# Spatial Decay and Spectral Properties of Rotating Waves in Evolution Equations

Patterns of Dynamics
Conference in Honor of Bernold Fiedler
Free University of Berlin, July 25-29, 2016

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July 28, 2016

Universität Bielefeld

CRC 701

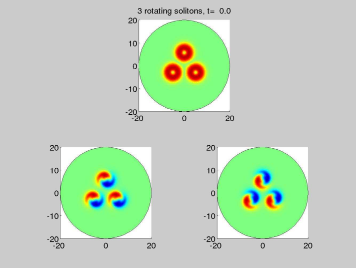
W.-J. Beyn, D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. *Dyn. Partial Differ. Equ.*, 13(3):191-240, 2016.

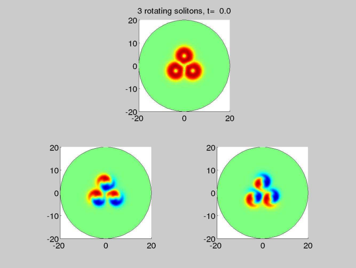
D. Otten. Spatial decay and spectral properties of rotating waves in parabolic systems. PhD thesis, Bielefeld University, *Shaker Verlag*, 2014.

# Congratulations to Bernold Fiedler


**Dynamics of Patterns**
MFO (Oberwohlfach)
December 16-22, 2012
**Organisors**:
Wolf-Jürgen Beyn
Björn Sandstede
Bernold Fiedler

Bernold, do you remember on our discussions about QCGL soliton interactions?





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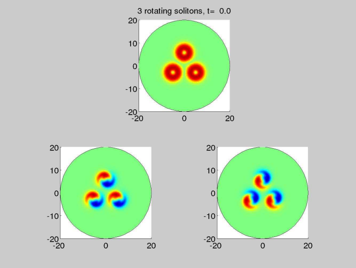
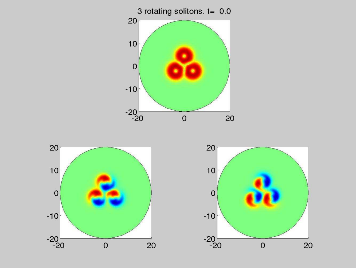
You ask me: Did you ever investigated soliton interactions for shifted phases?

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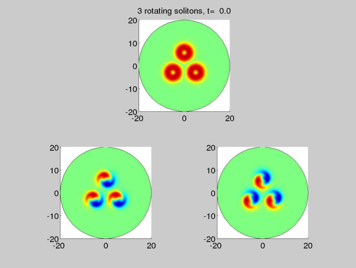
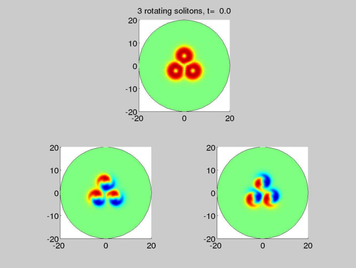
Good news: Yes, I did it!





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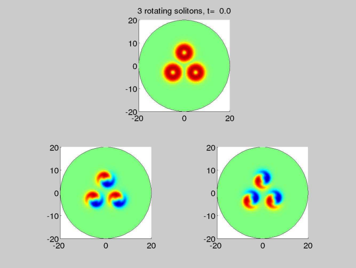
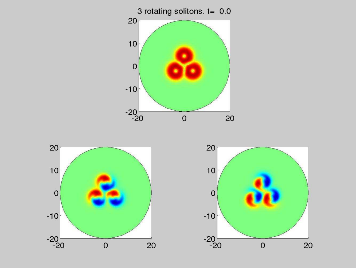
Better news: I never published these results!

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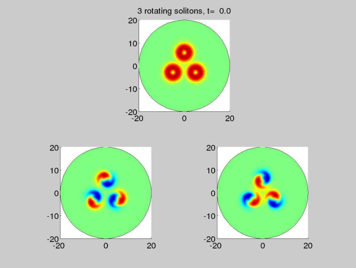
Best news: You now get a look into these results!

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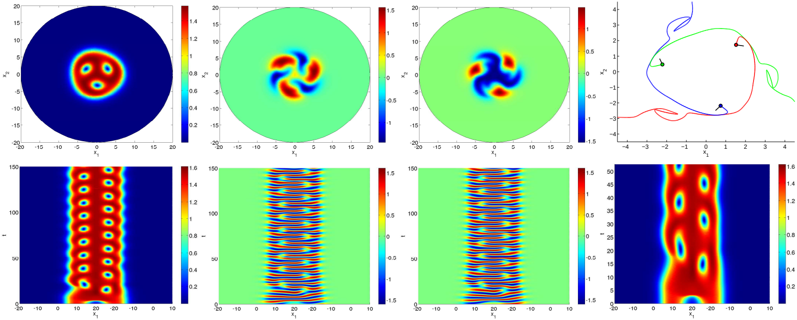
... I even applied the decompose and freeze method to it!



Happy Birthday, Bernold!

# Outline

# Outline

1. Rotating patterns in $\mathbb{R}^d$
2. Spatial decay of rotating waves
3. Spectral properties of linearization at rotating waves
4. Cubic-quintic complex Ginzburg-Landau equation

# Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$, $A\in\mathbb{R}^{N,N}$, $f:\mathbb{R}^N\to\mathbb{R}^N$, $u_0:\mathbb{R}^d\to\mathbb{R}^N$.

Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{R}^N$ of (1)

$$u_\star(x,t) = v_\star(e^{-tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{R}^N$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.

**Transformation (into a co-rotating frame)**: $v(x,t)=u(e^{tS}x,t)$ solves

$$(2)\qquad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx,\nabla v(x)\rangle = Dv(x)Sx = \underbrace{\sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_jD_iv(x)}_{\textbf{(drift term)}} \overset{-S=S^\top}{=} \underbrace{\sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_jD_i - x_iD_j\right)v(x)}_{\textbf{(rotational term)}}$$

# Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1)\qquad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

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$$\langle Sx,\nabla v(x)\rangle = Dv(x)Sx = \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_jD_iv(x) \overset{-S=S^\top}{=} \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_jD_i - x_iD_j\right)v(x)$$

**(drift term)** **(rotational term)**

# Rotating Patterns in $\mathbb{R}^d$

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Note: $v_\star$ is a stationary solution of (2), i.e. $v_\star$ solves the **rotating wave equation**

$$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

$A\triangle v_\star(x) + \langle Sx,\nabla v_\star(x)\rangle$: **Ornstein-Uhlenbeck operator**.

# Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

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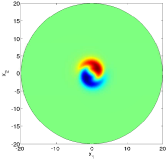
**Question:** How to show exponential decay of $v_\star$ at $|x|=\infty$?
**Consequence:** Exponentially small error by truncation to bounded domain.

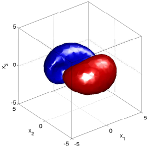
# Examples for rotating waves

**Cubic-quintic complex Ginzburg-Landau equation**: (spinning solitons)

$$u_t = \alpha \triangle u + u\left(\delta + \beta |u|^2 + \gamma |u|^4\right)$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $\delta \in \mathbb{R}$, $d \in \{2,3\}$.

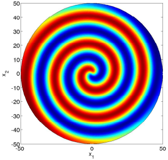


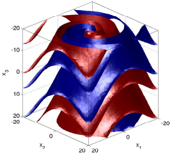




**$\lambda$-$\omega$ system**: (spiral waves, scroll waves)

$$u_t = \alpha \triangle u + \left(\lambda(|u|^2) + i\omega(|u|^2)\right) u$$

$u(x,t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $\lambda, \omega : [0,\infty[ \to \mathbb{R}$, $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $d \in \{2,3\}$.

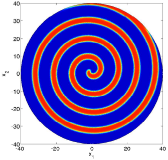



**Barkley model**: (spiral waves, also scroll waves)

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \triangle u + \begin{pmatrix} \frac{1}{\varepsilon} u_1 (1-u_1)(u_1 - \frac{u_2+b}{a}) \\ u_1 - u_2 \end{pmatrix}$$

with $u(x,t) \in \mathbb{R}^2$, $x \in \mathbb{R}^d$, $t \geqslant 0$, $0 \leqslant D \ll 1$, $\varepsilon, a, b > 0$.



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**Ginzburg-Landau equation:**

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**Barkley model:**

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# Outline

# Spatial decay of rotating waves

## Theorem 1: (Exponential decay of $v_\star$)

Let $f \in C^2 \quad (\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^2 \quad (\mathbb{R}^d, \mathbb{R}^N)$ of

(RWE) $$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

such that

(TC) $$\sup_{|x|\geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W^{1,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \left(\begin{array}{rcll} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{array}\right)$$

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satisfies

$$v_\star - v_\infty \in W^{2,p}_\theta(\mathbb{R}^d, \mathbb{R}^N)$$

for every exponential decay rate

$$0 \leqslant \mu \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p}. \qquad \left(\begin{array}{rcll} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A \\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty) \end{array}\right)$$

# Spatial decay of rotating waves

## Theorem 1: (Exponential decay of $v_\star$: higher regularity)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^N,\mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p\geqslant\frac{d}{2}$ (if $k\geqslant 3$).

Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ with the following property: Every classical solution $v_\star\in C^{k+1}(\mathbb{R}^d,\mathbb{R}^N)$ of

(RWE) $$A\triangle v_\star(x)+\langle Sx,\nabla v_\star(x)\rangle+f(v_\star(x))=0,\ x\in\mathbb{R}^d,$$

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# Spatial decay of rotating waves

## Theorem 1: (Exponential decay of $v_\star$: pointwise estimates)

Let $f \in C^{\max\{2,k-1\}}(\mathbb{R}^N,\mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta(x)=\exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu\in\mathbb{R}$, $k\in\mathbb{N}$, $p\geqslant\frac{d}{2}$ (if $k\geqslant 3$).
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such that

(TC) $$\sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|\leqslant K_1 \text{ for some } R_0>0$$

satisfies

$$v_\star-v_\infty\in W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^N),\ |D^\alpha(v_\star(x)-v_\infty)|\leqslant C\exp\left(-\mu\sqrt{|x|^2+1}\right)\ \forall\, x\in\mathbb{R}^d$$

for every exponential decay rate

$$0\leqslant\mu\leqslant\varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max}p}\qquad\left(\begin{array}{rcll} a_{\max} & = & \rho(A) & : \textbf{spectral radius of } A\\ -a_0 & = & s(-A) & : \textbf{spectral bound of } -A\\ -b_0 & = & s(Df(v_\infty)) & : \textbf{spectral bound of } Df(v_\infty)\end{array}\right)$$

and for every multiindex $\alpha\in\mathbb{N}_0^d$ satisfying $d<(k-|\alpha|)p$.

# Spatial decay of eigenfunctions at rotating waves

## Theorem 2: (Exponential decay of eigenfunctions $v$)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^N,\mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta_j(x)=\exp\left(\mu_j\sqrt{|x|^2+1}\right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j=1,2$, $k\in\mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k \geqslant 2$).
Then for every $0<\varepsilon<1$ there exists $K_1=K_1(\varepsilon)>0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^N)$ of (RWE) satisfying (TC) the following property holds: Every classical solution $v \in C^{k+1}(\mathbb{R}^d,\mathbb{C}^N)$ of

$$\text{(EVP)} \qquad A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x\in\mathbb{R}^d,$$

with $\lambda\in\mathbb{C}$, $\mathrm{Re}\,\lambda \geqslant -(1-\varepsilon)\beta_\infty$, such that

$$v \in L^p_{\theta_1}(\mathbb{R}^d,\mathbb{C}^N) \quad \text{for **some** exp. decay rate} \quad -\sqrt{\varepsilon\frac{\gamma_A\beta_\infty}{2d|A|^2}} \leqslant \mu_1 < 0$$

satisfies

$$v \in W^{k,p}_{\theta_2}(\mathbb{R}^d,\mathbb{C}^N) \quad \text{for **every** exp. decay rate} \quad 0 \leqslant \mu_2 \leqslant \varepsilon\frac{\sqrt{a_0 b_0}}{a_{\max} p}$$

and

$$|D^\alpha v(x)| \leqslant C\exp\left(-\mu_2\sqrt{|x|^2+1}\right)\ \forall\, x\in\mathbb{R}^d$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k-|\alpha|)p$.

# Exponentially weighted Sobolev spaces and assumptions

**Exponentially weighted Sobolev spaces**: For $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d,\mathbb{R}^N) := \left\{v \in L^1_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^N) \mid \|\theta v\|_{L^p} < \infty\right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d,\mathbb{R}^N) := \left\{v \in L^p_\theta(\mathbb{R}^d,\mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d,\mathbb{R}^N)\ \forall\,|\beta| \leqslant k\right\}.$$

**Assumptions**:

(A1) ($L^p$-dissipativity condition): For $A \in \mathbb{R}^{N,N}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w, z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2|w|^2\ \forall\, z, w \in \mathbb{R}^N$$

(A2) (System condition): $A, Df(v_\infty) \in \mathbb{R}^{N,N}$ simultaneously diagonalizable over $\mathbb{C}$

(A3) (Rotational condition): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

**Note**: Assumption (A1) is equivalent with

(A1') ($L^p$-antieigenvalue condition): $A \in \mathbb{R}^{N,N}$ is invertible and

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{R}^N\\ w\neq 0\\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p} \text{ for some } 1 < p < \infty$$

($\mu_1(A)$ : first antieigenvalue of $A$)

(to be read as $A > 0$ in case $N = 1$).

# Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

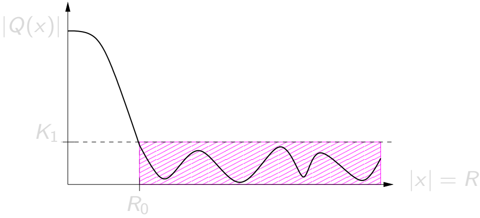
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

**1. Far-Field Linearization:** $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) := \int_0^1 Df(v_\infty + tw_\star(x))dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + a(x)w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



# Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

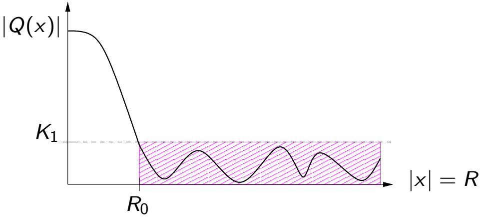
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

**2. Decomposition of $a$:** Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$



# Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

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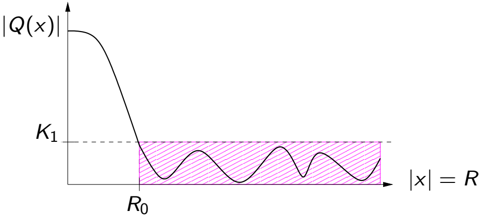
$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$




**3. Decomposition of $Q$:**

$Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)$,

$Q, Q_{\mathrm{s}}, Q_{\mathrm{c}} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

$Q_{\mathrm{s}}$ small, i.e. $\|Q_{\mathrm{s}}\|_{L^\infty} < K_1$,

$Q_{\mathrm{c}}$ compactly supported.

# Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

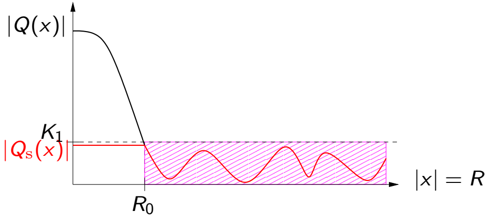
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

**2. Decomposition of $a$:** Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$




**3. Decomposition of $Q$:**

$$Q(x) = Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x),$$
$$Q, Q_{\mathrm{s}}, Q_{\mathrm{c}} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N}),$$
$Q_{\mathrm{s}}$ small, i.e. $\|Q_{\mathrm{s}}\|_{L^\infty} < K_1$,
$Q_{\mathrm{c}}$ compactly supported.

# Outline of proof: Theorem 1 (Exponential Decay of $v_\star$)

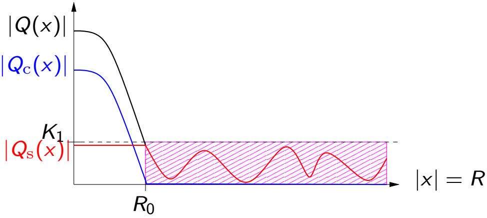
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

**2. Decomposition of $a$:** Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_\star(x)) - Df(v_\infty)dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_\mathrm{s}(x) + Q_\mathrm{c}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$




**3. Decomposition of $Q$:**

$Q(x) = Q_\mathrm{s}(x) + Q_\mathrm{c}(x)$,

$Q, Q_\mathrm{s}, Q_\mathrm{c} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

$Q_\mathrm{s}$ small, i.e. $\|Q_\mathrm{s}\|_{L^\infty} < K_1$,

$Q_\mathrm{c}$ compactly supported.

**Exponential Decay:** To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + (Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x))\, w_\star(x) = 0,\ x \in \mathbb{R}^d.$$

**Operators:** Study the following operators

$$\begin{aligned}
\mathcal{L}_{\mathrm{c}} v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v + Q_{\mathrm{c}} v, && \text{(exp. decay)}\\
\mathcal{L}_{\mathrm{s}} v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v, && \text{(exp. decay)}\\
\mathcal{L}_\infty v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, \quad \text{(far-field operator)} && \text{(exp. decay)}\\
\mathcal{L}_0 v &:= A\triangle v + \langle S\cdot, \nabla v\rangle. \quad \text{(Ornstein-Uhlenbeck operator)} && \text{(max. domain)}
\end{aligned}$$


D. Otten.
Exponentially weighted resolvent estimates for complex Ornstein-Uhlenbeck systems, 2015.
The identification problem for complex-valued Ornstein-Uhlenbeck operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, 2016.
A new $L^p$-antieigenvalue condition for Ornstein-Uhlenbeck operators, 2016.

W.-J. Beyn, D. Otten.
Spatial Decay of Rotating Waves in Reaction Diffusion Systems, 2016.

**Exponential Decay:** To show exponential decay for the solution $v_\star$ of

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,$$

investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\ x \in \mathbb{R}^d.$$

**Operators:** Study the following operators

$$\begin{aligned}
\mathcal{L}_{\mathrm{c}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v + Q_{\mathrm{c}} v, && && \text{(exp. decay)}\\
\mathcal{L}_{\mathrm{s}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}} v, && && \text{(exp. decay)}\\
\mathcal{L}_\infty v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, && \text{(far-field operator)} && \text{(exp. decay)}\\
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investigate the linear system $(w_\star(x) := v_\star(x) - v_\infty)$

$$A\triangle w_\star(x) + \langle Sx, \nabla w_\star(x)\rangle + \left(Df(v_\infty) + Q_{\mathrm{s}}(x) + Q_{\mathrm{c}}(x)\right) w_\star(x) = 0,\; x \in \mathbb{R}^d.$$

**Operators:** Study the following operators

$$\begin{aligned}
\mathcal{L}_{\mathrm{c}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}}v + Q_{\mathrm{c}}v, && && \text{(exp. decay)}\\
\mathcal{L}_{\mathrm{s}} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_{\mathrm{s}}v, && && \text{(exp. decay)}\\
\mathcal{L}_{\infty} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, && \text{(far-field operator)} && \text{(exp. decay)}\\
\mathcal{L}_{0} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle. && \text{(Ornstein-Uhlenbeck operator)} && \text{(max. domain)}
\end{aligned}$$

**Maximal domain of $\mathcal{L}_0$** given by

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) = \left\{v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^N) \cap L^p(\mathbb{R}^d, \mathbb{C}^N) :\; \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^N)\right\},\; 1 < p < \infty$$

satisfies $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$.

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# Outline

1. Rotating patterns in $\mathbb{R}^d$
2. Spatial decay of rotating waves
3. Spectral properties of linearization at rotating waves
4. Cubic-quintic complex Ginzburg-Landau equation

# Eigenvalue problem for linearization at rotating waves

**Motivation**: Stability is determined by spectral properties of linearization $\mathcal{L}$.
**Linearization** at the profile $v_\star$ of the rotating wave

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

**Eigenvalue problem**

$$[(\lambda I - \mathcal{L})v](x) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\ \lambda\in\mathbb{C}.$$

A **rotating wave** $u_\star(x,t) = v_\star\left(e^{-tS}x\right)$ is called **strongly spectrally stable**

$$:\Longleftrightarrow \begin{cases} \operatorname{Re}\sigma(\mathcal{L})\leqslant 0 \text{ (\textbf{spectrally stable})} \\ \text{and} \\ \forall\,\lambda\in\sigma(\mathcal{L}) \text{ with } \operatorname{Re}\lambda = 0:\ \lambda \text{ is caused by the } \mathrm{SE}(d)\text{-group action.}\end{cases}$$

Decomposition of the **spectrum** $\sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$ into

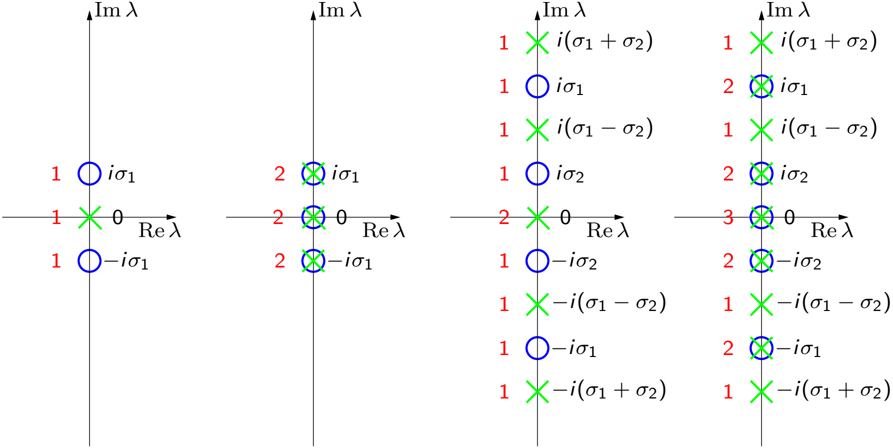
$$\sigma(\mathcal{L}) = \sigma_{\mathrm{ess}}(\mathcal{L})\,\dot\cup\,\sigma_{\mathrm{pt}}(\mathcal{L}),$$

with

$\sigma_{\mathrm{pt}}(\mathcal{L}) := \{\lambda\in\sigma(\mathcal{L})\mid \lambda \text{ isolated with finite multiplicity}\}$, **(point spectrum)**
$\sigma_{\mathrm{ess}}(\mathcal{L}) := \sigma(\mathcal{L})\backslash\sigma_{\mathrm{pt}}(\mathcal{L})$. **(essential spectrum)**

# Illustration: Point spectrum of $\mathcal{L}$ on the imaginary axis

$\lambda \in \left(\sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\}\right) \subseteq \sigma_{\mathrm{pt}}(\mathcal{L})$ & algebraic multiplicity




$d = 2$, $\dim \mathrm{SE}(2) = 3$ | $d = 3$, $\dim \mathrm{SE}(3) = 6$ | $d = 4$, $\dim \mathrm{SE}(4) = 10$ | $d = 5$, $\dim \mathrm{SE}(5) = 15$

# Point spectrum of $\mathcal{L}$ on the imaginary axis

## Theorem 3: (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$ and shape of eigenfunctions)

Let $S \in \mathbb{R}^{d,d}$, $S = -S^\top$, with eigenvalues $\lambda_1^S, \ldots, \lambda_d^S$ of $S$, and let $U \in \mathbb{C}^{d,d}$ be unitary satisfying $\Lambda_S = U^* S U$ with $\Lambda_S = \mathrm{diag}(\lambda_1^S, \ldots, \lambda_d^S)$. Moreover, let $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^N)$ be a classical solution of (RWE).
Then, $v : \mathbb{R}^d \to \mathbb{C}^N$ defined by

$$v(x) = \langle Qx + b, \nabla v_\star(x) \rangle = Dv_\star(x)(Qx + b),\ x \in \mathbb{R}^d,\ Q \in \mathbb{C}^{d,d},\ b \in \mathbb{C}^d$$

is a classical solution of $(\lambda I - \mathcal{L})v = 0$ if either

$$\lambda = -\lambda_l^S, \quad Q = 0, \quad b = Ue_l$$

for some $l = 1, \ldots, d$, or

$$\lambda = -(\lambda_i^S + \lambda_j^S), \quad Q = U(I_{ij} - I_{ji})U^\top, \quad b = 0$$

for some $i = 1, \ldots, d-1$ and $j = i+1, \ldots, d$.

- $\dim \mathrm{SE}(d) = \frac{d(d+1)}{2}$ eigenfunctions of $\mathcal{L}$ and their explicit representation,
- $\sigma_{\mathrm{pt}}^{\mathrm{part}}(\mathcal{L}) := \sigma(S) \cup \{\lambda_1 + \lambda_2 \mid \lambda_1, \lambda_2 \in \sigma(S),\ \lambda_1 \neq \lambda_2\} \subseteq \sigma(\mathcal{L})$,
- $v(x) = \langle Sx, \nabla v_\star(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geqslant 2$.
- point spectrum on imaginary axis is determined by the $\mathrm{SE}(d)-$group action,
- Theorem also valid for spiral waves, scroll waves, scroll rings.

# Properties of linearization at localized rotating waves

## Theorem 4: (Fredholm properties of $\mathcal{L}$ and decay of eigenfunctions)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j = 1, 2$, $k \in \mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k \geqslant 2$).
Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^N)$ of (RWE) satisfying (TC) the following properties hold:

1. (Fredholm properties). The operator
$$\lambda I - \mathcal{L} : (\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \to (L^p(\mathbb{R}^d, \mathbb{C}^N), \|\cdot\|_{L^p})$$
is Fredholm of index 0.

# Properties of linearization at localized rotating waves

## Theorem 4: (Fredholm properties of $\mathcal{L}$ and decay of eigenfunctions)

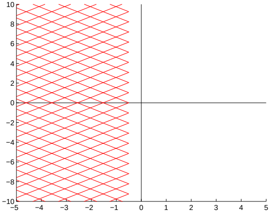
Let $f \in C^{\max\{2,k\}}(\mathbb{R}^N,\mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty)=0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, assume (A1)-(A3) for some $1<p<\infty$, and let $\theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2+1}\right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j=1,2$, $k \in \mathbb{N}$, $p \geqslant \frac{d}{2}$ (if $k \geqslant 2$).
Then for every $0<\varepsilon<1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d,\mathbb{R}^N)$ of (RWE) satisfying (TC) the following properties hold:

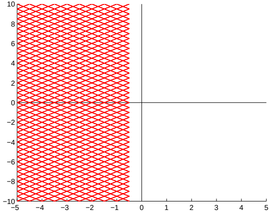
2. (Solvability of resolvent equation). There exist exactly $n$ (nontrivial lin. ind.) eigenfunctions $v_j \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ and adjoint eigenfunctions $\psi_j \in \mathcal{D}^q_{\mathrm{loc}}(\mathcal{L}_0^*)$ with
$$(\lambda I - \mathcal{L})v_j = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^* \psi_j = 0 \quad \text{for} \quad j = 1,\ldots,n.$$
Moreover,
$$(\lambda I - \mathcal{L})v = h, \quad h \in L^p(\mathbb{R}^d,\mathbb{C}^N)$$
has at least one (not necessarily unique) solution $v \in \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$ if and only if
$$h \in (\mathcal{N}(\lambda I - \mathcal{L})^*)^\perp, \quad \text{i.e.} \quad \langle \psi_j, h\rangle_{q,p} = 0,\, j=1,\ldots,n.$$

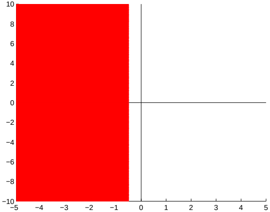
# Illustration: Essential spectrum of $\mathcal{L}$

$$\left\{-\lambda(\omega)+i\sum_{l=1}^{k}n_l\sigma_l \;\middle|\; \lambda(\omega) \text{ eigenvalue of } \omega^2 A - Df(v_\infty)\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L})$$

$\pm i\sigma_1,\ldots,\pm i\sigma_k$ nonzero eigenvalues of $S\in\mathbb{R}^{d,d}$, $-S=S^\top$, $n_l\in\mathbb{Z}$, $\omega\in\mathbb{R}$




$d=2$ or $3$ | $d=4$ (not dense) | $d=4$ (dense)

Parameters for illustration: $A=\frac{1}{2}+\frac{1}{2}i$, $Df(v_\infty)=-\frac{1}{2}$,

$\sigma_1=1.027$ | $\sigma_1=1$, $\sigma_2=1.5$ | $\sigma_1=1$, $\sigma_2=\frac{\exp(1)}{2}$

$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L})\subseteq\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda\leqslant s(Df(v_\infty))\}$ dense $\Longleftrightarrow\exists\,\sigma_n,\sigma_m$: $\sigma_n\sigma_m^{-1}\notin\mathbb{Q}$.

# Essential spectrum of $\mathcal{L}$

**Dispersion relation**: $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{L})$ if $\lambda \in \mathbb{C}$ satisfies

$$\text{(DR)}\quad \det\left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k} n_l \sigma_l I_N - Df(v_\infty)\right) = 0 \text{ for some } \omega \in \mathbb{R},\ n \in \mathbb{Z}^k.$$

## Theorem 5: (Essential spectrum of $\mathcal{L}$)

Assume $f \in C^{\max\{2,r-1\}}(\mathbb{R}^N, \mathbb{R}^N)$, $v_\infty \in \mathbb{R}^N$, $f(v_\infty) = 0$, $Df(v_\infty) \leqslant -\beta_\infty I < 0$, (A1)-(A3) for some $1 < p < \infty$, and $\frac{d}{p} \leqslant r$ (if $r \geqslant 2$) or $\frac{d}{p} \leqslant 2$ (if $r \geqslant 3$).
Moreover, let $\pm i\sigma_1, \ldots, \pm i\sigma_k$ denote the nonzero eigenvalues of $S$.
Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_\star \in C^{r+1}(\mathbb{R}^d, \mathbb{R}^N)$ of (RWE) satisfying (TC) the following property holds:

$$\sigma_{\mathrm{ess}}^{\mathrm{part}}(\mathcal{L}) := \left\{\lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR)}\right\} \subseteq \sigma_{\mathrm{ess}}(\mathcal{L}) \quad \text{in } L^p(\mathbb{R}^d, \mathbb{C}^N).$$

- essential spectrum is determined by the far-field linearization
- only for exponentially localized rotating waves, but **not** for nonlocalized waves (e.g. spiral waves, sroll waves)
- theory e.g. for spiral waves much more involved ($\rightarrow$ Floquet theory)

# References


**Spectrum at 2-dimensional localized rotating waves**:

W.-J. Beyn, J. Lorenz.
Nonlinear stability of rotating patterns, 2008.

**Spectrum of drift term**:

G. Metafune.
$L^p$-spectrum of Ornstein-Uhlenbeck operators, 2001.

**Spectrum at spiral and scroll waves**:

B. Sandstede, A. Scheel.
Absolute and convective instabilities of waves on unbounded and large bounded domains, 2000.

B. Fiedler, A. Scheel.
Spatio-temporal dynamics of reaction-diffusion patterns, 2003.

# Outline

1. Rotating patterns in $\mathbb{R}^d$
2. Spatial decay of rotating waves
3. Spectral properties of linearization at rotating waves
4. Cubic-quintic complex Ginzburg-Landau equation

# Example

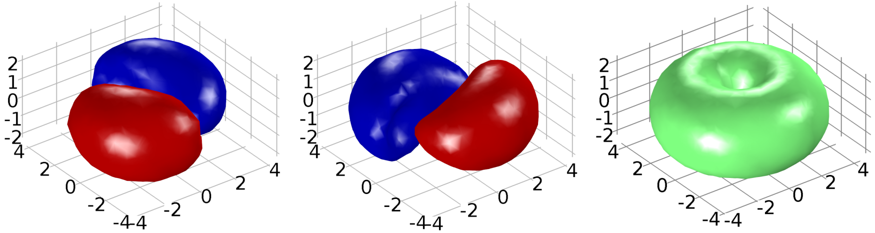
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \triangle u + u\left(\mu + \beta\,|u|^2 + \gamma\,|u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[ \to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.




$\mathrm{Re}\; v_\star(x) = \pm 0.5$ $\qquad$ $\mathrm{Im}\; v_\star(x) = \pm 0.5$ $\qquad$ $|v_\star(x)| = 0.5$

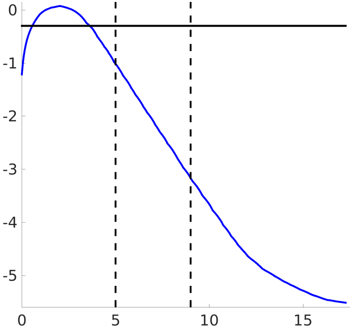
Freezing method implies numerical results for profile $v_\star$ and velocities $S$.

**Spatial decay** of a spinning soliton in QCGL for $d = 3$: Assume

$$\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\delta < 0, \quad p_{\min} = \frac{2|\alpha|}{|\alpha| + \operatorname{Re}\alpha} < p < \frac{2|\alpha|}{|\alpha| - \operatorname{Re}\alpha} = p_{\max}$$

**Decay rate of spinning soliton**:

$$0 \leqslant \mu < \frac{\sqrt{-\operatorname{Re}\alpha \operatorname{Re}\delta}}{|\alpha| p} =: \mu^{\mathrm{pro}}(p) < \frac{\sqrt{-\operatorname{Re}\alpha \operatorname{Re}\delta}}{|\alpha| \max\{p_{\min}, \frac{d}{2}\}} =: \mu^{\mathrm{pro}}_{\max}.$$




**Parameters**:

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,$$

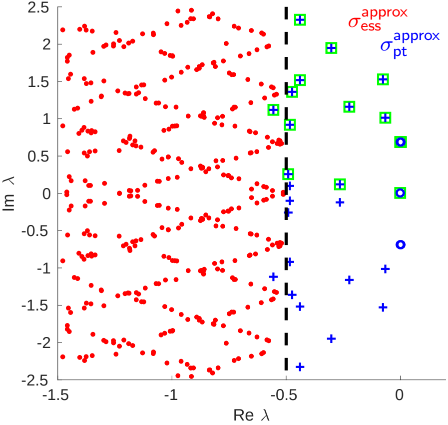
$$\mu = -\frac{1}{2},\ v_\infty = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ a_0 = \operatorname{Re}\alpha,$$

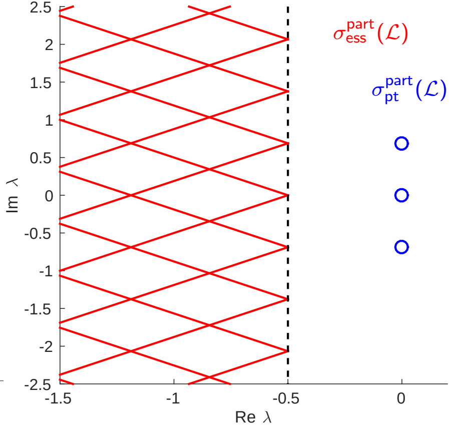
$$a_{\max} = |\alpha|,\ b_0 = \beta_\infty = -\operatorname{Re}\delta = -\frac{1}{2},$$

**Numerical vs. theoretical decay rate**: $(p = 2)$

$$\mathrm{NDR} \approx 0.5387, \quad \mathrm{TDR} = \mu^{\mathrm{pro}}_{\max} = \frac{\sqrt{2}}{4} \approx 0.4714.$$

**Spectrum** of QCGL for a spinning soliton with $d = 3$: (numerical vs. analytical)



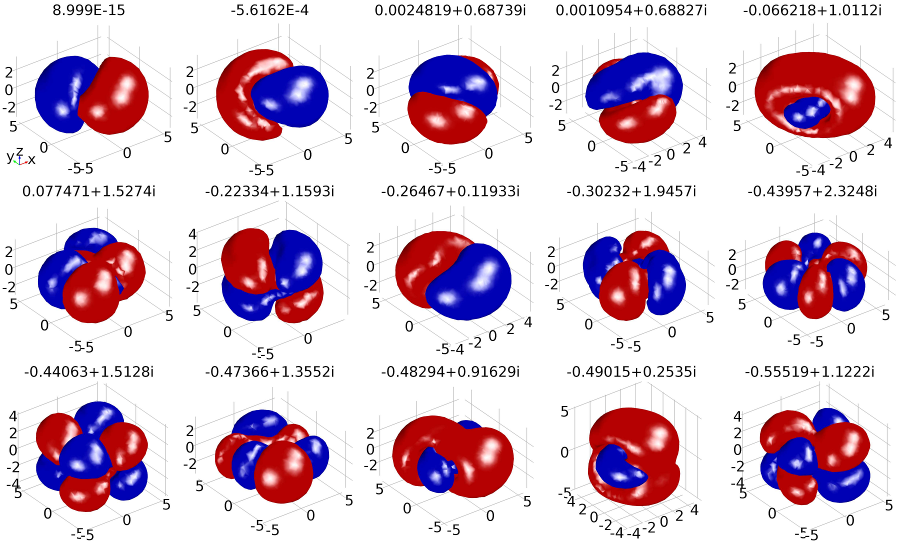

**Point spectrum** on $i\mathbb{R}$ and **essential spectrum** by dispersion relation:

$$\sigma_{\text{ess}}^{\text{part}}(\mathcal{L}) = \{\lambda = -\omega^2\alpha_1 + \delta_1 + i(\mp\omega^2\alpha_2 \pm \delta_2 - n\sigma_1) : \omega \in \mathbb{R},\ n \in \mathbb{Z}\},$$

$$\sigma_{\text{pt}}^{\text{part}}(\mathcal{L}) = \{0, \pm i\sigma_1\}, \quad \sigma_1 = 0.6888$$

for parameters $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{1}{10}i$, $\mu = -\frac{1}{2}$.

# **Eigenfunctions** of QCGL for a spinning soliton with $d = 3$: $\mathrm{Re}\, v(x) = \pm 0.8$

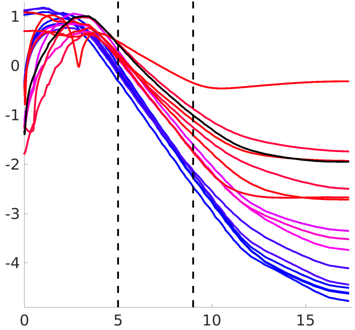


**Spatial decay** of eigenfunctions of QCGL at a spinning soliton for $d = 3$: Note

$$\mathrm{Re}\,\lambda \geqslant -(1-\varepsilon)\beta_\infty = -(1-\varepsilon)(-\mathrm{Re}\,\delta) \quad \Leftrightarrow \quad \varepsilon \leqslant \frac{\mathrm{Re}\,\lambda - \mathrm{Re}\,\delta}{-\mathrm{Re}\,\delta} =: \varepsilon(\lambda).$$

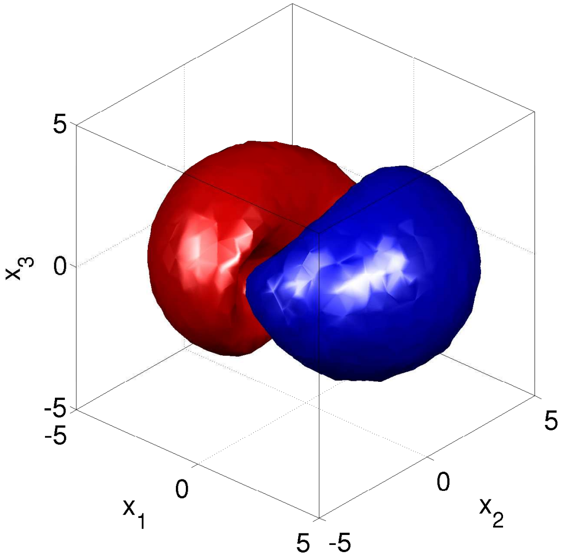
**Decay rate of eigenfunctions**:

$$0 \leqslant \mu \leqslant \frac{\varepsilon(\lambda)\sqrt{-\mathrm{Re}\,\alpha \mathrm{Re}\,\delta}}{|\alpha| p} =: \mu^{\mathrm{eig}}(p,\lambda) < \frac{\varepsilon(\lambda)\sqrt{-\mathrm{Re}\,\alpha \mathrm{Re}\,\delta}}{|\alpha| \max\{p_{\min}, \frac{d}{2}\}} =: \mu^{\mathrm{eig}}_{\max}(\lambda).$$




| eigenvalue | NDR | TDR |
|---|---|---|
| $8.999 \cdot 10^{-15}$ | 0.5387 | 0.4714 |
| $-5.6162 \cdot 10^{-4}$ | 0.5478 | 0.4714 |
| $0.00110 \pm 0.68827i$ | 0.5507 | 0.4714 |
| $0.00248 \pm 0.6874i$ | 0.5398 | 0.4714 |
| $-0.06622 \pm 1.0112i$ | 0.4899 | 0.4090 |
| $-0.07747 \pm 1.5274i$ | 0.5355 | 0.3984 |
| $-0.22334 \pm 1.1593i$ | 0.4756 | 0.2608 |
| $-0.26467 \pm 0.1193i$ | 0.4785 | 0.2219 |
| $-0.30232 \pm 1.9457i$ | 0.4649 | 0.1864 |
| $-0.43957 \pm 2.3248i$ | 0.3595 | 0.0570 |
| $-0.44063 \pm 1.5128i$ | 0.3310 | 0.0560 |
| $-0.47366 \pm 1.3552i$ | 0.4781 | 0.0248 |
| $-0.48294 \pm 0.9163i$ | 0.4145 | 0.0161 |
| $-0.48506 \pm 0.0991i$ | 0.2126 | 0.0141 |
| $-0.49015 \pm 0.2535i$ | 0.3307 | 0.0093 |
| $-0.55519 \pm 1.1222i$ | 0.3581 | — |

**Eigenfunction** $\langle Sx, \nabla v_\star(x)\rangle$ of QCGL for a spinning soliton with $d = 3$:



# Conclusion:

**Theoretical results**:

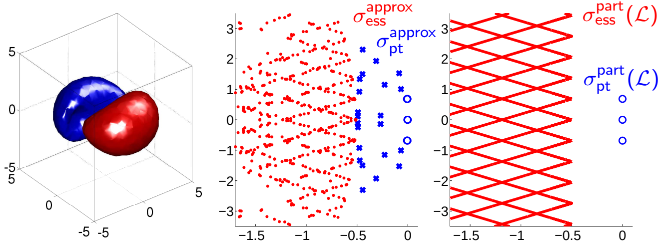
1. spatial decay of rotating waves
2. spectral properties of linearization at localized rotating waves
   - point spectrum on the imaginary axis, shape of eigenfunctions and spatial decay of eigenfunctions
   - essential spectrum

**Numerical results:**

3. approximation of rotating waves
4. approximation of spectra and eigenfunctions of linearization

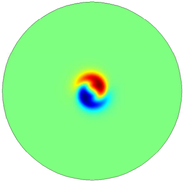
**Present to Bernold:**

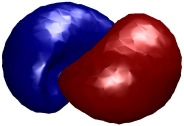
5. results on phase-shift interactions of multiple spinning solitons

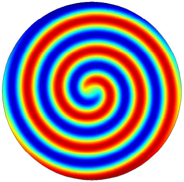


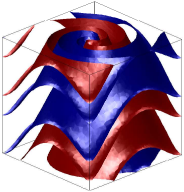
# Open problems and work in progress

- **Fredholm properties and $L^p$-spectra of localized rotating waves**
  (joint work with: W.-J. Beyn)
- **Fourier-Bessel method on $\mathbb{R}^d$ and on circular domains**
  (joint work with: W.-J. Beyn, C. Döding)
- **Freezing traveling waves in incompressible Navier-Stokes equations**
  (joint work with: W.-J. Beyn, C. Döding)
- **Rotating waves in systems of damped wave equations**
  (joint work with: W.-J. Beyn, J. Rottmann-Matthes)
- **Nonlinear stability of rotating waves for $d \geqslant 3$**
  (joint work with: W.-J. Beyn)
- **Approximation theorem for rotating waves**









# Nonlinear stability of rotating waves

## Problem 1: (Nonlinear stability of rotating waves)

For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any initial value $u_0 \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ the following property hold:
The reaction diffusion system

$$\begin{aligned} u_t(x,t) =& A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) =& u_0(x) \qquad\qquad\qquad\qquad\quad ,\ t=0,\ x\in\mathbb{R}^d, \end{aligned}$$

has a unique solution $u \in C^1(]0,\infty[, L^p(\mathbb{R}^d,\mathbb{K}^N)) \cap C([0,\infty[, W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N))$ and the solution $u$ satisfies

$$\inf_{\gamma\in\mathrm{SE}(d)} \|u(t) - a(\gamma)v_\star\|_2 \leqslant \varepsilon \quad \forall\, t \geqslant 0.$$

Moreover, there exists a $\delta_0 > 0$ such that for any initial value $u_0 \in W^{2,p}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N)$ with $\|u_0 - v_\star\|_1 \leqslant \delta$ there exists some asymptotic phase $\gamma_\infty \in \mathrm{SE}(d)$ such that the solution $u$ satisfies

$$\|u(t) - a(\gamma_\infty \circ \gamma_\star(t))v_\star\|_2 \to 0 \quad \text{as} \quad t\to\infty.$$

# Motivation 1: Nonlinear stability of rotating waves

Exponential decay and spectral properties are motivated by

nonlinear stability of rotating waves.

**Main Assumptions**: (Beyn, Lorenz, 2008)

1. (Localization condition). Pattern $v_\star$ is localized up to order 2, i.e.
   - $v_\star - v_\infty \in H^2(\mathbb{R}^2, \mathbb{R}^N)$,
   - $\sup_{|x|\geqslant R} |D^\alpha (v_\star(x) - v_\infty)| \to 0$ as $R \to \infty$, $\forall\, 0 \leqslant |\alpha| \leqslant 2$.
2. (Stability condition). $Df(v_\infty) \in \mathbb{R}^{N,N}$ is negative definite, i.e.
   - $Df(v_\infty) \leqslant -2\beta I < 0$, $\beta > 0$.
3. (Spectral condition).
   - eigenfunctions $D_1 v_\star, D_2 v_\star, D_\phi v_\star \in H^2_{\mathrm{Eucl}}(\mathbb{R}^2, \mathbb{R}^N)$ are nontrivial
   - corresponding eigenvalues $\pm ic, 0$ are algebraically simple
   - $\mathcal{L} : H^2_{\mathrm{Eucl}} \to L^2$ has no eigenvalues $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geqslant -2\beta$, except for the eigenvalues $\pm ic, 0$.

# Approximation theorem for rotating waves

## Problem 2: (Approximation theorem for rotating waves)

There exist some $\rho > 0$ and $R_0 > 0$ such that for every radius $R > R_0$ the boundary value problem

$$\begin{aligned}
0 &= A\triangle v_R(x) + \langle S_R x + \lambda_R, \nabla v_R(x)\rangle + f(v_R(x)) && , x \in B_R(0), \\
0 &= v_R(x) && , x \in \partial B_R(0), \\
0 &= \mathrm{Re}\, \langle v_R - \hat{v}, (x_j D_i - x_i D_j)\hat{v}\rangle_{L^2(B_R(0),\mathbb{K}^N)} && , i = 1,\ldots,d-1, \\
& && \ \ j = i+1,\ldots,d, \\
0 &= \mathrm{Re}\, \langle v_R - \hat{v}, D_l \hat{v}\rangle_{L^2(B_R(0),\mathbb{K}^N)} && , l = 1,\ldots,d,
\end{aligned}$$

has a unique solution $(v_R, (S_R, \lambda_R))$ in a neighborhood of

$$B_\rho(v_\star|_{B_R(0)}, (S_\star, \lambda_\star)) = \Big\{(v, (S,\lambda)) \in W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d, \mathbb{K}^N) \times \mathrm{se}(d) \mid \\ \big\| v_\star|_{B_R(0)} - v \big\|_{W^{2,2}_{\mathrm{Eucl}}(B_R(0),\mathbb{K}^N)} + d((S_\star,\lambda_\star),(S,\lambda)) \leqslant \rho \Big\}.$$

Moreover, there exist some $C > 0$ and $\eta > 0$ such that

$$\|v_R - v_\star\|_{W^{2,2}_{\mathrm{Eucl}}(\mathbb{R}^d,\mathbb{K}^N)} + d((S_R,\lambda_R),(S_\star,\lambda_\star)) \leqslant Ce^{-\eta R}.$$

# Outline

# Outline of proof: Theorem 2 (Decay of eigenfunctions)

Consider

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x) = \lambda v(x),\ x \in \mathbb{R}^d.$$

1. **Splitting off the stable part**:

$$Df(v_\star(x)) = Df(v_\infty) + (Df(v_\star(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x),\ x \in \mathbb{R}^d,$$

leads to

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q(x))\, v(x) = \lambda v(x),\ x \in \mathbb{R}^d.$$

2. **Decomposition of (the variable coefficient)** $Q$:

$$Q(x) = Q_\varepsilon(x) + Q_{\mathrm{c}}(x),\ Q_\varepsilon \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } \|\cdot\|_{C_{\mathrm{b}}},$$
$$Q_{\mathrm{c}} \in C_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d,$$

leads to

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_{\mathrm{c}}(x))\, v(x) = \lambda v(x),\ x \in \mathbb{R}^d.$$

($\rightarrow$ inhomogeneous Cauchy problem for $\mathcal{L}_{\mathrm{c}}$)

# Outline

# Outline of proof: Theorem 3 (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$)

Consider the **rotating wave equation**

(RWE) $$0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

and the $\mathrm{SE}(\mathbf{d})$**-group action**

$$[a(R,\tau)v](x) = v(R^{-1}(x-\tau)), \quad x \in \mathbb{R}^d, (R,\tau) \in \mathrm{SE}(d).$$

**1. Generators of group action**: Applying the generators

$$D^{(i,j)} := x_j D_i - x_i D_j \quad \text{and} \quad D_l = \frac{\partial}{\partial x_l}$$

to (RWE) leads to $\frac{d(d+1)}{2}$ equations

$$\begin{aligned} 0 =& (x_j D_i - x_i D_j)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \\ 0 =& D_l \left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \end{aligned}$$

for $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$, $l = 1, \ldots, d$.

**2. Commutator relations of generators**: Using commutator relations

$$D_l D_k = D_k D_l,$$

$$D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j,$$

# Outline of proof: Theorem 3 (Point spectrum of $\mathcal{L}$ on $i\mathbb{R}$)

Consider the **rotating wave equation**

$$\text{(RWE)} \qquad 0 = A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

and the $\mathrm{SE}(\mathbf{d})$**-group action**

$$[a(R,\tau)v](x) = v(R^{-1}(x-\tau)), \quad x \in \mathbb{R}^d, (R,\tau) \in \mathrm{SE}(d).$$

**1. Generators of group action**: Applying the generators

$$D^{(i,j)} := x_j D_i - x_i D_j \quad \text{and} \quad D_l = \frac{\partial}{\partial x_l}$$

to (RWE) leads to $\frac{d(d+1)}{2}$ equations

$$\begin{aligned} 0 =& (x_j D_i - x_i D_j)\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \\ 0 =& D_l\left(A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x))\right) \end{aligned}$$

for $i = 1, \ldots, d-1$, $j = i+1, \ldots, d$, $l = 1, \ldots, d$.

**2. Commutator relations of generators**: Using commutator relations

$$\begin{aligned} &D_l D_k = D_k D_l, \\ &D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j, \end{aligned}$$

# Outline

# Outline of proof: Theorem 4 (Fredholm properties of $\mathcal{L}$)

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\star(x))v(x),\ x \in \mathbb{R}^d.$$

1. **Splitting off the stable part**:

$$Df(v_\star(x)) = Df(v_\infty) + (Df(v_\star(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x),\ x \in \mathbb{R}^d,$$

leads to

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + (Df(v_\infty) + Q(x))\, v(x),\ x \in \mathbb{R}^d.$$

2. **Decomposition of (the variable coefficient)** $Q$:

$$Q(x) = Q_\varepsilon(x) + Q_\mathrm{c}(x),\ Q_\varepsilon \in C_\mathrm{b}(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } \|\cdot\|_{C_\mathrm{b}},$$
$$Q_\mathrm{c} \in C_\mathrm{b}(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d,$$

allows us to decompose the differential operator $\lambda I - \mathcal{L}$ into

$$\lambda I - \mathcal{L} = \lambda I - \mathcal{L}_\mathrm{c} = (I - Q_\mathrm{c}(\cdot)(\lambda I - \mathcal{L}_\mathrm{s})^{-1})(\lambda I - \mathcal{L}_\mathrm{s}).$$

3. **Fredholm properties of each factor**:
   - $\lambda I - \mathcal{L}_\mathrm{s}$ Fredholm of index 0: unique solvability of resolvent equation for $\mathcal{L}_\mathrm{s}$.
   - $I - Q_\mathrm{c}(\cdot)(\lambda I - \mathcal{L}_\mathrm{s})^{-1}$ Fredholm of index 0: compact perturbation of identity, unique solvability of resolvent equation for $\mathcal{L}_\mathrm{s}$ and $\mathcal{D}^p_\mathrm{loc}(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^N)$.
   - $\lambda I - \mathcal{L}$ Fredholm of index 0: Theorem on products of Fredholm operators

# Outline

# Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Linearization at the profile $v_\star$:**

$$[\mathcal{L}v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + Df(v_\infty)v(x) + Q(x)v(x)$$

$$Q(x) := Df(v_\star(x)) - Df(v_\infty), \qquad \sup_{|x|\geqslant R} |Q(x)|_2 \to 0 \text{ as } R \to \infty$$

**1. Orthogonal transformation:** $S \in \mathbb{R}^{d,d}$, $S^T = -S$, $S = P\Lambda^S_{\mathrm{block}}P^T$.
$T_1(x) = Px$ yields

$$[\mathcal{L}_1 v](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

with

$$\left\langle \Lambda^S_{\mathrm{block}}x, \nabla v(x)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l}D_{2l-1} - x_{2l-1}D_{2l}\right) v(x).$$

# Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Orthogonal transformation:**

$$\left[\mathcal{L}_1 v\right](x) = A\triangle v(x) + \left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x)\right\rangle + Df(v_\infty)v(x) + Q(T_1(x))v(x)$$

$$\left\langle \Lambda^S_{\mathrm{block}} x, \nabla v(x)\right\rangle = \sum_{l=1}^{k} \sigma_l \left(x_{2l}D_{2l-1} - x_{2l-1}D_{2l}\right) v(x)$$

**2. Several planar polar coordinates:** Transformation

$$\begin{pmatrix} x_{2l-1} \\ x_{2l} \end{pmatrix} = T(r_l,\phi_l) := \begin{pmatrix} r_l \cos\phi_l \\ r_l \sin\phi_l \end{pmatrix},\ l = 1,\ldots,k,\ \phi_l \in ]-\pi,\pi],\ r_l > 0.$$

yields for $\xi = (r_1,\phi_1,\ldots,r_k,\phi_k,x_{2k+1},\ldots,x_d)$ with total transformation $T_2(\xi)$, $Q(\xi) := Q(T_1(T_2(\xi)))$

$$\begin{aligned}\left[\mathcal{L}_2 v\right](x) =& A\left[\sum_{l=1}^{k}\left(\partial^2_{r_l} + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial^2_{\phi_l}\right) + \sum_{l=2k+1}^{d}\partial^2_{x_l}\right]v(\xi) \\ & - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),\end{aligned}$$

# Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Several planar polar coordinates:**

$$\begin{aligned}\left[\mathcal{L}_2 v\right](\xi) =& A\left[\sum_{l=1}^{k}\left(\partial_{r_l}^2 + \frac{1}{r_l}\partial_{r_l} + \frac{1}{r_l^2}\partial_{\phi_l}^2\right) + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) \\ &- \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi) + Q(\xi)v(\xi),\end{aligned}$$

$$\xi = (r_1,\phi_1,\ldots,r_k,\phi_k,x_{2k+1},\ldots,x_d), \quad Q(\xi) := Q(T_1(T_2(\xi)))$$

**3. Simplified operator (far-field linearization):** Neglecting $\mathcal{O}\left(\frac{1}{r}\right)$–terms yields

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](x) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi).$$

# Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Simplified operator (far-field linearization):**

$$\left[\mathcal{L}_2^{\mathrm{sim}} v\right](\xi) = A\left[\sum_{l=1}^{k}\partial_{r_l}^2 + \sum_{l=2k+1}^{d}\partial_{x_l}^2\right]v(\xi) - \sum_{l=1}^{k}\sigma_l\partial_{\phi_l}v(\xi) + Df(v_\infty)v(\xi)$$

**4. Angular Fourier decomposition:**

$$v(\xi) = \exp\left(i\omega\sum_{l=1}^{k} r_l\right)\exp\left(i\sum_{l=1}^{k} n_l\phi_l\right)\hat{v},\ n_l \in \mathbb{Z},\ \omega\in\mathbb{R},\ \hat{v}\in\mathbb{C}^N,\ |\hat{v}|=1$$

$$\phi_l \in ]-\pi,\pi],\ r_l>0,\ l=1,\ldots,k,$$

yields

$$\left[\left(\lambda I - \mathcal{L}_2^{\mathrm{sim}}\right)v\right](\xi) = \left(\lambda I_N + \omega^2 A + i\sum_{l=1}^{k} n_l\sigma_l I_N - Df(v_\infty)\right)v(\xi).$$

# Outline of proof: Theorem 5 (Essential spectrum of $\mathcal{L}$)

**Angular Fourier decomposition:**

$$\left[\left(\lambda I-\mathcal{L}_2^{\mathrm{sim}}\right)v\right](\xi)=\left(\lambda I_N+\kappa^2 A+i\sum_{l=1}^{k}n_l\sigma_l I_N-Df(v_\infty)\right)v(\xi).$$

$$n_l\in\mathbb{Z},\quad \kappa\in\mathbb{R},\quad \pm i\sigma_l \text{ nonzero eigenvalues of } S\in\mathbb{R}^{d,d}$$

**5. Finite-dimensional eigenvalue problem:** $\left[\left(\lambda I-\mathcal{L}_2^{\mathrm{sim}}\right)v\right](\xi)=0$ for every $\xi$ if $\lambda\in\mathbb{C}$ satisfies

$$\left(\omega^2 A-Df(v_\infty)\right)\hat{v}=-\left(\lambda+i\sum_{l=1}^{k}n_l\sigma_l\right)\hat{v},\ \text{for some } \omega\in\mathbb{R}.$$

# Outline

# The operator $\mathcal{L}_0$

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle, \; x \in \mathbb{R}^d, \; d \geqslant 2.$$

$$\downarrow$$

Heat kernel

$$H_0(x,\xi,t) = (4\pi t A)^{-\frac{d}{2}} \exp\left(-(4tA)^{-1}\left|e^{tS}x - \xi\right|^2\right), \; x,\xi \in \mathbb{R}^d, \; t > 0.$$

$$\downarrow$$

Semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, $1 \leqslant p \leqslant \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x,\xi,t)v(\xi)d\xi, \; t > 0.$$

strong $\downarrow$ continuity

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \; 1 \leqslant p < \infty.$$

semigroup theory $\swarrow$ $\searrow$ identification problem

| unique solv. of resolvent equ. for $A_p$, $1 \leqslant p < \infty$, $\mathrm{Re}\,\lambda > 0$ | A-priori $\rightarrow$ estimates | exponential decay, $1 \leqslant p < \infty$ | max. domain and max. realization, $1 < p < \infty$ |
|---|---|---|---|
| $(\lambda I - A_p)\, v_\star = g \in L^p$. | | $v_\star \in W^{1,p}_\theta$. | $A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$. |

# Identification problem of $\mathcal{L}_0$

$$\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) := \left\{ v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{C}^N) \cap L^p(\mathbb{R}^d,\mathbb{C}^N) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d,\mathbb{C}^N) \right\},\ 1<p<\infty.$$

Infinitesimal generator

$(A_p,\mathcal{D}(A_p))\,,\ 1 \leqslant p < \infty.$

$\downarrow$

$\mathcal{S}$ is a core for $(A_p,\mathcal{D}(A_p))$

$\downarrow$

Identification of $\mathcal{L}_0$

maximal domain and maximal realization for $1<p<\infty$:

$A_p = \mathcal{L}_0$ on $\mathcal{D}(A_p) = \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$

$\leftarrow$

Ornstein-Uhlenbeck operator

$[\mathcal{L}_0 v](x) = A\triangle v(x) + \langle Sx, \nabla v(x)\rangle\,,\ x\in\mathbb{R}^d,\ d \geqslant 2.$

$\downarrow$

$\mathcal{L}_0 : \mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0) \to L^p(\mathbb{R}^d,\mathbb{C}^N)$

is a closed operator, $1<p<\infty$

$\downarrow$

$L^p$-resolvent estimates

and

unique solv. of resolvent equ.

for $\mathcal{L}_0$ in $\mathcal{D}^p_{\mathrm{loc}}(\mathcal{L}_0)$,

$1<p<\infty$

$\uparrow$

$L^p$-dissipativity condition: $\exists\, \gamma_A > 0$

$$|z|^2 \mathrm{Re}\,\langle w, Aw\rangle + (p-2)\mathrm{Re}\,\langle w,z\rangle\,\mathrm{Re}\,\langle z, Aw\rangle \geqslant \gamma_A |z|^2|w|^2\ \forall\, z,w\in\mathbb{K}^N$$

$\updownarrow$

$L^p$-first antieigenvalue condition

$$\mu_1(A) := \inf_{\substack{w\in\mathbb{K}^N\\ w\neq 0\\ Aw\neq 0}} \frac{\mathrm{Re}\,\langle w, Aw\rangle}{|w||Aw|} > \frac{|p-2|}{p},\quad 1<p<\infty$$