Exponential decay of two-dimensional rotating waves (Part 1)

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Outline

Problem

Consider the stationary problem

$$\alpha\triangle u + cD_\phi u + f(u) = 0,\ x \in \mathbb{R}^2$$

where $u : \mathbb{R}^2 \to \mathbb{R}^N$ is unknown, $\alpha \in \mathbb{R}$ with $\alpha > 0$, $c \in \mathbb{R}$ with $c \neq 0$ and $f : \mathbb{R}^N \to \mathbb{R}^N$ are given and D_ϕ is defined as

$$D_\phi := -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$$

Under the assumptions

There exists a constant vector $u_\infty \in \mathbb{R}^N$ such that

(A1) $\lim_{R\to\infty}\sup_{|x|\geqslant R}|u(x) - u_\infty| = 0$,

(A2) $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $B_\infty := Df(u_\infty)$ is negative definite.

we want to show unique solvability and exponential decay, i.e.

$$|u(x) - u_\infty| \leqslant Ce^{-C|x|},$$

$$\left|D^\beta u(x)\right| \leqslant Ce^{-C|x|},\ 1 \leqslant |\beta| \leqslant 2.$$

Example

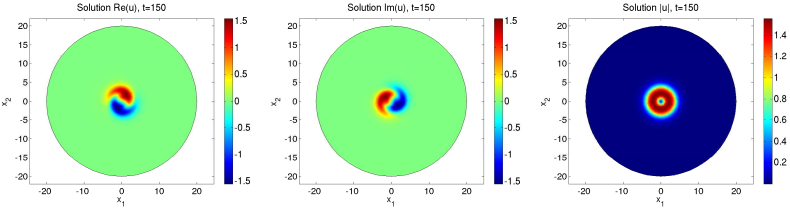
Consider the quintic complex Ginzburg-Landau equation (QCGL):

$$u_t = \alpha \triangle u + u \left(\mu + \beta |u|^2 + \gamma |u|^4 \right)$$

with $u : \mathbb{R}^2 \times [0, \infty[\to \mathbb{C}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2} i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10} i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called spinning soliton solutions.



Motivation

Consider the stationary problem

$$\alpha\triangle u + cD_\phi u + f(u) = 0,\ x \in \mathbb{R}^2.$$

Let $u_\infty \in \mathbb{R}^N$ be a stationary point (satisfying (A1) and (A2))

$$\alpha\triangle u_\infty + cD_\phi u_\infty + f(u_\infty) = 0$$

i.e. $f(u_\infty) = 0$. Since $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ by Taylor's theorem we obtain for every $u = u(x) \in \mathbb{R}^N$

$$f(u) = \underbrace{f(u_\infty)}_{=0} + \underbrace{\int_0^1 Df(u_\infty + t(u - u_\infty))dt}_{=:a(x)}\,(u - u_\infty)\,.$$

Using assumption (A1) we have

$$a(x) \to B_\infty,\ \text{as}\ |x| \to \infty$$

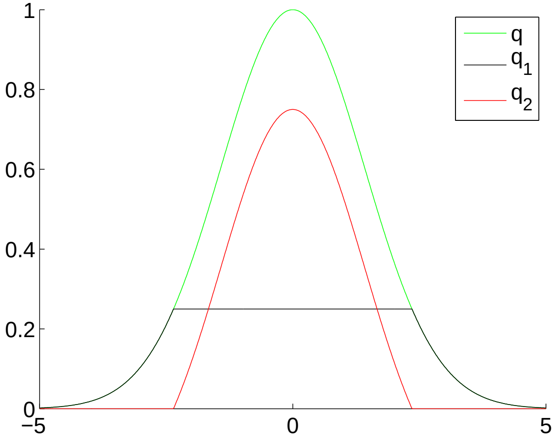
where $B_\infty := Df(u_\infty) \in \mathbb{R}^{N\times N}$. Define $q(x) := a(x) - B_\infty$, then

$$q(x) \to 0,\ \text{as}\ |x| \to \infty.$$

Now we decompose q in the following way

$$q(x) = q_1(x) + q_2(x)$$

where q_1 is a small perturbation and q_2 is compactly supported.



From the preliminary idea we obtain

$$\begin{aligned} 0 &= \alpha\triangle u + cD_\phi u + f(u) \\ &= \alpha\triangle u + cD_\phi u + au \\ &= \alpha\triangle u + cD_\phi u + B_\infty u + qu \\ &= \alpha\triangle u + cD_\phi u + B_\infty u + q_1 u + q_2 u. \end{aligned}$$

Therefore, we must study the following operators

$$\begin{aligned} \mathcal{L}_\infty u &:= \alpha\triangle u + cD_\phi u + B_\infty u, && \text{(const. coeff. operator)} \\ \mathcal{L}_{q_1} u &:= \alpha\triangle u + cD_\phi u + B_\infty u + q_1 u, && \text{(small pert. of } \mathcal{L}_\infty\text{)} \\ \mathcal{L}_q u &:= \alpha\triangle u + cD_\phi u + B_\infty u + qu. && \text{(compact pert. of } \mathcal{L}_{q_1}\text{)} \end{aligned}$$

Today we will only analyze the $\mathcal{L}_\infty$-operator.

The operator $\mathcal{L}_\infty$

Consider the operator

$$\mathcal{L}_\infty u := \alpha \triangle u + cD_\phi u + B_\infty u = g$$

where $B_\infty := Df(u_\infty) \in \mathbb{K}^{N\times N}$. To decouple the equation let us assume that B_∞ is diagonalizable over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, i. e.

$$\exists\, Y \in \mathbb{K}^{N\times N}:\; Y^{-1} B_\infty Y = \Lambda_\infty$$

where $\lambda_\infty = \mathrm{diag}(\lambda_1^\infty, \ldots, \lambda_N^\infty)$ and $\lambda_1^\infty, \ldots, \lambda_N^\infty \in \mathbb{K}$ are the eigenvalues of B_∞. Since B_∞ is assumed to be negative definite, $\mathrm{Re}\ \lambda_i^\infty < 0$ for every $i = 1, \ldots, N$. Substituting in $\mathcal{L}_\infty$ we obtain

$$\alpha \triangle u + cD_\phi u + Y\Lambda_\infty Y^{-1} u = g.$$

Multiplying from left by Y^{-1}

$$\alpha \triangle Y^{-1} u + cD_\phi Y^{-1} u + \Lambda_\infty Y^{-1} u = Y^{-1} g$$

and substituting $v := Y^{-1} u$ as well as $r := Y^{-1} g$ we finally arrive at

$$\alpha \triangle v + cD_\phi v + \Lambda_\infty v = r.$$

By this way we have decoupled the system. Therefore, it is sufficient to analyze the operator $\mathcal{L}_\infty$ in the scalar case with $N = 1$, i. e.

$$\mathcal{L}_\infty u = \alpha \triangle u + cD_\phi u - \delta u = g$$

where $\alpha, \delta \in \mathbb{K}$ with $\mathrm{Re}\ \alpha, \mathrm{Re}\ \delta > 0$. For convenience we discuss only the case $\mathbb{K} = \mathbb{R}$. To show exponential decay we consider this operator on an exponentially weighted function space.

Weight Functions

Let $\omega^{-1} \in C^1(\mathbb{R}^d,]0,\infty[) \cap L^1(\mathbb{R}^d,]0,\infty[)$ be a positive and integrable weight function with

$$|\nabla\omega(x)| \leqslant C\omega(x) \; \forall x \in \mathbb{R}^d.$$

Example

Let $\eta \in \mathbb{R}$ with $\eta \geqslant 0$

- $\omega_\eta(x) = e^{\eta|x|}$ with $C = \eta$ (lack of differentiability at $x = 0$)
- $\omega_\eta(x) = e^{\eta\sqrt{|x|^2+1}}$ with $C = \eta$ (smooth version of $e^{\eta|x|}$)
- $\omega_\eta(x) = \cosh(\eta\,|x|)$ with $C = \eta$ (smooth at $x = 0$)

Henceforth we will consider $\omega_\eta(x) = e^{\eta|x|}$.

Weighted Sobolev Spaces

Let $p \in \mathbb{R}$ with $1 \leqslant p < \infty$ and $d, N \in \mathbb{N}$. Define the exponentially weighted L^p-space

$$L^p_\eta(\mathbb{R}^d, \mathbb{R}^N) := \{u \in L_{loc}(\mathbb{R}^d, \mathbb{R}^N) \mid \|u\|_{L^p_\eta} < \infty\},$$

$$\|u\|_{L^p_\eta} := \left(\int_{\mathbb{R}^d} \left(e^{\eta|x|}\,|u(x)|\right)^p dx\right)^{\frac{1}{p}},$$

$$W^{k,p}_\eta(\mathbb{R}^d, \mathbb{R}^N) = \{u \in L^p_\eta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\eta(\mathbb{R}^d, \mathbb{R}^N)\ \forall\ |\beta| \leqslant k\},$$

$$\|u\|_{W^{k,p}_\eta} := \left(\sum_{|\beta|\leqslant k} \left\|D^\beta u\right\|^p_{L^p_\eta}\right)^{\frac{1}{p}}.$$

$\left(W^{k,p}_\eta(\mathbb{R}^d, \mathbb{R}^N), \|\bullet\|_{W^{k,p}_\eta}\right)$ is a Banach space for $1 \leqslant p < \infty$ and $\left(H^k_\eta(\mathbb{R}^d, \mathbb{R}^N) := W^{k,2}_\eta(\mathbb{R}^d, \mathbb{R}^N), \|\bullet\|_{H^k_\eta} := \|\bullet\|_{W^{k,2}_\eta}\right)$ is a Hilbert space for $k \in \mathbb{N}_0$. Remark: $W^{k,p}_0(\mathbb{R}^d, \mathbb{R}^N) = W^{k,p}(\mathbb{R}^d, \mathbb{R}^N)$.

Homogeneous Equation

Consider the initial value problem

$$\begin{aligned} u_t &= \alpha \triangle u + c D_\phi u - \delta u = \mathcal{L}_\infty u \\ u(0) &= u_0 \end{aligned} \tag{1}$$

where $u : \mathbb{R}^2 \times [0,\infty[\to \mathbb{R}$ is unknown, $\triangle$ denotes the Laplacian and D_ϕ denotes the angular derivative given by

$$\triangle := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \; D_\phi := -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}.$$

Assumptions:

(A3) $\alpha \in \mathbb{R}$ with $\alpha > 0$ (diffusion coefficient)

(A4) $c \in \mathbb{R}$ with $c \neq 0$ (angular velocity)

(A5) $\delta \in \mathbb{R}$ with $\delta > 0$ (propagation constant)

(A6) $\eta \in \mathbb{R}$ with $\eta \geqslant 0$ (decay rate)

(A7) $u_0 \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ with $p \in \mathbb{R}$ and $1 \leqslant p < \infty$ (initial data)

Definition

A heat kernel of (1) with respect to $(L^p_\eta(\mathbb{R}^2,\mathbb{R}), \|\bullet\|_{L^p_\eta})$ is a function

$$H : \mathbb{R}^2 \times \mathbb{R}^2 \times]0,\infty[\to \mathbb{R} \text{ with } (x,\xi,t) \mapsto H(x,\xi,t),$$

such that

(H1) $H \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times]0,\infty[, \mathbb{R})$

(H2) $H_t(\bullet,\xi,t) = \mathcal{L}_\infty H(\bullet,\xi,t) \quad \forall\, \xi \in \mathbb{R}^2 \;\forall\, t > 0$

(H3) $\forall\, u_0 \in L^p_\eta(\mathbb{R}^2,\mathbb{R}) : \lim\limits_{\substack{t\to 0\\ t>0}} \left\| \int_{\mathbb{R}^2} H(\bullet,\xi,t)u_0(\xi)d\xi - u_0(\bullet) \right\|_{L^p_\eta} = 0$

Let H be a heat kernel of (1), then the solution of (1) is given by

$$u(x,t) = \begin{cases} \int_{\mathbb{R}^2} H(x,\xi,t)u_0(\xi)d\xi & , t > 0 \\ u_0(x) & , t = 0 \end{cases} =: e^{t\mathcal{L}_\infty} u_0(x).$$

Hence, we get the stationary solution $\bar{u}$ of (1) by going to the limit $t \to \infty$.

Theorem

Under the assumptions (A3)–(A7) the heat kernel $H : \mathbb{R}^2 \times \mathbb{R}^2 \times]0, \infty[\to \mathbb{R}$ of (1) with respect to $(L^p_\eta(\mathbb{R}^2, \mathbb{R}), \|\bullet\|_{L^p_\eta})$ is given by

$$H(x, \xi, t) = \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2}$$

where

$$Q := \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix} \in \mathbb{R}^2,\ D_\phi := -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}.$$

Now the solution of (1) is given by

$$u(x, t) = \begin{cases} \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2} u_0(\xi) d\xi & , \ t > 0 \\ u_0(x) & , \ t = 0 \end{cases} =: e^{t\mathcal{L}_\infty} u_0(x).$$

Theorem

Let the assumptions (A3)–(A7) be satisfied with $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p^{\frac{1}{2}}}$, *then we have for* $t > 0$

$$\|u(t)\|_{L^p_\eta} \leqslant C(t) \|u_0\|_{L^p_\eta} e^{-\left(\delta - \eta^2 p \alpha\right) t},$$

$$\|D_i u(t)\|_{L^p_\eta} \leqslant C(t) \|u_0\|_{L^p_\eta} e^{-\left(\delta - \eta^2 p \alpha\right) t}, \ i = 1, 2.$$

Hence the stationary solution (of the homogeneous equation) is $\bar{u} \equiv 0$ (in L^p_η–sense).

Inhomogeneous Equation

Consider the initial value problem

$$\begin{aligned} u_t &= \alpha \triangle u + c D_\phi u - \delta u - g = \mathcal{L}_\infty u - g \\ u(0) &= u_0 \end{aligned} \tag{2}$$

Assumption:

(A8) $g \in L^p_\eta(\mathbb{R}^2, \mathbb{R})$ (inhomogenity)

By Duhamel's principle we obtain the solution

$$u(t) = e^{t\mathcal{L}_\infty} u_0 - \int_0^t e^{(t-s)\mathcal{L}_\infty} g ds.$$

Again, by going to the limit $t \to \infty$ we obtain the stationary solution of (2)

$$\bar{u}(x) = -\int_{\mathbb{R}^2} \int_0^\infty \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2} g(\xi) dt d\xi$$

Remark: The Green's function coincides with the integral over the heat kernel

$$G(x,\xi) = \int_0^\infty H(x,\xi,t) dt$$

Radial exponential decay and regularity estimates

Theorem

Let the assumptions (A3)–(A8) be satisfied with $0 \leqslant \eta < \frac{\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}} p}$ *and let* $\bar{u}(x)$ *denote the solution of* $\mathcal{L}_\infty u = g$*, then we have* $\bar{u} \in W^{1,p}_\eta(\mathbb{R}^2, \mathbb{R})$ *with*

$$\|\bar{u}\|_{L^p_\eta} \leqslant C_5 \|g\|_{L^p_\eta},$$
$$\|D_i\bar{u}\|_{L^p_\eta} \leqslant C_6 \|g\|_{L^p_\eta},\ i = 1, 2,$$

where $C_j = C_j(\alpha, \delta, \eta, p) > 0$*,* $j = 5, 6$*.*

Unfortunately, up to now the estimates on $|\alpha| \|\triangle\bar{u}\|_{L^p_\eta}$ and $|c| \|D_\phi\bar{u}\|_{L^p_\eta}$ have failed. Nevertheless as a consequence we have

$$\mathcal{L}_\infty : L^p_\eta(\mathbb{R}^2, \mathbb{R}) \supset \mathcal{D}(\mathcal{L}_\infty) \to L^p_\eta(\mathbb{R}^2, \mathbb{R})$$

is a linear, densely defined, closed operator.

Proof

For $1 < p < \infty$ we obtain

$$
\begin{aligned}
&\|\bar{u}\|_{L^p_\eta} = \left\| e^{\eta|\bullet|}\bar{u}(\bullet) \right\|_{L^p} \\
&= \left\| \int_0^\infty \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}\bullet - \xi\right|^2 + \eta|\bullet|} g(\xi) d\xi dt \right\|_{L^p} \\
&\leqslant \int_0^\infty \left\| \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}\bullet - \xi\right|^2 + \eta|\bullet|} g(\xi) d\xi \right\|_{L^p} dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2 + \eta|x|} g(\xi) d\xi \right|^p dx \right)^{\frac{1}{p}} dt \\
&\leqslant \int_0^\infty \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2 + \eta|x|} \left|g(\xi)\right| d\xi \right)^p dx \right)^{\frac{1}{p}} dt
\end{aligned}
$$

Transformation theorem: $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ open, $\Phi : \Omega \to \Phi(\Omega) \subset \mathbb{R}^d$ diffeomorphism, u integrable on $\Phi(\Omega)$, then it holds

$$\int_{\Phi(\Omega)} u(y)dy = \int_{\Omega} u(\Phi(x)) \left|\det(D\Phi(x))\right| dx$$

Using the transformation theorem in ξ with $\Phi(\xi) = e^{tQ}x - \xi =: \psi$, $\Omega = \mathbb{R}^2$ ($\Rightarrow \xi = e^{tQ}x - \psi$ and $\det(D\Phi(\xi)) = 1$, $\Phi(\mathbb{R}^2) = \mathbb{R}^2$)

$$\int_0^\infty \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}\left|e^{tQ}x - \xi\right|^2 + \eta|x|} \left|g(\xi)\right| d\xi\right)^p dx\right)^{\frac{1}{p}} dt$$

$$= \int_0^\infty \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2 + \eta|x|} \left|g(e^{tQ}x - \psi)\right| d\psi\right)^p dx\right)^{\frac{1}{p}} dt$$

Splitting the heat kernel and using Hölder's inequality in ψ with $1 < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ yields

$$= \int_0^\infty \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \left(\frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \right)^{\frac{1}{q}} \left(\frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \right)^{\frac{1}{p}} e^{\eta|x|} \left| g(e^{tQ}x - \psi) \right| d\psi \right)^p dx \right)^{\frac{1}{p}} dt$$

$$\leqslant \int_0^\infty \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} d\psi \right)^{\frac{p}{q}} \int_{\mathbb{R}^2} \left(\left(\frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \right)^{\frac{1}{p}} e^{\eta|x|} \left| g(e^{tQ}x - \psi) \right| \right)^p d\psi dx \right)^{\frac{1}{p}} dt$$

$$= \int_0^\infty \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} d\psi \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} e^{\eta p|x|} \left| g(e^{tQ}x - \psi) \right|^p d\psi dx \right)^{\frac{1}{p}} dt$$

By the transformation theorem it is easy to verify that

$$\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} d\psi = e^{-\delta t}.$$

Using this and Fubini's theorem we resume with (remark that $\frac{1}{q} = \frac{p-1}{p}$)

$$= \int_0^\infty \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} d\psi \right)^{\frac{1}{q}}$$

$$\left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} e^{\eta p|x|} \left| g(e^{tQ}x - \psi) \right|^p d\psi dx \right)^{\frac{1}{p}} dt$$

$$= \int_0^\infty e^{-\frac{1}{q}\delta t} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \right.$$

$$\left. \int_{\mathbb{R}^2} e^{\eta p|x|} \left| g(e^{tQ}x - \psi) \right|^p dx d\psi \right)^{\frac{1}{p}} dt$$

Transformation theorem: $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ open, $\Phi : \Omega \to \Phi(\Omega) \subset \mathbb{R}^d$ diffeomorphism, u integrable on $\Phi(\Omega)$, then it holds

$$\int_{\Phi(\Omega)} u(y)dy = \int_{\Omega} u(\Phi(x)) \left|\det(D\Phi(x))\right| dx$$

Using the transformation theorem in x with $\Phi(x) = e^{tQ}x - \psi =: y$, $\Omega = \mathbb{R}^2$ ($\Rightarrow x = e^{-tQ}(y + \psi)$ and $\det(D\Phi(x)) = 1$, $\Phi(\mathbb{R}^2) = \mathbb{R}^2$) and remark that $\left|e^{tQ}\zeta\right| = |\zeta|$

$$= \int_0^\infty e^{-\frac{1}{q}\delta t} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \int_{\mathbb{R}^2} \left(e^{\eta|x|} \left| g(e^{tQ}x - \psi) \right| \right)^p dx d\psi \right)^{\frac{1}{p}} dt$$

$$= \int_0^\infty e^{-\frac{1}{q}\delta t} \left(\int_{\mathbb{R}^2} \frac{1}{4\pi\alpha t} e^{-\delta t - \frac{1}{4\alpha t}|\psi|^2} \int_{\mathbb{R}^2} \left(e^{\eta|y+\psi|} \left| g(y) \right| \right)^p dy d\psi \right)^{\frac{1}{p}} dt$$

Once again, using Hölder's inequality in y with $\frac{1}{p}+\frac{1}{\infty}=\frac{1}{p}$ we obtain

$$\int_0^\infty e^{-\frac{1}{q}\delta t}\left(\int_{\mathbb{R}^2}\frac{1}{4\pi\alpha t}e^{-\delta t-\frac{1}{4\alpha t}|\psi|^2}\int_{\mathbb{R}^2}\left(e^{\eta|y+\psi|}\left|g(y)\right|\right)^p dyd\psi\right)^{\frac{1}{p}}dt$$

$$=\int_0^\infty e^{-\frac{1}{q}\delta t}\left(\int_{\mathbb{R}^2}\frac{1}{4\pi\alpha t}e^{-\delta t-\frac{1}{4\alpha t}|\psi|^2}\right.$$

$$\left.\int_{\mathbb{R}^2}\left(e^{\eta(|y+\psi|-|y|)}e^{\eta|y|}\left|g(y)\right|\right)^p dyd\psi\right)^{\frac{1}{p}}dt$$

$$\leqslant\int_0^\infty e^{-\frac{1}{q}\delta t}\left(\int_{\mathbb{R}^2}\frac{1}{4\pi\alpha t}e^{-\delta t-\frac{1}{4\alpha t}|\psi|^2}\right.$$

$$\left.\left(\operatorname*{ess\,sup}_{y\in\mathbb{R}^2}e^{\eta(|y+\psi|-|y|)}\right)^p\int_{\mathbb{R}^2}\left(e^{\eta|y|}\left|g(y)\right|\right)^p dyd\psi\right)^{\frac{1}{p}}dt$$

$$=\|g\|_{L^p_\eta}\int_0^\infty e^{-\frac{1}{q}\delta t}\left(\frac{1}{4\pi\alpha t}e^{-\delta t}\int_{\mathbb{R}^2}e^{-\frac{1}{4\alpha t}|\psi|^2+\eta p|\psi|}d\psi\right)^{\frac{1}{p}}dt$$

Integral 1: Let $A, B \in \mathbb{R}$, then

$$\int_0^\infty s e^{-As^2+Bs} ds = \frac{1}{2A} + \frac{B\pi^{\frac{1}{2}}}{4A^{\frac{3}{2}}} e^{\frac{B^2}{4A}} \left(\operatorname{erf}\left(\left(\frac{B^2}{4A} \right)^{\frac{1}{2}} \right) + 1 \right)$$

Using the transformation theorem to transform the ψ-integral into polar coordinates and using the above mentioned Integral 1 with $A = \frac{1}{4\alpha t}$ and $B = \eta p$

$$= \|g\|_{L^p_\eta} \int_0^\infty e^{-\frac{1}{q}\delta t} \left(\frac{1}{4\pi\alpha t} e^{-\delta t} \int_{\mathbb{R}^2} e^{-\frac{1}{4\alpha t}|\psi|^2 + \eta p|\psi|} d\psi \right)^{\frac{1}{p}} dt$$

$$= \|g\|_{L^p_\eta} \int_0^\infty e^{-\frac{1}{q}\delta t} \left(\frac{1}{2\alpha t} e^{-\delta t} \int_0^\infty s e^{-\frac{1}{4\alpha t}s^2 + \eta p s} ds \right)^{\frac{1}{p}} dt$$

$$= \|g\|_{L^p_\eta} \int_0^\infty e^{-\frac{1}{q}\delta t} \left(e^{-\delta t} + \pi^{\frac{1}{2}} \left(\eta^2 p^2 \alpha t \right)^{\frac{1}{2}} e^{-\left(\delta - \eta^2 p^2 \alpha\right) t} \right.$$
$$\left. \left(\operatorname{erf}\left(\left(\eta^2 p^2 \alpha t \right)^{\frac{1}{2}} \right) + 1 \right) \right)^{\frac{1}{p}} dt$$

Finally, using Hölder's inequality in t with $\frac{1}{p} + \frac{1}{q} = 1$ we obtain

$$= \|g\|_{L^p_\eta} \int_0^\infty e^{-\frac{1}{q}\delta t} \left(e^{-\delta t} + \pi^{\frac{1}{2}} \left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}} e^{-\left(\delta t - \eta^2 p^2 \alpha\right)t} \right.$$

$$\left. \left(\mathrm{erf}\left(\left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}} \right) + 1 \right) \right)^{\frac{1}{p}} dt$$

$$\leqslant \|g\|_{L^p_\eta} \left(\int_0^\infty e^{-\delta t} dt \right)^{\frac{1}{q}} \left(\int_0^\infty e^{-\delta t} \right.$$

$$\left. + \pi^{\frac{1}{2}} \left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}} e^{-\left(\delta - \eta^2 p^2 \alpha\right)t} \left(\mathrm{erf}\left(\left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}} \right) + 1 \right) dt \right)^{\frac{1}{p}}$$

Integral 2: Let $A, B \in \mathbb{R}$ with $0 \leqslant B < A$, then

$$\int_0^\infty t^{\frac{1}{2}} e^{-(A-B)t} \left(\operatorname{erf}\left(B^{\frac{1}{2}} t^{\frac{1}{2}}\right) + 1\right) dt$$

$$= \pi^{-\frac{1}{2}} A^{-1} (A-B)^{-\frac{3}{2}} \left(\pi A + (A-B)^{\frac{1}{2}} B^{\frac{1}{2}} - \arctan\left(\left(\frac{A-B}{B}\right)^{\frac{1}{2}}\right)\right)$$

Using the above mentioned Integral 2 with $A = \delta$ and $B = \eta^2 p^2 \alpha$

$$\|g\|_{L^p_\eta} \left(\int_0^\infty e^{-\delta t} dt\right)^{\frac{1}{q}} \left(\int_0^\infty e^{-\delta t}\right.$$

$$\left. + \pi^{\frac{1}{2}} \left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}} e^{-\left(\delta - \eta^2 p^2 \alpha\right) t} \left(\operatorname{erf}\left(\left(\eta^2 p^2 \alpha t\right)^{\frac{1}{2}}\right) + 1\right) dt\right)^{\frac{1}{p}}$$

$$= \|g\|_{L^p_\eta} \left(\frac{1}{\delta}\right)^{\frac{1}{q}} \left(\frac{1}{\delta} + \pi \eta p \alpha \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} + \frac{\eta^2 p^2 \alpha^{-\frac{3}{2}}}{\delta} \left(\delta - \eta^2 p^2 \alpha\right)^{-1}\right.$$

$$\left. - \frac{\eta p \alpha}{\delta} \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} \arctan\left(\left(\frac{\delta - \eta^2 p^2 \alpha}{\eta^2 p^2 \alpha}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{p}}$$

Because of $\frac{1}{q} = \frac{p-1}{p}$ we finally get

$$= \|g\|_{L^p_\eta} \left(\frac{1}{\delta}\right)^{\frac{1}{q}} \left(\frac{1}{\delta} + \pi\eta p\alpha \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} + \frac{\eta^2 p^2 \alpha^{-\frac{3}{2}}}{\delta} \left(\delta - \eta^2 p^2 \alpha\right)^{-1} - \frac{\eta p \alpha}{\delta} \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} \arctan\left(\left(\frac{\delta - \eta^2 p^2 \alpha}{\eta^2 p^2 \alpha}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{p}}$$

$$= \|g\|_{L^p_\eta} \left(\frac{1}{\delta^p} + \frac{\pi\eta p\alpha}{\delta^{p-1}} \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} + \frac{\eta^2 p^2 \alpha^{-\frac{3}{2}}}{\delta^p} \left(\delta - \eta^2 p^2 \alpha\right)^{-1} - \frac{\eta p \alpha}{\delta^p} \left(\delta - \eta^2 p^2 \alpha\right)^{-\frac{3}{2}} \arctan\left(\left(\frac{\delta - \eta^2 p^2 \alpha}{\eta^2 p^2 \alpha}\right)^{\frac{1}{2}}\right)\right)^{\frac{1}{p}}$$

$$=: C_5 \|g\|_{L^p_\eta}$$

Open Problems

- Solvability in complex case (i. e. $\alpha, \delta \in \mathbb{C}$ with $\mathrm{Re}\ \alpha, \mathrm{Re}\ \delta > 0$)
- Estimates of $|\alpha| \, \|\triangle \bar{u}\|_{L^p_\eta}$ and $|c| \, \|D_\phi \bar{u}\|_{L^p_\eta}$
- Analytical representation of $\mathcal{D}(\mathcal{L}_\infty)$
- Heat kernel w. r. t. $\left(BC_{unif}(\mathbb{R}^2, \mathbb{R}), \|\bullet\|_{\infty,\eta}\right)$

Why we can choose w.l.o.g. $u_\infty = 0$?

Let $u_\infty \in \mathbb{R}^N$ be a stationary point (i.e. $\alpha \triangle u_\infty + cD_\phi u_\infty + f(u_\infty) = 0$). Defining $v = u - u_\infty$ and $g(v) = f(u_\infty + v)$ we obtain by Taylor's theorem for every $u = u(x) \in \mathbb{R}^N$

$$
\begin{aligned}
v_t &= (u - u_\infty)_t = u_t \\
&= \alpha \triangle u + cD_\phi u + f(u) \\
&= \alpha \triangle u + cD_\phi u + f(u_\infty) + \int_0^1 Df(u_\infty + t(u - u_\infty))dt(u - u_\infty) \\
&= \alpha \triangle (u - u_\infty) + cD_\phi (u - u_\infty) + \int_0^1 Df(u_\infty + t(u - u_\infty))dt(u - u_\infty) \\
&= \alpha \triangle v + cD_\phi v + \int_0^1 Df(u_\infty + tv)dtv \\
&= \alpha \triangle v + cD_\phi v + g(v)
\end{aligned}
$$

where $v_\infty := 0 \in \mathbb{R}^N$ is a stationary point of the v-equation. Hence, we can assume w.l.o.g. $u_\infty = 0$.

The operator $\mathcal{L}_\infty$

Consider the operator

$$\mathcal{L}_\infty u := A\triangle u + cD_\phi u + B_\infty u = g$$

where $B_\infty := Df(u_\infty) \in \mathbb{K}^{N\times N}$ and $A \in \mathbb{K}^{N\times N}$. To decouple the equation let us assume that A and B_∞ are simultaneously diagonalizable over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, i. e.

$$\exists\, Y \in \mathbb{K}^{N\times N}:\; Y^{-1}AY = \Lambda_A \text{ and } Y^{-1}B_\infty Y = \Lambda_\infty$$

where $\lambda_A = \mathrm{diag}(\lambda_1^A, \ldots, \lambda_N^A)$, $\lambda_\infty = \mathrm{diag}(\lambda_1^\infty, \ldots, \lambda_N^\infty)$, $\lambda_1^A, \ldots, \lambda_N^A \in \mathbb{K}$ and $\lambda_1^\infty, \ldots, \lambda_N^\infty \in \mathbb{K}$ are the eigenvalues of A and B_∞, respectively. Since A and B_∞ are assumed to be positive and negative definite, respectively, $\mathrm{Re}\;\lambda_i^A > 0$ and $\mathrm{Re}\;\lambda_i^\infty < 0$ for every $i = 1, \ldots, N$. Substituting in $\mathcal{L}_\infty$ we obtain

$$Y\Lambda_A Y^{-1}\triangle u + cD_\phi u + Y\Lambda_\infty Y^{-1}u = g.$$

Multiplying from left by Y^{-1}

$$\Lambda_A \triangle Y^{-1}u + cD_\phi Y^{-1}u + \Lambda_\infty Y^{-1}u = Y^{-1}g$$

and substituting $v := Y^{-1}u$ as well as $r := Y^{-1}g$ we finally arrive at

$$\Lambda_A \triangle v + cD_\phi v + \Lambda_\infty v = r.$$

By this way we have decoupled the system. Therefore it is sufficient to analyse the operator $\mathcal{L}_\infty$ in the scalar case with $N = 1$, i.e.

$$\mathcal{L}_\infty u := \alpha \triangle u + cD_\phi u - \delta u = g$$

where $\alpha, \delta \in \mathbb{K}$ with $\mathrm{Re}\ \alpha > 0$ and $\mathrm{Re}\ \delta > 0$.
Remark: $A, B_\infty \in \mathbb{K}^{N \times N}$ are simultaneously diagonalizable over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ if and only if A and B_∞ are diagonalizable over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $AB_\infty = B_\infty A$.