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$

\[
\begin{pmatrix}
\ a_{\max} &=& \rho(A) \\
- a_0 &=& s(-A) \\
- b_0 &=& s(Df(v_\infty))
\end{pmatrix}
\]

: spectral radius of $A$

: spectral bound of $-A$

: spectral bound of $Df(v_\infty)$
Spatial decay of rotating waves

**Theorem 1: (Exponential decay of \(v_\star\))**

Let \(f \in C^2(\mathbb{R}^N, \mathbb{R}^N), v_\infty \in \mathbb{R}^N, f(v_\infty) = 0, Df(v_\infty) \leq -\beta_\infty I < 0,\)

assume (A1)-(A3) for some \(1 < p < \infty,\)

and let \(\theta(x) = \exp(\mu \sqrt{|x|^2 + 1})\) be a weight function for \(\mu \in \mathbb{R}.\)

Then for every \(0 < \varepsilon < 1\) there exists \(K_1 = K_1(\varepsilon) > 0\) with the following property:

Every classical solution \(v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)\) of

\[
A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,
\]

such that

\[
\text{(TC)} \quad \sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \quad \text{for some } R_0 > 0
\]

satisfies

\[
v_\star - v_\infty \in W^{2,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)
\]

for every exponential decay rate

\[
0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.
\]

\[
\begin{pmatrix}
a_{\max} \\
-a_0 \\
-b_0
\end{pmatrix} = \begin{pmatrix}
\rho(A) \\
s(-A) \\
s(Df(v_\infty))
\end{pmatrix} : \text{spectral radius of } A, s(-A), s(Df(v_\infty)) : \text{spectral bound of } -A, Df(v_\infty) \]
Spatial decay of rotating waves

Theorem 1: (Exponential decay of $v_\star$: higher regularity)

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,$$

such that

$$\sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W_0^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}.$$

\[
\begin{pmatrix}
  a_{\max} & = & \rho(A) \\
  -a_0 & = & s(-A) \\
  -b_0 & = & s(Df(v_\infty))
\end{pmatrix}
\]

: spectral radius of $A$

: spectral bound of $-A$

: spectral bound of $Df(v_\infty)$
**Theorem 1: (Exponential decay of $v_*$: pointwise estimates)**

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_* \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ of

(RWE) \hspace{1cm} A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, x \in \mathbb{R}^d,

such that

(TC) \hspace{1cm} \sup_{|x| \geq R_0} |v_*(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0

satisfies

$v_* - v_\infty \in W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$, $|D^\alpha (v_*(x) - v_\infty)| \leq C \exp\left(-\mu \sqrt{|x|^2 + 1}\right) \forall x \in \mathbb{R}^d$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$$

$$\left( \begin{array}{c} a_{\max} = \rho(A) \text{ : spectral radius of } A \\ -a_0 = s(-A) \text{ : spectral bound of } -A \\ -b_0 = s(Df(v_\infty)) \text{ : spectral bound of } Df(v_\infty) \end{array} \right)$$

and for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$. 
Theorem 2: (Exponential decay of eigenfunctions $v$)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta_j(x) = \exp \left( \mu_j \sqrt{|x|^2 + 1} \right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j = 1, 2$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 2$).

Then for every $0 < \epsilon < 1$ there exists $K_1 = K_1(\epsilon) > 0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ of (RWE) satisfying (TC) the following property holds: Every classical solution $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^N)$ of

\[
A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\star(x))v(x) = \lambda v(x), \; x \in \mathbb{R}^d,
\]

with $\lambda \in \mathbb{C}$, $\Re \lambda \geq -(1 - \epsilon)\beta_\infty$, such that

\[
v \in L^p_{\theta_1}(\mathbb{R}^d, \mathbb{C}^N) \quad \text{for some exp. decay rate} \quad -\sqrt{\epsilon \frac{\gamma A \beta_\infty}{2d |A|^2}} \leq \mu_1 < 0
\]
satisfies

\[
v \in W^{k,p}_{\theta_2}(\mathbb{R}^d, \mathbb{C}^N) \quad \text{for every exp. decay rate} \quad 0 \leq \mu_2 \leq \epsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}
\]

and

\[
|D^\alpha v(x)| \leq C \exp \left( -\mu_2 \sqrt{|x|^2 + 1} \right) \; \forall \; x \in \mathbb{R}^d
\]

for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$. 

Denny Otten
Spatial decay and spectral properties of rotating waves
Berlin 2016
Exponentially weighted Sobolev spaces and assumptions

**Exponentially weighted Sobolev spaces:** For $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp \left( \mu \sqrt{|x|^2 + 1} \right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^N) \mid \| \theta v \|_{L^p} < \infty \right\},$$

$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \forall |\beta| \leq k \right\}.$$

**Assumptions:**

**(A1)** (*$L^p$*-dissipativity condition): For $A \in \mathbb{R}^{N,N}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \text{Re} \langle w, Aw \rangle + (p-2) \text{Re} \langle w, z \rangle \text{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \forall z, w \in \mathbb{R}^N$$

**(A2)** (*System condition*): $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ simultaneously diagonalizable over $\mathbb{C}$

**(A3)** (*Rotational condition*): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^T$

**Note:** Assumption (A1) is equivalent with

**(A1')** (*$L^p$*-antieigenvalue condition): $A \in \mathbb{R}^{N,N}$ is invertible and

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{R}^N \\text{Re} \langle w, Aw \rangle \neq 0 \\text{Re} \langle w, Aw \rangle \neq 0 \\text{Re} \langle w, Aw \rangle \neq 0}} \frac{\text{Re} \langle w, Aw \rangle}{|w||Aw|} > \frac{|p-2|}{p} \text{ for some } 1 < p < \infty$$

($\mu_1(A)$: first antieigenvalue of $A$)

(to be read as $A > 0$ in case $N = 1$).
Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$ 

1. **Far-Field Linearization:** $f \in C^1$, Taylor’s theorem, $f(v_\infty) = 0$

$$a(x) := \int_0^1 Df(v_\infty + tw_\star(x)) dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + a(x)w_\star(x) = 0, \quad x \in \mathbb{R}^d.$$
Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

Consider the nonlinear problem

$$A \Delta v_\star(x) + \langle Sv_\star(x), \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d, \ d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty) dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A \Delta w_\star(x) + \langle Sv_\star(x), \nabla w_\star(x) \rangle + (Df(v_\infty) + Q(x)) w_\star(x) = 0, \quad x \in \mathbb{R}^d.$$
Outline of proof: Theorem 1 (Exponential Decay of $v_*$)

Consider the nonlinear problem

$$A \triangle v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A \triangle w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \ x \in \mathbb{R}^d.$$ 

3. Decomposition of $Q$:

$$Q(x) = Q_s(x) + Q_c(x),$$

$Q, Q_s, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$, $Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1$, $Q_c$ compactly supported.
Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

Consider the nonlinear problem

$$A \Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d, \ d \geq 2.$$  

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A \Delta w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$  

3. Decomposition of $Q$:  

$$Q(x) = Q_s(x) + Q_c(x),$$

$Q, Q_s, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N}),$

$Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1,$

$Q_c$ compactly supported.
Outline of proof: Theorem 1 (Exponential Decay of $v_*$)

Consider the nonlinear problem

$$A \Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$ 

2. Decomposition of $a$: Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A \Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$ 

3. Decomposition of $Q$:

$$Q(x) = Q_s(x) + Q_c(x),$$

$Q, Q_s, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

$Q_s$ small, i.e. $\|Q_s\|_{L^\infty} < K_1$,

$Q_c$ compactly supported.
Exponential Decay: To show exponential decay for the solution $v_*$ of

$$A\nabla v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \ x \in \mathbb{R}^d,$$

investigate the linear system $(w_*(x) := v_*(x) - v_\infty)$

$$A\nabla w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \ x \in \mathbb{R}^d.$$

Operators: Study the following operators

$$\mathcal{L}_c v := A\nabla v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v + Q_c v, \quad \text{(exp. decay)}$$

$$\mathcal{L}_s v := A\nabla v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v, \quad \text{(exp. decay)}$$

$$\mathcal{L}_\infty v := A\nabla v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v, \quad \text{(far-field operator)} \quad \text{(exp. decay)}$$

$$\mathcal{L}_0 v := A\nabla v + \langle S \cdot, \nabla v \rangle. \quad \text{(Ornstein-Uhlenbeck operator)} \quad \text{(max. domain)}$$

D. Otten.
- The identification problem for complex-valued Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, 2016.

W.-J. Beyn, D. Otten.
Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A \triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.$$

Operators: Study the following operators

$$\mathcal{L}_c v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v + Q_c v,$$  
(exp. decay)

$$\mathcal{L}_s v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v,$$  
(exp. decay)

$$\mathcal{L}_\infty v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v, \quad \text{(far-field operator)}$$  
(exp. decay)

$$\mathcal{L}_0 v := A \triangle v + \langle S \cdot, \nabla v \rangle. \quad \text{(Ornstein-Uhlenbeck operator)}$$  
(max. domain)

---

D. Otten.
The identification problem for complex-valued Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, 2016.

W.-J. Beyn, D. Otten.
Exponential Decay: To show exponential decay for the solution \( v_\star \) of

\[
A \triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \ x \in \mathbb{R}^d,
\]

investigate the linear system \((w_\star(x) := v_\star(x) - v_\infty))

\[
A \triangle w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_\star(x) = 0, \ x \in \mathbb{R}^d.
\]

Operators: Study the following operators

\[
\mathcal{L}_c v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v + Q_c v, \quad (\text{exp. decay})
\]

\[
\mathcal{L}_s v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v + Q_s v, \quad (\text{exp. decay})
\]

\[
\mathcal{L}_\infty v := A \triangle v + \langle S \cdot, \nabla v \rangle + Df(v_\infty) v, \quad (\text{far-field operator}) \quad (\text{exp. decay})
\]

\[
\mathcal{L}_0 v := A \triangle v + \langle S \cdot, \nabla v \rangle. \quad (\text{Ornstein-Uhlenbeck operator}) \quad (\text{max. domain})
\]

Maximal domain of \( \mathcal{L}_0 \) given by

\[
\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) = \{ v \in W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) : \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \}, \ 1 < p < \infty
\]

satisfies \( \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N) \).
References

Nonlinear stability of rotating waves for $d = 2$:
- W.-J. Beyn, J. Lorenz.

Exponential decay of traveling waves:
- M. Shub.

Ornstein-Uhlenbeck operator in $L^p(\mathbb{R}^d, \mathbb{R})$ and its identification problem:
  $L^p$-estimates for a class of elliptic operators with unbounded coefficients in $\mathbb{R}^N$, 2005.
- G. Metafune.
  Feller semigroups on $\mathbb{R}^N$, 2002.

Ornstein-Uhlenbeck operator in $C_b(\mathbb{R}^d, \mathbb{R})$ and its identification problem:
- G. Da Prato, A. Lunardi.

Weight function of exponential growth rate:
  Multi-pulse evolution and space-time chaos in dissipative systems, 2009.

Semigroup theory:
- K.-J. Engel, R. Nagel.
References

$L^p$-dissipativity:

A. Cialdea, V. Maz'ya.

A. Cialdea

Antieigenvalues:

K. Gustafson.

K. Gustafson, M. Seddighin.
On the eigenvalues which express antieigenvalues, 2005.
A note on total antieigenvectors, 1993.
Antieigenvalue bounds, 1989.

Rotating waves:

C. Wulff.
Theory of meandering and drifting spiral waves in reaction-diffusion systems, 1996.

B. Fiedler, A. Scheel.

B. Fiedler, B. Sandstede, A. Scheel, C. Wulff.
Bifurcation from relative equilibria of noncompact group actions: skew products, meanders, and shifts, 1996.
Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. Spectral properties of linearization at rotating waves

4. Cubic-quintic complex Ginzburg-Landau equation
**Eigenvalue problem for linearization at rotating waves**

**Motivation:** Stability is determined by spectral properties of linearization $\mathcal{L}$.

**Linearization** at the profile $v_\star$ of the rotating wave

$$[\mathcal{L} v](x) = A \Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\star(x))v(x), \ x \in \mathbb{R}^d, \ d \geq 2.$$  

**Eigenvalue problem**

$$[(\lambda I - \mathcal{L}) v](x) = 0, \ x \in \mathbb{R}^d, \ d \geq 2, \ \lambda \in \mathbb{C}.$$  

A rotating wave $u_\star(x, t) = v_\star(e^{-tS}x)$ is called **strongly spectrally stable**

$$\left\{ \begin{array}{ll} \text{Re } \sigma(\mathcal{L}) \leq 0 \text{ (spectrally stable)} \\ \text{and} \\ \forall \lambda \in \sigma(\mathcal{L}) \text{ with Re } \lambda = 0 : \ \lambda \text{ is caused by the SE}(d)\text{-group action.} \end{array} \right.$$  

Decomposition of the **spectrum** $\sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$ into

$$\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_{\text{pt}}(\mathcal{L}),$$

with

$$\sigma_{\text{pt}}(\mathcal{L}) := \{ \lambda \in \sigma(\mathcal{L}) \mid \lambda \text{ isolated with finite multiplicity} \},$$

$$\sigma_{\text{ess}}(\mathcal{L}) := \sigma(\mathcal{L}) \setminus \sigma_{\text{pt}}(\mathcal{L}).$$
Illustration: Point spectrum of $\mathcal{L}$ on the imaginary axis

$\lambda \in (\sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S), \lambda_1 \neq \lambda_2\}) \subseteq \sigma_{pt}(\mathcal{L})$ & algebraic multiplicity

$\operatorname{dim} \text{SE}(2) = 3$
$\operatorname{dim} \text{SE}(3) = 6$
$\operatorname{dim} \text{SE}(4) = 10$
$\operatorname{dim} \text{SE}(5) = 15$
Theorem 3: (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$ and shape of eigenfunctions)

Let $S \in \mathbb{R}^{d,d}$, $S = -S^\top$, with eigenvalues $\lambda_1^S, \ldots, \lambda_d^S$ of $S$, and let $U \in \mathbb{C}^{d,d}$ be unitary satisfying $\Lambda_S = U^* S U$ with $\Lambda_S = \text{diag}(\lambda_1^S, \ldots, \lambda_d^S)$. Moreover, let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of (RWE). Then, $v : \mathbb{R}^d \to \mathbb{C}^N$ defined by

$$v(x) = \langle Qx + b, \nabla v_\star(x) \rangle = Dv_\star(x)(Qx + b), \ x \in \mathbb{R}^d, \ Q \in \mathbb{C}^{d,d}, \ b \in \mathbb{C}^d$$

is a classical solution of $(\lambda I - \mathcal{L})v = 0$ if either

$$\lambda = -\lambda_i^S, \ Q = 0, \ b = U e_l$$

for some $l = 1, \ldots, d$, or

$$\lambda = -(\lambda_i^S + \lambda_j^S), \ Q = U(I_{ij} - I_{ji})U^\top, \ b = 0$$

for some $i = 1, \ldots, d - 1$ and $j = i + 1, \ldots, d$.

- $\dim \text{SE}(d) = \frac{d(d+1)}{2}$ eigenfunctions of $\mathcal{L}$ and their explicit representation,
- $\sigma_{\text{pt}}\text{SE}(\mathcal{L}) := \sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S), \ \lambda_1 \neq \lambda_2\} \subseteq \sigma(\mathcal{L})$,
- $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geq 2$.
- point spectrum on imaginary axis is determined by the $\text{SE}(d)$–group action,
- Theorem also valid for spiral waves, scroll waves, scroll rings.

Denny Otten
Spatial decay and spectral properties of rotating waves
Berlin 2016
Properties of linearization at localized rotating waves

Theorem 4: (Fredholm properties of \( \mathcal{L} \) and decay of eigenfunctions)

Let \( f \in C^{\max\{2,k\}}(\mathbb{R}^N, \mathbb{R}^N) \), \( v_\infty \in \mathbb{R}^N \), \( f(v_\infty) = 0 \), \( Df(v_\infty) \leq -\beta_\infty I < 0 \), assume (A1)-(A3) for some \( 1 < p < \infty \), and let \( \theta_j(x) = \exp \left( \mu_j \sqrt{|x|^2 + 1} \right) \) be a weight function for \( \mu_j \in \mathbb{R}, j = 1, 2, k \in \mathbb{N}, p \geq \frac{d}{2} \) (if \( k \geq 2 \)). Then for every \( 0 < \varepsilon < 1 \) there exists \( K_1 = K_1(\varepsilon) > 0 \) such that for every classical solution \( v_* \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N) \) of (RWE) satisfying (TC) the following properties hold:

1. (Fredholm properties). The operator

\[
\lambda I - \mathcal{L} : (\mathcal{D}_{0}\mathcal{L}_0, \| \cdot \|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^N), \| \cdot \|_{L^p})
\]

is Fredholm of index 0.
Properties of linearization at localized rotating waves

Theorem 4: (Fredholm properties of $\mathcal{L}$ and decay of eigenfunctions)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^N, \mathbb{R}^N)$, $\nu_\infty \in \mathbb{R}^N$, $f(\nu_\infty) = 0$, $Df(\nu_\infty) \leq -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta_j(x) = \exp \left( \mu_j \sqrt{|x|^2 + 1} \right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j = 1, 2$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 2$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $\nu_\star \in C^{k+1}((\mathbb{R}^d, \mathbb{R}^N)$ of (RWE) satisfying (TC) the following properties hold:

- (Solvability of resolvent equation). There exist exactly $n$ (nontrivial lin. ind.) eigenfunctions $v_j \in D^p_{loc}(\mathcal{L}_0)$ and adjoint eigenfunctions $\psi_j \in D^q_{loc}(\mathcal{L}_0^*)$ with $(\lambda I - \mathcal{L})v_j = 0$ and $(\lambda I - \mathcal{L})^*\psi_j = 0$ for $j = 1, \ldots, n$.

Moreover,

$$(\lambda I - \mathcal{L})v = h, \quad h \in L^p(\mathbb{R}^d, \mathbb{C}^N)$$

has at least one (not necessarily unique) solution $v \in D^p_{loc}(\mathcal{L}_0)$ if and only if $h \in (\mathcal{N}(\lambda I - \mathcal{L})^*)^\perp$, i.e. $\langle \psi_j, h \rangle_{q,p} = 0$, $j = 1, \ldots, n$. 

Denny Otten
Spatial decay and spectral properties of rotating waves
Berlin 2016
Illustration: Essential spectrum of \( \mathcal{L} \)

\[
\left\{ -\lambda(\omega) + i \sum_{l=1}^{k} n_l \sigma_l \mid \lambda(\omega) \text{ eigenvalue of } \omega^2 A - Df(v_{\infty}) \right\} \subseteq \sigma_{\text{ess}}(\mathcal{L})
\]

\( \pm i \sigma_1, \ldots, \pm i \sigma_k \) nonzero eigenvalues of \( S \in \mathbb{R}^{d,d} \), \( -S = S^\top \), \( n_l \in \mathbb{Z} \), \( \omega \in \mathbb{R} \)

\[\text{Parameters for illustration: } A = \frac{1}{2} + \frac{1}{2} i, \ Df(v_{\infty}) = -\frac{1}{2}, \]

\[\sigma_1 = 1.027 \quad \sigma_1 = 1 \quad \sigma_1 = \frac{1}{\exp(1)} \]

\( \sigma_2 = 1.5 \quad \sigma_2 = \frac{\exp(1)}{2} \)

\( \sigma_{\text{part}}(\mathcal{L}) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \leq s(Df(v_{\infty})) \} \) dense \iff \exists \sigma_n, \sigma_m: \sigma_n \sigma_m^{-1} \notin \mathbb{Q} \).
Essential spectrum of $\mathcal{L}$

Dispersion relation: $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$ if $\lambda \in \mathbb{C}$ satisfies

$$(\text{DR}) \quad \det \left( \lambda I_N + \omega^2 A + i \sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty) \right) = 0 \text{ for some } \omega \in \mathbb{R}, \; n \in \mathbb{Z}^k.$$ 

Theorem 5: (Essential spectrum of $\mathcal{L}$)

Assume $f \in C^{\max\{2, r-1\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I < 0$, (A1)-(A3) for some $1 < p < \infty$, and $\frac{d}{p} \leq r$ (if $r \geq 2$) or $\frac{d}{p} \leq 2$ (if $r \geq 3$). Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of $S$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_* \in C^{r+1}(\mathbb{R}^d, \mathbb{R}^N)$ of (RWE) satisfying (TC) the following property holds:

$$\sigma_{\text{part}}(\mathcal{L}) := \left\{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR)} \right\} \subseteq \sigma_{\text{ess}}(\mathcal{L}) \text{ in } L^p(\mathbb{R}^d, \mathbb{C}^N).$$

- essential spectrum is determined by the far-field linearization
- only for exponentially localized rotating waves, but not for nonlocalized waves (e.g. spiral waves, sroll waves)
- theory e.g. for spiral waves much more involved ($\rightarrow$ Floquet theory)
References

Spectrum at 2-dimensional localized rotating waves:
- W.-J. Beyn, J. Lorenz.

Spectrum of drift term:
- G. Metafune.

Spectrum at spiral and scroll waves:
- B. Sandstede, A. Scheel.
  Absolute and convective instabilities of waves on unbounded and large bounded domains, 2000.
- B. Fiedler, A. Scheel.
Outline

1. Rotating patterns in $\mathbb{R}^d$

2. Spatial decay of rotating waves

3. Spectral properties of linearization at rotating waves

4. Cubic-quintic complex Ginzburg-Landau equation
Example
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

\[ u_t = \alpha \triangle u + u \left( \mu + \beta |u|^2 + \gamma |u|^4 \right), \quad u = u(x, t) \in \mathbb{C} \]

with \( u : \mathbb{R}^d \times [0, \infty[ \rightarrow \mathbb{C} \), \( d \in \{2, 3\} \). For the parameters

\[ \alpha = \frac{1}{2} + \frac{1}{2} i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10} i, \quad \mu = -\frac{1}{2} \]

this equation exhibits so called **spinning soliton** solutions.

Freezing method implies numerical results for profile \( v_* \) and velocities \( S \).
Spatial decay of a spinning soliton in QCGL for $d = 3$: Assume

\[ \text{Re} \alpha > 0, \quad \text{Re} \delta < 0, \quad p_{\min} = \frac{2|\alpha|}{|\alpha| + \text{Re} \alpha} < p < \frac{2|\alpha|}{|\alpha| - \text{Re} \alpha} = p_{\max} \]

Decay rate of spinning soliton:

\[ 0 \leq \mu < \frac{\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha|p} =: \mu^{\text{pro}}(p) < \frac{\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha| \max\{p_{\min}, \frac{d}{2}\}} =: \mu^{\text{pro}}_{\max} \]

Parameters:

\[ \alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \]

\[ \mu = -\frac{1}{2}, \quad v_{\infty} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_0 = \text{Re} \alpha, \]

\[ a_{\max} = |\alpha|, \quad b_0 = \beta_{\infty} = -\text{Re} \delta = -\frac{1}{2}, \]

Numerical vs. theoretical decay rate: ($p = 2$)

\[ \text{NDR} \approx 0.5387, \quad \text{TDR} = \mu^{\text{pro}}_{\max} = \frac{\sqrt{2}}{4} \approx 0.4714. \]
**Spectrum** of QCGL for a spinning soliton with \( d = 3 \): (numerical vs. analytical)

**Point spectrum** on \( i\mathbb{R} \) and **essential spectrum** by dispersion relation:

\[
\sigma_{\text{ess}}(\mathcal{L}) = \{\lambda = -\omega^2\alpha_1 + \delta_1 + i(\mp \omega^2\alpha_2 \pm \delta_2 - n\sigma_1) : \omega \in \mathbb{R}, n \in \mathbb{Z}\},
\]

\[
\sigma_{\text{pt}}(\mathcal{L}) = \{0, \pm i\sigma_1\}, \quad \sigma_1 = 0.6888
\]

for parameters \( \alpha = \frac{1}{2} + \frac{1}{2}i, \beta = \frac{5}{2} + i, \gamma = -1 - \frac{1}{10}i, \mu = -\frac{1}{2} \).
Eigenfunctions of QCGL for a spinning soliton with $d = 3$: $\Re \nu(x) = \pm 0.8$
Spatial decay of eigenfunctions of QCGL at a spinning soliton for $d = 3$: Note

$$\text{Re} \lambda \geq -(1 - \varepsilon)\beta_\infty = -(1 - \varepsilon)(-\text{Re} \delta) \iff \varepsilon \leq \frac{\text{Re} \lambda - \text{Re} \delta}{-\text{Re} \delta} =: \varepsilon(\lambda).$$

Decay rate of eigenfunctions:

$$0 \leq \mu \leq \frac{\varepsilon(\lambda)\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha|p} =: \mu_{\text{eig}}(p, \lambda) < \frac{\varepsilon(\lambda)\sqrt{-\text{Re} \alpha \text{Re} \delta}}{|\alpha| \max\{p_{\text{min}}, \frac{d}{2}\}} =: \mu_{\text{max}}^{\text{eig}}(\lambda).$$

<table>
<thead>
<tr>
<th>eigenvalue</th>
<th>NDR</th>
<th>TDR</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8.999 \cdot 10^{-15}$</td>
<td>0.5387</td>
<td>0.4714</td>
</tr>
<tr>
<td>$-5.6162 \cdot 10^{-4}$</td>
<td>0.5478</td>
<td>0.4714</td>
</tr>
<tr>
<td>$0.00110 \pm 0.68827i$</td>
<td>0.5507</td>
<td>0.4714</td>
</tr>
<tr>
<td>$0.00248 \pm 0.6874i$</td>
<td>0.5398</td>
<td>0.4714</td>
</tr>
<tr>
<td>$-0.06622 \pm 1.0112i$</td>
<td>0.4899</td>
<td>0.4090</td>
</tr>
<tr>
<td>$-0.07747 \pm 1.5274i$</td>
<td>0.5355</td>
<td>0.3984</td>
</tr>
<tr>
<td>$-0.22334 \pm 1.1593i$</td>
<td>0.4756</td>
<td>0.2608</td>
</tr>
<tr>
<td>$-0.26467 \pm 0.1193i$</td>
<td>0.4785</td>
<td>0.2219</td>
</tr>
<tr>
<td>$-0.30232 \pm 1.9457i$</td>
<td>0.4649</td>
<td>0.1864</td>
</tr>
<tr>
<td>$-0.43957 \pm 2.3248i$</td>
<td>0.3595</td>
<td>0.0570</td>
</tr>
<tr>
<td>$-0.44063 \pm 1.5128i$</td>
<td>0.3310</td>
<td>0.0560</td>
</tr>
<tr>
<td>$-0.47366 \pm 1.3552i$</td>
<td>0.4781</td>
<td>0.0248</td>
</tr>
<tr>
<td>$-0.48294 \pm 0.9163i$</td>
<td>0.4145</td>
<td>0.0161</td>
</tr>
<tr>
<td>$-0.48506 \pm 0.0991i$</td>
<td>0.2126</td>
<td>0.0141</td>
</tr>
<tr>
<td>$-0.49015 \pm 0.2535i$</td>
<td>0.3307</td>
<td>0.0093</td>
</tr>
<tr>
<td>$-0.55519 \pm 1.1222i$</td>
<td>0.3581</td>
<td>—</td>
</tr>
</tbody>
</table>
Eigenfunction $\langle S x, \nabla v_\ast(x) \rangle$ of QCGL for a spinning soliton with $d = 3$: 
Conclusion:

Theoretical results:

1. spatial decay of rotating waves
2. spectral properties of linearization at localized rotating waves
   ▶ point spectrum on the imaginary axis, shape of eigenfunctions and spatial decay of eigenfunctions
   ▶ essential spectrum

Numerical results:

3. approximation of rotating waves
4. approximation of spectra and eigenfunctions of linearization

Present to Bernold:

5. results on phase-shift interactions of multiple spinning solitons
Open problems and work in progress

- **Fredholm properties and $L^p$-spectra of localized rotating waves**  
  (joint work with: W.-J. Beyn)

- **Fourier-Bessel method on $\mathbb{R}^d$ and on circular domains**  
  (joint work with: W.-J. Beyn, C. Döding)

- **Freezing traveling waves in incompressible Navier-Stokes equations**  
  (joint work with: W.-J. Beyn, C. Döding)

- **Rotating waves in systems of damped wave equations**  
  (joint work with: W.-J. Beyn, J. Rottmann-Matthes)

- **Nonlinear stability of rotating waves for $d \geq 3$**  
  (joint work with: W.-J. Beyn)

- **Approximation theorem for rotating waves**
Nonlinear stability of rotating waves

Problem 1: (Nonlinear stability of rotating waves)

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in W^{2,p}_{\text{Eucl}}(\mathbb{R}^d, K^N)$ with $\|u_0 - v_*\|_1 \leq \delta$ the following property hold:

The reaction diffusion system

$$u_t(x, t) = A \triangle u(x, t) + f(u(x, t)), \quad t > 0, \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$u(x, 0) = u_0(x), \quad t = 0, \quad x \in \mathbb{R}^d,$$

has a unique solution $u \in C^1([0, \infty[, L^p(\mathbb{R}^d, K^N)) \cap C([0, \infty[, W^{2,p}_{\text{Eucl}}(\mathbb{R}^d, K^N))$ and the solution $u$ satisfies

$$\inf_{\gamma \in \text{SE}(d)} \|u(t) - a(\gamma)v_*\|_2 \leq \varepsilon \quad \forall \ t \geq 0.$$

Moreover, there exists a $\delta_0 > 0$ such that for any initial value $u_0 \in W^{2,p}_{\text{Eucl}}(\mathbb{R}^d, K^N)$ with $\|u_0 - v_*\|_1 \leq \delta$ there exists some asymptotic phase $\gamma_\infty \in \text{SE}(d)$ such that the solution $u$ satisfies

$$\|u(t) - a(\gamma_\infty \circ \gamma_*(t))v_*\|_2 \to 0 \quad \text{as} \quad t \to \infty.$$
Motivation 1: Nonlinear stability of rotating waves

Exponential decay and spectral properties are motivated by nonlinear stability of rotating waves.

**Main Assumptions:** (Beyn, Lorenz, 2008)

1. **(Localization condition).** Pattern $v_\star$ is localized up to order 2, i.e.
   - $v_\star - v_\infty \in H^2(\mathbb{R}^2, \mathbb{R}^N)$,
   - $\sup_{|x| \geq R} |D^\alpha (v_\star(x) - v_\infty)| \to 0$ as $R \to \infty$, $\forall 0 \leq |\alpha| \leq 2$.

2. **(Stability condition).** $Df(v_\infty) \in \mathbb{R}^{N,N}$ is negative definite, i.e.
   - $Df(v_\infty) \leq -2\beta I < 0$, $\beta > 0$.

3. **(Spectral condition).**
   - eigenfunctions $D_1 v_\star, D_2 v_\star, D_\phi v_\star \in H^2_{\text{Eucl}}(\mathbb{R}^2, \mathbb{R}^N)$ are nontrivial
   - corresponding eigenvalues $\pm ic, 0$ are algebraically simple
   - $\mathcal{L} : H^2_{\text{Eucl}} \to L^2$ has no eigenvalues $s \in \mathbb{C}$ with $\Re s \geq -2\beta$, except for the eigenvalues $\pm ic, 0$. 

Denny Otten
Spatial decay and spectral properties of rotating waves
Berlin 2016
Approximation theorem for rotating waves

Problem 2: (Approximation theorem for rotating waves)

There exist some $\rho > 0$ and $R_0 > 0$ such that for every radius $R > R_0$ the boundary value problem

$$
0 = A \Delta v_R(x) + \langle S_R x + \lambda_R, \nabla v_R(x) \rangle + f(v_R(x)) \quad , \quad x \in B_R(0),
$$

$$
0 = v_R(x) \quad , \quad x \in \partial B_R(0),
$$

$$
0 = \text{Re} \left\langle v_R - \hat{v}, (x_j D_i - x_i D_j) \hat{v} \right\rangle_{L^2(B_R(0), \mathbb{K}^N)} \quad , \quad i = 1, \ldots, d - 1,
$$

$$
0 = \text{Re} \left\langle v_R - \hat{v}, D_i \hat{v} \right\rangle_{L^2(B_R(0), \mathbb{K}^N)} \quad , \quad j = i + 1, \ldots, d,
$$

$$
0 = \text{Re} \left\langle v_R - \hat{v}, D_l \hat{v} \right\rangle_{L^2(B_R(0), \mathbb{K}^N)} \quad , \quad l = 1, \ldots, d,
$$

has a unique solution $(v_R, (S_R, \lambda_R))$ in a neighborhood of

$$
B_\rho(v_\star|_{B_R(0)}, (S_\star, \lambda_\star)) = \left\{ (v, (S, \lambda)) \in W^{2,2}_{\text{Eucl}}(\mathbb{R}^d, \mathbb{K}^N) \times \text{se}(d) \mid \left\| v_\star|_{B_R(0)} - v \right\|_{W^{2,2}_{\text{Eucl}}(B_R(0), \mathbb{K}^N)} + d((S_\star, \lambda_\star), (S, \lambda)) \leq \rho \right\}.
$$

Moreover, there exist some $C > 0$ and $\eta > 0$ such that

$$
\left\| v_R - v_\star \right\|_{W^{2,2}_{\text{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)} + d((S_R, \lambda_R), (S_\star, \lambda_\star)) \leq Ce^{-\eta R}.
$$
Outline

Outline of proof: Theorem 2

Outline of proof: Theorem 3

Outline of proof: Theorem 4

Outline of proof: Theorem 5

Overview: Semigroup approach
Outline of proof: Theorem 2 (Decay of eigenfunctions)

Consider

\[ A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

1. Splitting off the stable part:

\[ Df(v_*(x)) = Df(v_\infty) + (Df(v_*(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x), \ x \in \mathbb{R}^d, \]

leads to

\[ [L_0 v](x) + (Df(v_\infty) + Q(x)) v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

2. Decomposition of (the variable coefficient) \( Q \):

\[ Q(x) = Q_\varepsilon(x) + Q_c(x), \ Q_\varepsilon \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } ||\cdot||_{C_b}, \]

\[ Q_c \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d, \]

leads to

\[ [L_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)) v(x) = \lambda v(x), \ x \in \mathbb{R}^d. \]

(→ inhomogeneous Cauchy problem for \( L_c \))
Outline

5 Outline of proof: Theorem 2

6 Outline of proof: Theorem 3

7 Outline of proof: Theorem 4

8 Outline of proof: Theorem 5

9 Overview: Semigroup approach
Outline of proof: Theorem 3 (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$)

Consider the **rotating wave equation**

\[(\text{RWE}) \quad 0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)), \quad x \in \mathbb{R}^d, \quad d \geq 2.\]

and the **SE(d)-group action**

\[ [a(R,\tau)v](x) = v(R^{-1}(x - \tau)), \quad x \in \mathbb{R}^d, (R,\tau) \in \text{SE}(d). \]

1. **Generators of group action**: Applying the generators

\[ D^{(i,j)} := x_j D_i - x_i D_j \quad \text{and} \quad D_l = \frac{\partial}{\partial x_l} \]

leads to \( \frac{d(d+1)}{2} \) equations

\[ 0 = (x_j D_i - x_i D_j) \left( A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) \right) \]
\[ 0 = D_l \left( A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) \right) \]

for \( i = 1, \ldots, d - 1, \quad j = i + 1, \ldots, d, \quad l = 1, \ldots, d \).

2. **Commutator relations of generators**: Using commutator relations

\[ D_l D_k = D_k D_l, \]
\[ D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j, \]
\[ D^{(i,j)} D^{(r,s)} = D^{(r,s)} D^{(i,j)} + \delta_{is} D^{(r,j)} - \delta_{ir} D^{(s,j)} - \delta_{js} D^{(r,i)} + \delta_{jr} D^{(s,i)}. \]
Outline of proof: Theorem 3 (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$)

Consider the rotating wave equation

\[(\text{RWE}) \quad 0 = A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)), \quad x \in \mathbb{R}^d, \quad d \geq 2.\]

and the $\text{SE}(d)$-group action

\[a(R, \tau)v(x) = v(R^{-1}(x - \tau)), \quad x \in \mathbb{R}^d, (R, \tau) \in \text{SE}(d).\]

1. Generators of group action: Applying the generators

\[D^{(i,j)} := x_j D_i - x_i D_j \quad \text{and} \quad D_l = \frac{\partial}{\partial x_l}\]

to (RWE) leads to \(\frac{d(d+1)}{2}\) equations

\[
\begin{align*}
0 &= (x_j D_i - x_i D_j) (A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x))) \\
0 &= D_l (A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)))
\end{align*}
\]

for \(i = 1, \ldots, d - 1, \quad j = i + 1, \ldots, d, \quad l = 1, \ldots, d.\)

2. Commutator relations of generators: Using commutator relations

\[
\begin{align*}
D_l D_k &= D_k D_l, \\
D_l D^{(i,j)} &= D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j, \\
D_l (x_j D_i - x_i D_j) &= (x_j D_i - x_i D_j) D_l - \delta_{lj} (x_j D_i - x_i D_j) + \delta_{li} (x_j D_i - x_i D_j), \\
D_l (x_j D_i - x_i D_j) &= (x_j D_i - x_i D_j) D_l - \delta_{lj} (x_j D_i - x_i D_j) + \delta_{li} (x_j D_i - x_i D_j)
\end{align*}
\]
Outline

5 Outline of proof: Theorem 2

6 Outline of proof: Theorem 3

7 Outline of proof: Theorem 4

8 Outline of proof: Theorem 5

9 Overview: Semigroup approach
Outline of proof: Theorem 4 (Fredholm properties of $\mathcal{L}$)

$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\star(x))v(x), \ x \in \mathbb{R}^d.$

1. Splitting off the stable part:

$$Df(v_\star(x)) = Df(v_\infty) + (Df(v_\star(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x), \ x \in \mathbb{R}^d,$$

leads to

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + (Df(v_\infty) + Q(x))v(x), \ x \in \mathbb{R}^d.$$

2. Decomposition of (the variable coefficient) $Q$:

$$Q(x) = Q_\epsilon(x) + Q_c(x), \ Q_\epsilon \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } \|\cdot\|_{C_b},$$

$$Q_c \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d,$$

allows us to decompose the differential operator $\lambda I - \mathcal{L}$ into

$$\lambda I - \mathcal{L} = \lambda I - \mathcal{L}_c = (I - Q_c(\cdot)(\lambda I - \mathcal{L}_s)^{-1})(\lambda I - \mathcal{L}_s).$$

3. Fredholm properties of each factor:

- $\lambda I - \mathcal{L}_s$ Fredholm of index 0: unique solvability of resolvent equation for $\mathcal{L}_s$.
- $I - Q_c(\cdot)(\lambda I - \mathcal{L}_s)^{-1}$ Fredholm of index 0: compact perturbation of identity, unique solvability of resolvent equation for $\mathcal{L}_s$ and $D^p_{loc}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$.
- $\lambda I - \mathcal{L}$ Fredholm of index 0: Theorem on products of Fredholm operators.
Outline

5 Outline of proof: Theorem 2

6 Outline of proof: Theorem 3

7 Outline of proof: Theorem 4

8 Outline of proof: Theorem 5

9 Overview: Semigroup approach
Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Linearization at the profile $v_*$:**

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\infty)v(x) + Q(x)v(x)$$

$$Q(x) := Df(v_*(x)) - Df(v_\infty), \quad \sup_{|x| \geq R} |Q(x)|_2 \to 0 \text{ as } R \to \infty$$

1. **Orthogonal transformation:** $S \in \mathbb{R}^{d,d}$, $S^T = -S$, $S = P\Lambda_{\text{block}}^SP^T$. $T_1(x) = Px$ yields

$$[\mathcal{L}_1v](x) = A\triangle v(x) + \langle \Lambda_{\text{block}}^Sx, \nabla v(x) \rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

with

$$\langle \Lambda_{\text{block}}^Sx, \nabla v(x) \rangle = \sum_{l=1}^{k} \sigma_l (x_{2l}D_{2l-1} - x_{2l-1}D_{2l}) v(x).$$
Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

Orthogonal transformation:

$$[\mathcal{L}_1 \nu](x) = A \Delta \nu(x) + \langle \Lambda^S_{\text{block}} x, \nabla \nu(x) \rangle + Df(\nu_\infty) \nu(x) + Q(T_1(x)) \nu(x)$$

$$\langle \Lambda^S_{\text{block}} x, \nabla \nu(x) \rangle = \sum_{l=1}^{k} \sigma_l (x_{2l} D_{2l-1} - x_{2l-1} D_{2l}) \nu(x)$$

2. Several planar polar coordinates: Transformation

$$\begin{pmatrix} x_{2l-1} \\ x_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l \cos \phi_l \\ r_l \sin \phi_l \end{pmatrix}, \quad l = 1, \ldots, k, \quad \phi_l \in ]-\pi, \pi], \quad r_l > 0.$$

yields for $\xi = (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d)$ with total transformation $T_2(\xi)$, $Q(\xi) := Q(T_1(T_2(\xi)))$

$$\begin{aligned} [\mathcal{L}_2 \nu](x) & = A \left[ \sum_{l=1}^{k} \left( \frac{\partial^2}{\partial r_l^2} + \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^{d} \partial_{x_l}^2 \right] \nu(\xi) \\ & \quad - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} \nu(\xi) + Df(\nu_\infty) \nu(\xi) + Q(\xi) \nu(\xi), \end{aligned}$$
Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

Several planar polar coordinates:

$$[\mathcal{L}_2 v] (\xi) = A \left[ \sum_{l=1}^{k} \left( \partial_{r_l}^2 + \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^{d} \partial_{x_l}^2 \right] v(\xi)$$

$$- \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),$$

$$\xi = (r_1, \phi_1, \ldots, r_k, \phi_k, x_{2k+1}, \ldots, x_d), \quad Q(\xi) := Q(T_1(T_2(\xi)))$$

3. Simplified operator (far-field linearization): Neglecting $O\left(\frac{1}{r}\right)$–terms yields

$$[\mathcal{L}_2^{\text{sim}} v] (x) = A \left[ \sum_{l=1}^{k} \partial_{r_l}^2 + \sum_{l=2k+1}^{d} \partial_{x_l}^2 \right] v(\xi) - \sum_{l=1}^{k} \sigma_l \partial_{\phi_l} v(\xi) + Df(v_\infty)v(\xi).$$
Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

Simplified operator (far-field linearization):

$$[\mathcal{L}_{2}^{\text{sim}} \nu] (\xi) = A \left[ \sum_{l=1}^{k} \partial_{r_l}^{2} + \sum_{l=2k+1}^{d} \partial_{x_{l'}}^{2} \right] \nu(\xi) - \sum_{l=1}^{k} \sigma_{l} \partial_{\phi_{l}} \nu(\xi) + Df(\nu_{\infty}) \nu(\xi).$$

4. Angular Fourier decomposition:

$$\nu(\xi) = \exp \left( i \omega \sum_{l=1}^{k} r_{l} \right) \exp \left( i \sum_{l=1}^{k} n_{l} \phi_{l} \right) \hat{\nu}, \ n_{l} \in \mathbb{Z}, \ \omega \in \mathbb{R}, \ \hat{\nu} \in \mathbb{C}^{N}, \ |\hat{\nu}| = 1$$

$$\phi_{l} \in ] - \pi, \pi], \ r_{l} > 0, \ l = 1, \ldots, k,$$

yields

$$\left[ (\lambda I - \mathcal{L}_{2}^{\text{sim}}) \nu \right] (\xi) = \left( \lambda I_{N} + \omega^{2} A + i \sum_{l=1}^{k} n_{l} \sigma_{l} I_{N} - Df(\nu_{\infty}) \right) \nu(\xi).$$
Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

Angular Fourier decomposition:

$$[(\lambda I - \mathcal{L}_2^{\text{sim}}) \nu](\xi) = \left(\lambda I_N + \kappa^2 A + i \sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty)\right)\nu(\xi).$$

$n_l \in \mathbb{Z}$, $\kappa \in \mathbb{R}$, $\pm i \sigma_l$ nonzero eigenvalues of $S \in \mathbb{R}^{d,d}$

5. Finite-dimensional eigenvalue problem: $[(\lambda I - \mathcal{L}_2^{\text{sim}}) \nu](\xi) = 0$ for every $\xi$ if $\lambda \in \mathbb{C}$ satisfies

$$(\omega^2 A - Df(v_\infty)) \hat{\nu} = -\left(\lambda + i \sum_{l=1}^{k} n_l \sigma_l\right) \hat{\nu}, \text{ for some } \omega \in \mathbb{R}.$$
Outline

5 Outline of proof: Theorem 2

6 Outline of proof: Theorem 3

7 Outline of proof: Theorem 4

8 Outline of proof: Theorem 5

9 Overview: Semigroup approach
The operator $\mathcal{L}_0$

**Ornstein-Uhlenbeck operator**

$$[\mathcal{L}_0 v](x) = A \Delta v(x) + \langle S x, \nabla v(x) \rangle, \ x \in \mathbb{R}^d, \ d \geq 2.$$  

↓

**Heat kernel**

$$H_0(x, \xi, t) = (4\pi t A)^{-\frac{d}{2}} \exp \left( - (4tA)^{-1} \|e^{tS}x - \xi\|^2 \right), \ x, \xi \in \mathbb{R}^d, \ t > 0.$$  

↓

**Semigroup** in $L^p(\mathbb{R}^d, \mathbb{C}^N), \ 1 \leq p \leq \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x, \xi, t)v(\xi)d\xi, \ t > 0.$$  

**Strong** ↓ **continuity**

**Infinitesimal generator**

$$(A_p, \mathcal{D}(A_p)), \ 1 \leq p < \infty.$$  

**Semigroup theory** √

**Identification problem**

unique solv. of

resolvent equ. for $A_p, \ 1 \leq p < \infty, \ \text{Re} \lambda > 0$

$$\lambda I - A_p \nu_\ast = g \in L^p.$$  

A-priori

exponential decay,  

max. domain and  

max. realization,

$$1 \leq p < \infty \quad 1 < p < \infty$$  

$$\nu_\ast \in W^{1,p}_\theta. \quad A_p = \mathcal{L}_0 \text{ on } \mathcal{D}(A_p) = \mathcal{D}^{p}_{\text{loc}}(\mathcal{L}_0).$$
Identification problem of $\mathcal{L}_0$

$$
\mathcal{D}^p_{\text{loc}}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N) \right\}, 1 < p < \infty.
$$

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), 1 \leq p < \infty.
$$

\[ S \text{ is a core}\]

\[ \text{for } (A_p, \mathcal{D}(A_p))\]

\[ \text{Identification of } \mathcal{L}_0\]

maximal domain and maximal realization for $1 < p < \infty$: $A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\text{loc}}(\mathcal{L}_0)$

Ornstein-Uhlenbeck operator

$$
[\mathcal{L}_0 v](x) = A \Delta v(x) + \langle S x, \nabla v(x) \rangle, x \in \mathbb{R}^d, d \geq 2.
$$

\[ \mathcal{L}_0 : \mathcal{D}^p_{\text{loc}}(\mathcal{L}_0) \to L^p(\mathbb{R}^d, \mathbb{C}^N)\]

is a closed operator, $1 < p < \infty$

\[ \text{L}^p\text{-resolvent estimates}\]

and

\[ \text{unique solv. of resolvent equ.}\]

for $\mathcal{L}_0$ in $\mathcal{D}^p_{\text{loc}}(\mathcal{L}_0)$,

$1 < p < \infty$

\[ \text{L}^p\text{-dissipativity condition: } \exists \, \gamma_A > 0
$$

$$
|z|^2 \text{Re} \left< w, Aw \right> + (p - 2) \text{Re} \left< w, z \right> \text{Re} \left< z, Aw \right> \geq \gamma_A |z|^2 |w|^2 \quad \forall \, z, w \in K^N
$$

\[ \text{L}^p\text{-first antieigenvalue condition}\]

$$
\mu_1(A) := \inf_{\begin{subarray}{l} w \in K^N \end{subarray}} \frac{\text{Re} \left< w, Aw \right>}{|w||Aw|} > \frac{|p - 2|}{p}, \quad 1 < p < \infty
$$

Denny Otten

Spatial decay and spectral properties of rotating waves

Berlin 2016