

Fredholm Properties and L^p -Spectra of Localized Rotating Waves in Parabolic Systems

Bielefeld University, November 18, 2016

Denny Otten

Department of Mathematics
Bielefeld University
Germany



CRC 701

joint work with: **Wolf-Jürgen Beyn** (Bielefeld University)



W.-J. Beyn, D. Otten. Fredholm Properties and L^p -Spectra of Localized Rotating waves in Parabolic Systems. Preprint to appear, 2016.



W.-J. Beyn, D. Otten. Spatial Decay of Rotating Waves in Reaction Diffusion Systems. *Dyn. Partial Differ. Equ.*, 13(3):191-240, 2016.



D. Otten. Spatial decay and spectral properties of rotating waves in parabolic systems. PhD thesis, Bielefeld University, *Shaker Verlag*, 2014.

Outline

- 1 Rotating patterns in \mathbb{R}^d
- 2 Spatial decay of rotating waves
- 3 Eigenvalue problem for rotating waves and some basic definitions
- 4 Fredholm properties of linearization in L^p
- 5 Essential L^p -spectrum and dispersion relation
- 6 Point L^p -spectrum and shape of eigenfunctions
- 7 Cubic-quintic complex Ginzburg-Landau equation

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Rotating Patterns in \mathbb{R}^d

Consider a **reaction diffusion system**

$$(1) \quad \begin{aligned} u_t(x, t) &= A\Delta u(x, t) + f(u(x, t)), \quad t > 0, x \in \mathbb{R}^d, d \geq 2, \\ u(x, 0) &= u_0(x), \quad t = 0, x \in \mathbb{R}^d. \end{aligned}$$

where $u : \mathbb{R}^d \times [0, \infty[\rightarrow \mathbb{R}^m$, $A \in \mathbb{R}^{m,m}$, $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$, $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}^m$.

Assume a **rotating wave** solution $u_* : \mathbb{R}^d \times [0, \infty[\rightarrow \mathbb{R}^m$ of (1)

$$u_*(x, t) = v_*(e^{-tS}x)$$

$v_* : \mathbb{R}^d \rightarrow \mathbb{R}^m$ profile (pattern), $0 \neq S \in \mathbb{R}^{d,d}$ skew-symmetric.

Transformation (into a co-rotating frame): $v(x, t) = u(e^{tS}x, t)$ solves

$$(2) \quad \begin{aligned} v_t(x, t) &= A\Delta v(x, t) + \langle Sx, \nabla v(x, t) \rangle + f(v(x, t)), \quad t > 0, x \in \mathbb{R}^d, d \geq 2, \\ v(x, 0) &= u_0(x), \quad t = 0, x \in \mathbb{R}^d. \end{aligned}$$

$$\langle Sx, \nabla v(x) \rangle = Dv(x)Sx = \sum_{i=1}^d \sum_{j=1}^d S_{ij}x_j D_i v(x) \stackrel{-S = S^T}{=} \sum_{i=1}^{d-1} \sum_{j=i+1}^d S_{ij} (x_j D_i - x_i D_j) v(x)$$

(drift term) (rotational term)

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Note: v_* is a stationary solution of (2), i.e. v_* solves the **rotating wave equation**

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, x \in \mathbb{R}^d, d \geq 2.$$

$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle$: **Ornstein-Uhlenbeck operator.**

Rotating Patterns in \mathbb{R}^d

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Questions and Ingredients: I1: **exp. decay of v_*** , I2: **spectral properties**

Q1: **Nonlinear stability** of rotating waves on \mathbb{R}^d ? (**Tools:** I1+I2)

Q2: **Truncations** of rotating waves to bounded domains? (**Tools:** I1+...)

Q3: **Spatial approximation** (e.g. with finite element method)? (**open problem**)

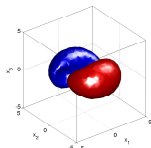
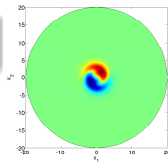
Q4: **Temporal approximation** (e.g. with Euler or BDF)? (**open problem**)

Examples for rotating waves

Cubic-quintic complex Ginzburg-Landau equation: (spinning solitons)

$$u_t = \alpha \Delta u + u \left(\delta + \beta |u|^2 + \gamma |u|^4 \right)$$

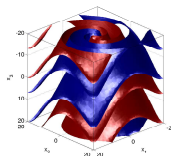
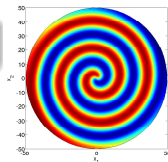
$u(x, t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geq 0$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$,
 $\delta \in \mathbb{R}$, $d \in \{2, 3\}$.



λ - ω system: (spiral waves, scroll waves)

$$u_t = \alpha \Delta u + \left(\lambda(|u|^2) + i\omega(|u|^2) \right) u$$

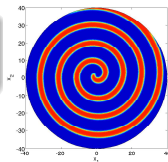
$u(x, t) \in \mathbb{C}$, $x \in \mathbb{R}^d$, $t \geq 0$, $\lambda, \omega : [0, \infty[\rightarrow \mathbb{R}$,
 $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$, $d \in \{2, 3\}$.



Barkley model: (spiral waves, also scroll waves)

$$u_t = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \Delta u + \begin{pmatrix} \frac{1}{\varepsilon} u_1 (1 - u_1) (u_1 - \frac{u_2 + b}{a}) \\ u_1 - u_2 \end{pmatrix}$$

$u(x, t) \in \mathbb{R}^2$, $x \in \mathbb{R}^d$, $t \geq 0$, $0 \leq D \ll 1$,
 $\varepsilon, a, b > 0$, $d \in \{2, 3\}$.



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Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile v_\star)

Let $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of

$$(RWE) \quad A\Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,$$

such that

$$(TC) \quad \sup_{|x| \geq R_0} |v_\star(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^m)$$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \quad \left(\begin{array}{lll} a_{\max} & = & \rho(A) & : \text{spectral radius of } A \\ -a_0 & = & s(-A) & : \text{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \text{spectral bound of } Df(v_\infty) \end{array} \right)$$

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Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_\star \in C^3(\mathbb{R}^d, \mathbb{R}^m)$ of

$$\text{(RWE)} \quad A\Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d,$$

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$$v_\star - v_\infty \in W_{\theta}^{2,p}(\mathbb{R}^d, \mathbb{R}^m)$$

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Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile v_* : higher regularity)

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

Every classical solution $v_* \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of

$$\text{(RWE)} \quad A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d,$$

such that

$$\text{(TC)} \quad \sup_{|x| \geq R_0} |v_*(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_* - v_\infty \in W_\theta^{k,p}(\mathbb{R}^d, \mathbb{R}^m)$$

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$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}. \quad \left(\begin{array}{lll} a_{\max} & = & \rho(A) & : \text{spectral radius of } A \\ -a_0 & = & s(-A) & : \text{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \text{spectral bound of } Df(v_\infty) \end{array} \right)$$

Spatial decay of rotating waves

Theorem 1: (Exponential decay of profile v_* : pointwise estimates)

Let $f \in C^{\max\{2, k-1\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta(x) = \exp\left(\mu\sqrt{|x|^2+1}\right)$ be a weight function for $\mu \in \mathbb{R}$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 3$).

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such that

$$\text{(TC)} \quad \sup_{|x| \geq R_0} |v_*(x) - v_\infty| \leq K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_* - v_\infty \in W_{\theta}^{k,p}(\mathbb{R}^d, \mathbb{R}^m), \quad |D^\alpha(v_*(x) - v_\infty)| \leq C \exp\left(-\mu\sqrt{|x|^2+1}\right) \quad \forall x \in \mathbb{R}^d$$

for every exponential decay rate

$$0 \leq \mu \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p} \quad \left(\begin{array}{lll} a_{\max} & = & \rho(A) & : \text{spectral radius of } A \\ -a_0 & = & s(-A) & : \text{spectral bound of } -A \\ -b_0 & = & s(Df(v_\infty)) & : \text{spectral bound of } Df(v_\infty) \end{array} \right)$$

and for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$.

Spatial decay of eigenfunctions

Theorem 2: (Exponential decay of eigenfunctions v)

Let $f \in C^{\max\{2,k\}}(\mathbb{R}^m, \mathbb{R}^m)$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $Df(v_\infty) \leq -\beta_\infty I_m < 0$, assume (A1)-(A3) for some $1 < p < \infty$, and let $\theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2 + 1}\right)$ be a weight function for $\mu_j \in \mathbb{R}$, $j = 1, 2$, $k \in \mathbb{N}$, $p \geq \frac{d}{2}$ (if $k \geq 2$).

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ such that for every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following property holds: Every classical solution $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$ of

$$\text{(EVP)} \quad A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_\star(x))v(x) = \lambda v(x), \quad x \in \mathbb{R}^d,$$

with $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda \geq -(1 - \varepsilon)\beta_\infty$, such that

$v \in L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^m)$ for **some** exp. growth rate $-\sqrt{\varepsilon \frac{\gamma_A \beta_\infty}{2d|A|^2}} \leq \mu_1 < 0$
satisfies

$v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^m)$ for **every** exp. decay rate $0 \leq \mu_2 \leq \varepsilon \frac{\sqrt{a_0 b_0}}{a_{\max} p}$

and

$$|D^\alpha v(x)| \leq C \exp\left(-\mu_2 \sqrt{|x|^2 + 1}\right) \quad \forall x \in \mathbb{R}^d$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ satisfying $d < (k - |\alpha|)p$.

Exponentially weighted Sobolev spaces and assumptions

Exponentially weighted Sobolev spaces: For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $1 \leq p \leq \infty$, $k \in \mathbb{N}_0$, and weight function $\theta(x) = \exp\left(\mu\sqrt{|x|^2 + 1}\right)$ with $\mu \in \mathbb{R}$ we define

$$L^p_\theta(\mathbb{R}^d, \mathbb{K}^m) := \{v \in L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{K}^m) \mid \|\theta v\|_{L^p} < \infty\},$$

$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{K}^m) := \{v \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^m) \mid D^\beta v \in L^p_\theta(\mathbb{R}^d, \mathbb{K}^m) \forall |\beta| \leq k\}.$$

Assumptions:

(A1) (L^p -dissipativity condition): For $A \in \mathbb{R}^{m,m}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \operatorname{Re} \langle w, Aw \rangle + (p-2) \operatorname{Re} \langle w, z \rangle \operatorname{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \quad \forall z, w \in \mathbb{R}^m$$

(A2) (System condition): $A, Df(v_\infty) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over \mathbb{C}

(A3) (Rotational condition): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^\top$

Note: Assumption (A1) is equivalent with

(A1') (L^p -antieigenvalue condition): $A \in \mathbb{R}^{m,m}$ is invertible and

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{R}^m \\ w \neq 0 \\ Aw \neq 0}} \frac{\operatorname{Re} \langle w, Aw \rangle}{|w| |Aw|} > \frac{|p-2|}{p} \quad \text{for some } 1 < p < \infty$$

($\mu_1(A)$): first antieigenvalue of A)

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$$L_{\theta}^p(\mathbb{R}^d, \mathbb{K}^m) := \{v \in L_{\text{loc}}^1(\mathbb{R}^d, \mathbb{K}^m) \mid \|\theta v\|_{L^p} < \infty\},$$

$$W_{\theta}^{k,p}(\mathbb{R}^d, \mathbb{K}^m) := \{v \in L_{\theta}^p(\mathbb{R}^d, \mathbb{K}^m) \mid D^{\beta} u \in L_{\theta}^p(\mathbb{R}^d, \mathbb{K}^m) \forall |\beta| \leq k\}.$$

Assumptions:

(A1) (L^p -dissipativity condition): For $A \in \mathbb{R}^{m,m}$, $1 < p < \infty$, there is $\gamma_A > 0$ with

$$|z|^2 \operatorname{Re} \langle w, Aw \rangle + (p-2) \operatorname{Re} \langle w, z \rangle \operatorname{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \quad \forall z, w \in \mathbb{R}^m$$

(A2) (System condition): $A, Df(v_{\infty}) \in \mathbb{R}^{m,m}$ simultaneously diagonalizable over \mathbb{C}

(A3) (Rotational condition): $0 \neq S \in \mathbb{R}^{d,d}$, $-S = S^{\top}$

Additionally:

(A4) (L^q -dissipativity condition): For $A \in \mathbb{R}^{m,m}$, $q = \frac{p}{p-1}$, there is $\delta_A > 0$ with

$$|z|^2 \operatorname{Re} \langle w, A^H w \rangle + (q-2) \operatorname{Re} \langle w, z \rangle \operatorname{Re} \langle z, A^H w \rangle \geq \delta_A |z|^2 |w|^2 \quad \forall z, w \in \mathbb{R}^m$$

Outline of proof: Theorem 1 (Exponential decay of v_*)

Exponential Decay: To show exponential decay for the solution v_* of

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d,$$

investigate the linear system ($w_*(x) := v_*(x) - v_\infty$)

$$A\Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$

Operators: Study the following operators

$$\mathcal{L}_c v := A\Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v + Q_s v + Q_c v, \quad (\text{exp. decay})$$

$$\mathcal{L}_s v := A\Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v + Q_s v, \quad (\text{exp. decay})$$

$$\mathcal{L}_\infty v := A\Delta v + \langle S \cdot, \nabla v \rangle + Df(v_\infty)v, \quad (\text{far-field operator}) \quad (\text{exp. decay})$$

$$\mathcal{L}_0 v := A\Delta v + \langle S \cdot, \nabla v \rangle. \quad (\text{Ornstein-Uhlenbeck operator}) \quad (\text{max. domain})$$



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$$\mathcal{L}_0 v := A\Delta v + \langle S \cdot, \nabla v \rangle. \quad (\text{Ornstein-Uhlenbeck operator}) \quad (\text{max. domain})$$

Maximal domain of \mathcal{L}_0 given by

$$\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) = \{v \in W_{\text{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) : \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m)\}, \quad 1 < p < \infty$$

satisfies $\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^m)$.

The operator \mathcal{L}_0

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A\Delta v(x) + \langle Sx, \nabla v(x) \rangle, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

↓

$$H_0(x, \xi, t) = (4\pi tA)^{-\frac{d}{2}} \exp\left(- (4tA)^{-1} \left| e^{tS} x - \xi \right|^2\right), \quad x, \xi \in \mathbb{R}^d, \quad t > 0.$$

↓

Semigroup in $L^p(\mathbb{R}^d, \mathbb{C}^m)$, $1 \leq p \leq \infty$

$$[T_0(t)v](x) = \int_{\mathbb{R}^d} H_0(x, \xi, t)v(\xi)d\xi, \quad t > 0.$$

strong ↓ continuity

Infinitesimal generator

$(A_p, \mathcal{D}(A_p))$, $1 \leq p < \infty$.

semigroup theory ✓

↘ identification problem

unique solv. of
resolvent equ. for A_p ,
 $1 \leq p < \infty$, $\operatorname{Re}\lambda > 0$

A-priori
→
estimates

exponential
decay,
 $1 \leq p < \infty$

max. domain and
max. realization,
 $1 < p < \infty$

$$(\lambda I - A_p)v_* = g \in L^p.$$

$$v_* \in W_\theta^{1,p}.$$

$$A_p = \mathcal{L}_0 \text{ on } \mathcal{D}(A_p) = \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0).$$

Identification problem of \mathcal{L}_0

$$\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) := \left\{ v \in W_{\text{loc}}^{2,p}(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) \mid \mathcal{L}_0 v \in L^p(\mathbb{R}^d, \mathbb{C}^m) \right\}, \quad 1 < p < \infty.$$

Infinitesimal generator

$$(A_p, \mathcal{D}(A_p)), \quad 1 \leq p < \infty.$$

↓
 \mathcal{S} is a **core**
 for $(A_p, \mathcal{D}(A_p))$

↓

Identification of \mathcal{L}_0

maximal domain and maximal realization for $1 < p < \infty$:

$$A_p = \mathcal{L}_0 \text{ on } \mathcal{D}(A_p) = \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$$

Ornstein-Uhlenbeck operator

$$[\mathcal{L}_0 v](x) = A \Delta v(x) + \langle Sx, \nabla v(x) \rangle, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

↓

$\mathcal{L}_0 : \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \rightarrow L^p(\mathbb{R}^d, \mathbb{C}^m)$
 is a **closed** operator, $1 < p < \infty$

↓

L^p -resolvent estimates

and

unique solv. of resolvent equ.

for \mathcal{L}_0 in $\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$,
 $1 < p < \infty$

←

L^p -dissipativity condition: $\exists \gamma_A > 0$

$$|z|^2 \operatorname{Re} \langle w, Aw \rangle + (p-2) \operatorname{Re} \langle w, z \rangle \operatorname{Re} \langle z, Aw \rangle \geq \gamma_A |z|^2 |w|^2 \quad \forall z, w \in \mathbb{K}^m$$

↕

L^p -first antieigenvalue condition

$$\mu_1(A) := \inf_{\substack{w \in \mathbb{K}^m \\ w \neq 0 \\ Aw \neq 0}} \frac{\operatorname{Re} \langle w, Aw \rangle}{|w| |Aw|} > \frac{|p-2|}{p}, \quad 1 < p < \infty$$

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Eigenvalue problem for linearization at rotating waves

Motivation: **Stability** is determined by **spectral properties** of **linearization** \mathcal{L} .

Eigenvalue problem:

$$(\lambda I - \mathcal{L})v(x) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2, \quad \lambda \in \mathbb{C}.$$

$$\mathcal{L}v(x) = A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

Definition 3: (Strongly spectrally stable)

A rotating wave $u_*(x, t) = v_*(e^{-tS}x)$ is called **strongly spectrally stable** iff

- 1 $\operatorname{Re} \sigma(\mathcal{L}) \leq 0$ (**spectrally stable**) and
- 2 $\forall \lambda \in \sigma(\mathcal{L}) \cap i\mathbb{R}$: $\lambda \in \sigma_{\text{pt}}(\mathcal{L})$, λ is caused by the $\operatorname{SE}(d)$ -group action **and**

$$\sum_{\lambda \in \sigma(\mathcal{L}) \cap i\mathbb{R}} \operatorname{alg}(\lambda) = \frac{d(d+1)}{2} = \dim \operatorname{SE}(d), \quad \operatorname{alg}(\lambda) := \text{algebraic mult. of } \lambda.$$

Recall from spectral theory

Linearized operator is **closed** and **densely defined**

$$\mathcal{L}v(x) = A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) = \{v \in W_{\text{loc}}^{2,p} \cap L^p \mid \mathcal{L}_0 v \in L^p\}, \quad \|v\|_{\mathcal{L}_0} := \|v\|_{L^p} + \|\mathcal{L}_0 v\|_{L^p}.$$

Definition 4: (Spectrum of \mathcal{L})

1 Resolvent set

$$\rho(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid (\lambda I - \mathcal{L})^{-1} : L^p \rightarrow \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \text{ exists and is bounded}\}.$$

2 Spectrum $\sigma(\mathcal{L}) := \mathbb{C} \setminus \rho(\mathcal{L})$. $0 \neq v \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$ is an **eigenfunction** of \mathcal{L} with **eigenvalue** $\lambda \in \sigma(\mathcal{L})$ if $(\lambda I - \mathcal{L})v = 0$. An eigenvalue $\lambda \in \sigma(\mathcal{L})$

- ▶ is **isolated** if $\exists \varepsilon > 0 \forall \lambda_0 \in \mathbb{C}$ with $0 < |\lambda - \lambda_0| < \varepsilon : \lambda_0 \in \rho(\mathcal{L})$.
- ▶ has **finite (algebraic) multiplicity** if $\dim(\mathcal{N}(\lambda I - \mathcal{L})) < \infty$ and $\exists n_\lambda \in \mathbb{N} \forall y \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$ s.t. $y(\lambda_0) = \sum_{j=0}^r (\lambda_0 - \lambda)^j y_j$ with $y_0 \neq 0$:

$$[(\lambda I - \mathcal{L})y]^{(\nu)}(\lambda) = 0 \text{ for } \nu = 0, \dots, n-1 \text{ and } [(\lambda I - \mathcal{L})y]^{(n)}(\lambda) \neq 0.$$

3 Point spectrum

$$\sigma_{\text{pt}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of finite alg. multiplicity}\}.$$

$\lambda \in \rho(\mathcal{L}) \cup \sigma_{\text{pt}}(\mathcal{L})$ is called a **normal point** of \mathcal{L} .

4 Essential spectrum

$$\sigma_{\text{ess}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is not a normal point of } \mathcal{L}\}.$$

Note: $\mathbb{C} = \rho(\mathcal{L}) \dot{\cup} \sigma(\mathcal{L})$, $\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \dot{\cup} \sigma_{\text{point}}(\mathcal{L})$.

Recall from spectral theory

Linearized operator is **closed** and **densely defined**

$$\mathcal{L}v(x) = A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) = \{v \in W_{\text{loc}}^{2,p} \cap L^p \mid \mathcal{L}_0 v \in L^p\}, \quad \|v\|_{\mathcal{L}_0} := \|v\|_{L^p} + \|\mathcal{L}_0 v\|_{L^p}.$$

Definition 5: (Fredholm operator)

The linear operator $\lambda I - \mathcal{L} : \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \rightarrow L^p$ is called **Fredholm** iff

- 1 $\lambda I - \mathcal{L}$ is closed,
- 2 $\dim(\mathcal{N}(\lambda I - \mathcal{L})) < \infty$ and
- 3 $\text{codim}(\mathcal{R}(\lambda I - \mathcal{L})) < \infty$.

The **index** κ of the Fredholm operator $\lambda I - \mathcal{L}$ is defined by

$$\kappa := \dim(\mathcal{N}(\lambda I - \mathcal{L})) - \text{codim}(\mathcal{R}(\lambda I - \mathcal{L}))$$

with $\text{codim}(\mathcal{R}(\lambda I - \mathcal{L})) := \dim(\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)/\mathcal{R}(\lambda I - \mathcal{L}))$.

Adjoint operator: Let $q = \frac{p}{p-1}$ for $1 < p < \infty$

$$\mathcal{L}^* v(x) = A^H \Delta v(x) + \langle S^T x, \nabla v(x) \rangle + Df(v_*(x))^H v(x), \quad x \in \mathbb{R}^d, \quad d \geq 2,$$

$$\mathcal{D}_{\text{loc}}^q(\mathcal{L}_0^*) = \{v \in W_{\text{loc}}^{2,q} \cap L^q \mid \mathcal{L}_0^* v \in L^q\}, \quad \|v\|_{\mathcal{L}_0^*} := \|v\|_{L^q} + \|\mathcal{L}_0^* v\|_{L^q}.$$

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Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of \mathcal{L})

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

• (Fredholm properties).

$\lambda I - \mathcal{L} : (\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0), \|\cdot\|_{\mathcal{L}_0}) \rightarrow (L^p(\mathbb{R}^d, \mathbb{C}^N), \|\cdot\|_{L^p})$ is Fredholm of index 0.

Properties of linearization at localized rotating waves

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Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_* \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

- ② **(Fredholm alternative)**. Let in addition to ①, (A4) hold for $q = \frac{p}{p-1}$ and $\lambda \in \sigma_{\text{pt}}(\mathcal{L})$ with geom. mult. $1 \leq n = \dim \mathcal{N}(\lambda I - \mathcal{L}) < \infty$. Then, there are exactly n linearly indep. eigenfunctions $v_j \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$ and adjoint eigenfunctions $\psi_j \in \mathcal{D}_{\text{loc}}^q(\mathcal{L}_0^*)$ with

$$(\lambda I - \mathcal{L})v_j = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^* \psi_j = 0 \quad \text{for } j = 1, \dots, n.$$

Moreover,

$$\text{(IP)} \quad (\lambda I - \mathcal{L})v = g, \quad g \in L^p(\mathbb{R}^d, \mathbb{C}^N)$$

has at least one (not necessarily unique) solution $v \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$ iff

$$g \in (\mathcal{N}(\lambda I - \mathcal{L})^*)^\perp, \quad \text{i.e.} \quad \langle \psi_j, g \rangle_{q,p} = 0, \quad j = 1, \dots, n.$$

In this case, one can select a solution $v \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$ of (IP) with

$$\|v\|_{\mathcal{L}_0} \leq C \|g\|_{L^p} \quad \text{and} \quad \|v\|_{W^{1,p}} \leq C \|g\|_{L^p}.$$

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of \mathcal{L})

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

③ (Exponential decay). Let in addition to ②:

$$\theta_j(x) = \exp\left(\mu_j \sqrt{|x|^2 + 1}\right), \quad x \in \mathbb{R}^d, \quad \mu_j \in \mathbb{R}, \quad j = 1, \dots, 4.$$

Then, every classical solution $v \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in C^2(\mathbb{R}^d, \mathbb{C}^m)$ of

$$(\lambda I - \mathcal{L})v = 0 \quad \text{and} \quad (\lambda I - \mathcal{L})^* \psi = 0$$

such that $v \in L_{\theta_1}^p(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in L_{\theta_3}^q(\mathbb{R}^d, \mathbb{C}^m)$ for some exp. growth rate

$$-\sqrt{\varepsilon \frac{\gamma_A(\beta_\infty - b_0 + \gamma)}{2d|A|^2}} \leq \mu_1 \leq 0 \quad \text{and} \quad -\sqrt{\varepsilon \frac{\delta_A(\beta_\infty - b_0 + \gamma)}{2d|A|^2}} \leq \mu_3 \leq 0$$

satisfies $v \in W_{\theta_2}^{1,p}(\mathbb{R}^d, \mathbb{C}^m)$ and $\psi \in W_{\theta_4}^{1,q}(\mathbb{R}^d, \mathbb{C}^m)$ for every exp. decay rate

$$0 \leq \mu_2 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} p} \quad \text{and} \quad 0 \leq \mu_4 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} q}.$$

Properties of linearization at localized rotating waves

Theorem 6: (Fredholm properties of \mathcal{L})

Assume (A1)-(A3) for some $1 < p < \infty$, $v_\infty \in \mathbb{R}^m$, $f(v_\infty) = 0$, $f \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda \geq -b_0 + \gamma$ for some $\gamma > 0$ and $-b_0 = s(Df(v_\infty))$.

Then, for any $0 < \varepsilon < 1$ there is $K_1 = K_1(\varepsilon) > 0$ such that for any classical solution $v_\star \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) the following properties hold:

④ **(Pointwise estimates for v).** Let in addition to ③:

$p \geq \frac{d}{2}$, $f \in C^k(\mathbb{R}^m, \mathbb{R}^m)$, $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$, $v \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$, $2 \leq k \in \mathbb{N}$.

Then, $v \in W_{\theta_2}^{k,p}(\mathbb{R}^d, \mathbb{C}^m)$ and

$$|D^\alpha v(x)| \leq C \exp\left(-\mu_2 \sqrt{|x|^2 + 1}\right), \quad x \in \mathbb{R}^d$$

for any $\mu_2 \in \mathbb{R}$, $0 \leq \mu_2 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} p}$ and $\alpha \in \mathbb{N}_0^d$, $d < (k - |\alpha|)p$.

⑤ **(Pointwise estimates for ψ).** Let in addition to ④:

$\min\{p, q\} \geq \frac{d}{2}$, $\psi \in C^{k+1}(\mathbb{R}^d, \mathbb{C}^m)$.

Then, $\psi \in W_{\theta_4}^{k,q}(\mathbb{R}^d, \mathbb{C}^m)$ and

$$|D^\alpha \psi(x)| \leq C \exp\left(-\mu_4 \sqrt{|x|^2 + 1}\right), \quad x \in \mathbb{R}^d$$

for any $\mu_4 \in \mathbb{R}$, $0 \leq \mu_4 \leq \varepsilon \frac{\sqrt{a_0 \gamma}}{a_{\max} q}$ and $\alpha \in \mathbb{N}_0^d$, $d < (k - |\alpha|)q$.

Outline of proof: Theorem 6 (Fredholm properties of \mathcal{L})

$$\mathcal{L}v = A\Delta v + \langle Sx, \nabla v \rangle + Df(v_*(x))v.$$

1. **Splitting off the stable part:** $Q(x) = Df(v_*(x)) - Df(v_\infty)$ implies

$$\mathcal{L}v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v$$

$$v_*(x) \rightarrow v_\infty \text{ as } |x| \rightarrow \infty \quad \Rightarrow \quad \sup_{|x| \geq R} |Q(x)| \rightarrow 0 \text{ as } R \rightarrow \infty$$

2. **Decomposition of Q :**

$$\mathcal{L}v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x))v$$

$Q(x) = Q_s(x) + Q_c(x)$, $Q_s, Q_c \in L^\infty$, Q_s small w.r.t. $\|\cdot\|_{L^\infty}$, Q_c comp. supported

3. **Decomposition of λ :** $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geq -b_0 + \gamma$ for some $\gamma > 0$, then

$$\lambda = \lambda_1 + \lambda_2 \quad \text{with} \quad \lambda_2 := -b_0 + \gamma, \quad \lambda_1 := \lambda - \lambda_2.$$

4. **Decomposition of $\lambda I - \mathcal{L}$:**

$$\lambda I - \mathcal{L} = (I - Q_c(\cdot)(\lambda_1 - \tilde{\mathcal{L}}_s)^{-1})(\lambda_1 I - \tilde{\mathcal{L}}_s)$$

$$\tilde{\mathcal{L}}_s = \mathcal{L}_s - \lambda_2 I, \quad \mathcal{L}_s v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x))v$$

Outline of proof: Theorem 6 (Fredholm properties of \mathcal{L})

Decomposition of $\lambda I - \mathcal{L}$:

$$\lambda I - \mathcal{L} = (I - Q_c(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_s)^{-1})(\lambda_1 I - \tilde{\mathcal{L}}_s)$$

$$\tilde{\mathcal{L}}_s = \mathcal{L}_s - \lambda_2 I, \quad \mathcal{L}_s v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q_s(x))v$$

5. Fredholm properties:

- $\lambda_1 I - \tilde{\mathcal{L}}_s$ is Fredholm of index 0:
 - ▶ unique solvability of resolvent equation for $\tilde{\mathcal{L}}_s$
- $I - Q_c(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_s)^{-1}$ Fredholm of index 0:
 - ▶ $Q_c(\cdot)(\lambda_1 I - \tilde{\mathcal{L}}_s)^{-1}$ is compact
 - ▶ compact perturbation of identity
 - ▶ unique solvability of resolvent equation for $\tilde{\mathcal{L}}_s$
 - ▶ $\mathcal{D}_{\text{loc}}^p(\mathcal{L}_0) \subseteq W^{1,p}(\mathbb{R}^d, \mathbb{C}^m)$
- $\lambda I - \mathcal{L}$ Fredholm of index 0:
 - ▶ Theorem on products of Fredholm operators

Outline

- 1 Rotating patterns in \mathbb{R}^d
- 2 Spatial decay of rotating waves
- 3 Eigenvalue problem for rotating waves and some basic definitions
- 4 Fredholm properties of linearization in L^p
- 5 Essential L^p -spectrum and dispersion relation**
- 6 Point L^p -spectrum and shape of eigenfunctions
- 7 Cubic-quintic complex Ginzburg-Landau equation

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Eigenvalue problem:

$$(\lambda I - \mathcal{L})v = 0, x \in \mathbb{R}^d$$

$$\mathcal{L}v = A\Delta v + \langle Sx, \nabla v \rangle + Df(v_*(x))v$$

1. Splitting off the stable part: $Q(x) = Df(v_*(x)) - Df(v_\infty)$ implies

$$(\lambda I - \mathcal{L}_Q)v = 0, x \in \mathbb{R}^d$$

$$\mathcal{L}_Q v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v = \mathcal{L}v$$

$$v_*(x) \rightarrow v_\infty \text{ as } |x| \rightarrow \infty \quad \Rightarrow \quad \sup_{|x| \geq R} |Q(x)| \rightarrow 0 \text{ as } R \rightarrow \infty$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Splitting off the stable part:

$$(\lambda I - \mathcal{L}_Q)v = 0, x \in \mathbb{R}^d$$

$$\mathcal{L}_Q v = A\Delta v + \langle Sx, \nabla v \rangle + (Df(v_\infty) + Q(x))v$$

$$Q(x) = Df(v_*(x)) - Df(v_\infty), \quad \sup_{|x| \geq R} |Q(x)| \rightarrow 0 \text{ as } R \rightarrow \infty$$

2. Orthogonal transformation: $S \in \mathbb{R}^{d,d}$, $S = -S^\top$, implies $S = P\Lambda_b^S P^\top$ with

$$P \in \mathbb{R}^{d,d} \text{ orth.}, \quad \Lambda_b^S = \text{diag}(\Lambda_1^S, \dots, \Lambda_k^S, \mathbf{0}), \quad \Lambda_j^S = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \quad \pm i\sigma_j \in \sigma(S).$$

Then, $\tilde{v}(y) = v(T_1(y))$ with $x = T_1(y) = Py$ yields

$$(\lambda I - \mathcal{L}_1)\tilde{v} = 0, y \in \mathbb{R}^d$$

$$\mathcal{L}_1 \tilde{v} = A\Delta \tilde{v} + \langle \Lambda_b^S y, \nabla \tilde{v} \rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}$$

$$\langle \Lambda_b^S y, \nabla \tilde{v} \rangle = \sum_{l=1}^k \sigma_l (y_{2l} \partial_{y_{2l-1}} - y_{2l-1} \partial_{y_{2l}}) \tilde{v}$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Orthogonal transformation:

$$(\lambda I - \mathcal{L}_1)\tilde{v} = 0, y \in \mathbb{R}^d$$

$$\mathcal{L}_1\tilde{v} = A\Delta\tilde{v} + \langle \Lambda_b^S y, \nabla\tilde{v} \rangle + (Df(v_\infty) + Q(T_1(y)))\tilde{v}$$

$$\langle \Lambda_b^S y, \nabla\tilde{v} \rangle = \sum_{l=1}^k \sigma_l (y_{2l}\partial_{y_{2l-1}} - y_{2l-1}\partial_{y_{2l}})\tilde{v}$$

3. Several planar polar coordinates: For $\phi \in (-\pi, \pi]^k$, $r \in (0, \infty)^k$ define

$$\begin{pmatrix} y_{2l-1} \\ y_{2l} \end{pmatrix} = T(r_l, \phi_l) := \begin{pmatrix} r_l \cos \phi_l \\ r_l \sin \phi_l \end{pmatrix}, l = 1, \dots, k,$$

$T_2(\xi) = (T(r_1, \phi_1), \dots, T(r_k, \phi_k), \tilde{y})$, $\xi = (r_1, \phi_1, \dots, r_k, \phi_k, \tilde{y})$, $\tilde{y} = (y_{2k+1}, \dots, y_d)$.
Then, $\hat{v}(\xi) = \tilde{v}(T_2(\xi))$ with $y = T_2(\xi)$ and $\mathbf{Q}(\xi) = \mathbf{Q}(T_1(T_2(\xi)))$ yields

$$(\lambda I - \mathcal{L}_2)\hat{v} = 0, \xi \in \Omega$$

$$\mathcal{L}_2\hat{v} = A \left[\sum_{l=1}^k \left(\partial_{r_l}^2 + \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^d \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^k \sigma_l \partial_{\phi_l} \hat{v} + (Df(v_\infty) + \mathbf{Q}(\xi))\hat{v}.$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Several planar polar coordinates: $\Omega = ((0, \infty) \times (-\pi, \pi])^k \times \mathbb{R}^{d-2k}$

$$(\lambda I - \mathcal{L}_2)\hat{v} = 0, \xi \in \Omega$$

$$\mathcal{L}_2\hat{v} = A \left[\sum_{l=1}^k \left(\partial_{r_l}^2 + \frac{1}{r_l} \partial_{r_l} + \frac{1}{r_l^2} \partial_{\phi_l}^2 \right) + \sum_{l=2k+1}^d \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^k \sigma_l \partial_{\phi_l} \hat{v} + (Df(v_\infty) + \mathbf{Q}(\xi))\hat{v}.$$

$$\mathbf{Q}(\xi) = \mathbf{Q}(T_1(T_2(\xi)))$$

4. Limit operator (far-field operator, simplified operator):

Let formally $|x| \rightarrow \infty$ (i.e. $r_l \rightarrow \infty$) and use $|Q(x)| \rightarrow 0$ as $|x| \rightarrow \infty$

$$(\lambda I - \mathcal{L}_\infty^{\text{sim}})\hat{v} = 0, \xi \in \Omega$$

$$\mathcal{L}_\infty^{\text{sim}}\hat{v} = A \left[\sum_{l=1}^k \partial_{r_l}^2 + \sum_{l=2k+1}^d \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^k \sigma_l \partial_{\phi_l} \hat{v} + Df(v_\infty)\hat{v}$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Limit operator: $\Omega = ((0, \infty) \times (-\pi, \pi])^k \times \mathbb{R}^{d-2k}$
 $(\lambda I - \mathcal{L}_{\infty}^{\text{sim}})\hat{v} = 0, \xi \in \Omega$

$$\mathcal{L}_{\infty}^{\text{sim}} \hat{v} = A \left[\sum_{l=1}^k \partial_{r_l}^2 + \sum_{l=2k+1}^d \partial_{y_l}^2 \right] \hat{v} - \sum_{l=1}^k \sigma_l \partial_{\phi_l} \hat{v} + Df(v_{\infty}) \hat{v}$$

5. Angular Fourier transform:

For $n \in \mathbb{Z}^k, \omega \in \mathbb{R}^k, \rho, \tilde{y} \in \mathbb{R}^{d-2k}, \underline{v} \in \mathbb{C}^m, |\underline{v}| = 1, \phi \in (-\pi, \pi]^k, r \in (0, \infty)^k$.

Inserting

$$\begin{aligned} \hat{v}(\xi) &= \exp\left(i \sum_{l=1}^k \omega_l r_l\right) \exp\left(i \sum_{l=1}^k n_l \phi_l\right) \exp\left(i \sum_{l=2k+1}^d \rho_l y_l\right) \underline{v}, \\ &= \exp(i \langle \omega, r \rangle + i \langle n, \phi \rangle + i \langle \rho, \tilde{y} \rangle) \underline{v} \end{aligned}$$

yields the m -dimensional eigenvalue problem

$$\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^k n_l \sigma_l I_m - Df(v_{\infty}) \right) \underline{v} = 0.$$

Essential Spectrum: Derivation of dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

Angular Fourier transform: $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$, $n \in \mathbb{Z}^k$, $\underline{v} \in \mathbb{C}^m$, $|\underline{v}| = 1$

$$\left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^k n_l \sigma_l I_m - Df(v_\infty) \right) \underline{v} = 0.$$

6. Dispersion relation: Every $\lambda \in \mathbb{C}$ satisfying

$$\text{(DR)} \quad \det \left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^k n_l \sigma_l I_m - Df(v_\infty) \right) = 0$$

for some $\omega \in \mathbb{R}^k$, $\rho \in \mathbb{R}^{d-2k}$, $n \in \mathbb{Z}^k$ belongs to $\sigma_{\text{ess}}(\mathcal{L})$.

Dispersion set:

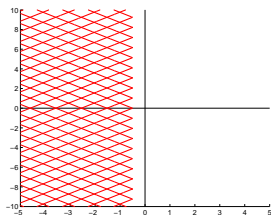
$$\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}.$$

Illustration: Dispersion set $\sigma_{\text{disp}}(\mathcal{L})$

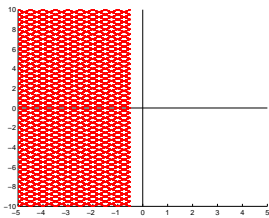
$$(DR) \quad \det \left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^k n_l \sigma_l I_m - Df(v_\infty) \right) = 0$$

$$\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}$$

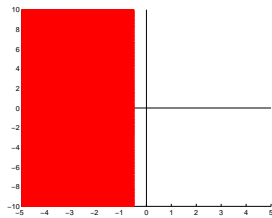
$S \in \mathbb{R}^{d,d}$, $S = -S^T$, $\pm i\sigma_1, \dots, \pm i\sigma_k$ nonzero eigenvalues of S , $\sigma_1, \dots, \sigma_k \in \mathbb{R}$.



$d = 2$ or 3



$d = 4$ (not dense)



$d = 4$ (dense)

Parameters for illustration: $A = \frac{1}{2} + \frac{1}{2}i$, $Df(v_\infty) = -\frac{1}{2}$,

$$\sigma_1 = 1.027$$

$$\sigma_1 = 1, \sigma_2 = 1.5$$

$$\sigma_1 = 1, \sigma_2 = \frac{\exp(1)}{2}$$

$$\sigma_{\text{disp}}(\mathcal{L}) \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \leq s(Df(v_\infty)) \} \text{ dense} \iff \exists \sigma_n, \sigma_m: \sigma_n \sigma_m^{-1} \notin \mathbb{Q}.$$

Essential L^p -spectrum of \mathcal{L}

$$(DR) \quad \det \left(\lambda I_m + (|\omega|^2 + |\rho|^2)A + i \sum_{l=1}^k n_l \sigma_l I_m - Df(v_\infty) \right) = 0$$

$$\sigma_{\text{disp}}(\mathcal{L}) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ satisfies (DR) for some } \omega \in \mathbb{R}^k, \rho \in \mathbb{R}^{d-2k}, n \in \mathbb{Z}^k \}$$

$S \in \mathbb{R}^{d,d}$, $S = -S^T$, $\pm i\sigma_1, \dots, \pm i\sigma_k$ nonzero eigenvalues of S , $\sigma_1, \dots, \sigma_k \in \mathbb{R}$.

Theorem 7: (Essential L^p -spectrum of \mathcal{L})

Let that assumptions of [Theorem 1](#) (pointwise estimates) be satisfied.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:

For every classical solution $v_\star \in C^{k+1}(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\text{disp}}(\mathcal{L}) \subseteq \sigma_{\text{ess}}(\mathcal{L}) \quad \text{in} \quad L^p(\mathbb{R}^d, \mathbb{C}^N).$$

- **essential spectrum** is determined by the **far-field linearization**
- Thm. 7 holds only for exponentially **localized** rotating waves, but **not** for **nonlocalized** rotating waves (e.g. **spiral waves**, **scroll waves**)
- essential spectrum for **spiral waves** much more involved (\rightarrow **Floquet theory**)

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Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Rotating wave equation:

$$(RWE) \quad 0 = A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)), \quad x \in \mathbb{R}^d$$

SE(d)-group action:

$$[a(R, \tau)v](x) = v(R^{-1}(x - \tau)), \quad x \in \mathbb{R}^d, (R, \tau) \in \text{SE}(d).$$

1. Generators of SE(d)-group action: Applying the generators

$$D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j$$

to (RWE) leads to $\frac{d(d+1)}{2} = d + \frac{d(d-1)}{2}$ equations

$$0 = D_l (A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)))$$

$$0 = D^{(i,j)} (A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)))$$

for $l = 1, \dots, d, i = 1, \dots, d-1, j = i+1, \dots, d$.

Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Generators of SE(d)-group action:

$$D_l = \partial_{x_l} \quad \text{and} \quad D^{(i,j)} = x_j D_i - x_i D_j$$

$$0 = D_l (A \Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)))$$

$$0 = D^{(i,j)} (A \Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)))$$

for $l = 1, \dots, d$, $i = 1, \dots, d-1$, $j = i+1, \dots, d$.

2. Commutator relations of generators:

$$D_l D_k = D_k D_l,$$

$$D_l D^{(i,j)} = D^{(i,j)} D_l + \delta_{lj} D_i - \delta_{li} D_j,$$

$$D^{(i,j)} D^{(r,s)} = D^{(r,s)} D^{(i,j)} + \delta_{is} D^{(r,j)} - \delta_{ir} D^{(s,j)} - \delta_{js} D^{(r,i)} + \delta_{jr} D^{(s,i)},$$

$$0 = \mathcal{L}(D_l v_*) - \sum_{n=1}^d S_{ln} D_n v_*,$$

$$0 = \mathcal{L}(D^{(i,j)} v_*) - \sum_{n=1}^d S_{jn} D^{(i,n)} v_* - \sum_{n=1}^d S_{in} D^{(n,j)} v_*.$$

Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Commutator relations of generators: $l = 1, \dots, d, i = 1, \dots, d-1, j = i+1, \dots, d$

$$0 = \mathcal{L}(D_l v_\star) - \sum_{n=1}^d S_{ln} D_n v_\star,$$

$$0 = \mathcal{L}(D^{(i,j)} v_\star) - \sum_{n=1}^d S_{jn} D^{(i,n)} v_\star - \sum_{n=1}^d S_{in} D^{(n,j)} v_\star.$$

3. Finite-dimensional eigenvalue problem: Linear combination of generators

$$v(x) = \sum_{i=1}^{d-1} \sum_{j=i+1}^d C_{ij}^{\text{rot}} D^{(i,j)} v_\star(x) + \sum_{l=1}^d C_l^{\text{tra}} D_l v_\star(x) = \langle C^{\text{rot}} x + C^{\text{tra}}, \nabla v_\star(x) \rangle$$

reduces $\mathcal{L}v = \lambda v$ to the following $\frac{d(d+1)}{2}$ -dimensional eigenvalue problem

$$\lambda C^{\text{tra}} = -S C^{\text{tra}},$$

$$\lambda C^{\text{rot}} = S^\top C^{\text{rot}} + C^{\text{rot}} S.$$

- **Unknowns:** $\lambda \in \mathbb{C}$, $C^{\text{rot}} \in \mathbb{C}^{d,d}$ skew-symmetric, $C^{\text{tra}} \in \mathbb{C}^d$
- EVP appears in **block diagonal form** \Rightarrow solve EVPs **separately**

Point spectrum: Derivation of symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

Finite-dimensional eigenvalue problem: $S \in \mathbb{R}^{d,d}$, $S = -S^\top$

$$(1) \quad \lambda C^{\text{tra}} = -S C^{\text{tra}},$$

$$(2) \quad \lambda C^{\text{rot}} = S^\top C^{\text{rot}} + C^{\text{rot}} S.$$

Unknowns: $\lambda \in \mathbb{C}$, $C^{\text{rot}} \in \mathbb{C}^{d,d}$ skew-symmetric, $C^{\text{tra}} \in \mathbb{C}^d$.

4. Solution of (1)-(2): S is **unitary diagonalizable**, i.e.

$$\Lambda_S = U^H S U, \quad U \in \mathbb{C}^{d,d} \text{ unitary}, \quad \Lambda_S = \text{diag}(\lambda_1^S, \dots, \lambda_d^S), \quad \sigma(S) = \{\lambda_1^S, \dots, \lambda_d^S\}$$

A transformation of (1)-(2) implies

$$\lambda = -\lambda_i^S, \quad C^{\text{rot}} = 0, \quad C^{\text{tra}} = U e_i, \quad (d \text{ solutions}),$$

$$\lambda = -(\lambda_i^S + \lambda_j^S), \quad C^{\text{rot}} = U(I_{ij} - I_{ji})U^\top, \quad C^{\text{tra}} = 0, \quad \left(\frac{d(d-1)}{2} \text{ solutions} \right)$$

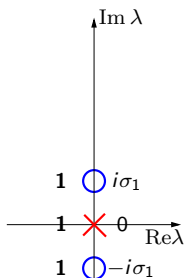
Symmetry set:

$$\sigma_{\text{sym}}(\mathcal{L}) = \sigma(S) \cup \{ \lambda_i^S + \lambda_j^S \mid 1 \leq i < j \leq d \}$$

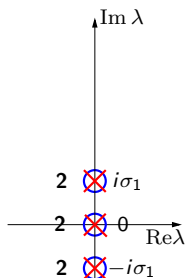
Illustration: Symmetry set $\sigma_{\text{sym}}(\mathcal{L})$

$$\sigma_{\text{sym}}(\mathcal{L}) = \sigma(S) \cup \{\lambda_i^S + \lambda_j^S \mid 1 \leq i < j \leq d\} \text{ \& algebraic multiplicities}$$

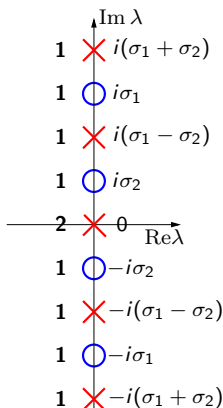
Number of elements $\frac{d(d+1)}{2} = d + \frac{d(d-1)}{2}$ equals $\dim \text{SE}(d)$.



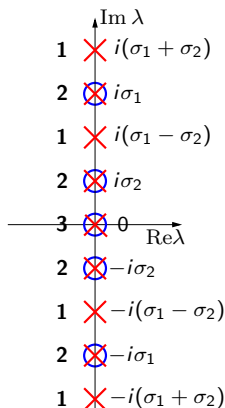
$d = 2$
 $\dim \text{SE}(2) = 3$



$d = 3$
 $\dim \text{SE}(3) = 6$



$d = 4$
 $\dim \text{SE}(4) = 10$



$d = 5$
 $\dim \text{SE}(5) = 15$

Point L^p -spectrum of \mathcal{L}

Theorem 8: (Point L^p -spectrum of \mathcal{L})

Let that assumptions of [Theorem 6](#) ① be satisfied.

Then for every $0 < \varepsilon < 1$ there exists $K_1 = K_1(\varepsilon) > 0$ with the following property:
For every classical solution $v_* \in C^2(\mathbb{R}^d, \mathbb{R}^m)$ of (RWE) satisfying (TC) it holds

$$\sigma_{\text{sym}}(\mathcal{L}) \subseteq \sigma_{\text{pt}}(\mathcal{L}) \quad \text{in } L^p(\mathbb{R}^d, \mathbb{C}^N).$$

In particular, [Theorem 6](#) ②- ⑤ implies exponential decay of eigenfunctions and adjoint eigenfunctions.

- [point spectrum](#) is determined by the [group action](#)
- Thm. 8 even holds for [nonlocalized](#) rotating waves ([spiral waves](#), [scroll waves](#))
- $v(x) = \langle Sx, \nabla v_*(x) \rangle$ eigenfunction of $\lambda = 0$ for every $d \geq 2$

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Example

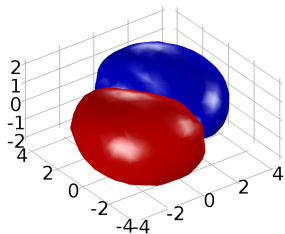
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

$$u_t = \alpha \Delta u + u \left(\mu + \beta |u|^2 + \gamma |u|^4 \right), \quad u = u(x, t) \in \mathbb{C}$$

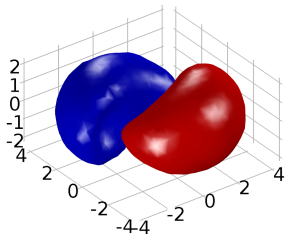
with $u : \mathbb{R}^d \times [0, \infty[\rightarrow \mathbb{C}$, $d \in \{2, 3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i, \quad \mu = -\frac{1}{2}$$

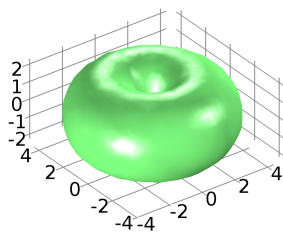
this equation exhibits so called **spinning soliton** solutions.



$$\operatorname{Re} v_*(x) = \pm 0.5$$



$$\operatorname{Im} v_*(x) = \pm 0.5$$



$$|v_*(x)| = 0.5$$

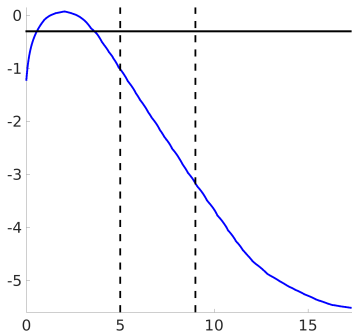
Freezing method implies numerical results for **profile** v_* and **velocities** S .

Spatial decay of a spinning soliton in QCGL for $d = 3$: Assume

$$\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\delta < 0, \quad \rho_{\min} = \frac{2|\alpha|}{|\alpha| + \operatorname{Re}\alpha} < \rho < \frac{2|\alpha|}{|\alpha| - \operatorname{Re}\alpha} = \rho_{\max}$$

Decay rate of spinning soliton:

$$0 \leq \mu < \frac{\sqrt{-\operatorname{Re}\alpha \operatorname{Re}\delta}}{|\alpha| \rho} =: \mu^{\text{pro}}(\rho) < \frac{\sqrt{-\operatorname{Re}\alpha \operatorname{Re}\delta}}{|\alpha| \max\{\rho_{\min}, \frac{d}{2}\}} =: \mu_{\max}^{\text{pro}}.$$



Parameters:

$$\alpha = \frac{1}{2} + \frac{1}{2}i, \quad \beta = \frac{5}{2} + i, \quad \gamma = -1 - \frac{1}{10}i,$$

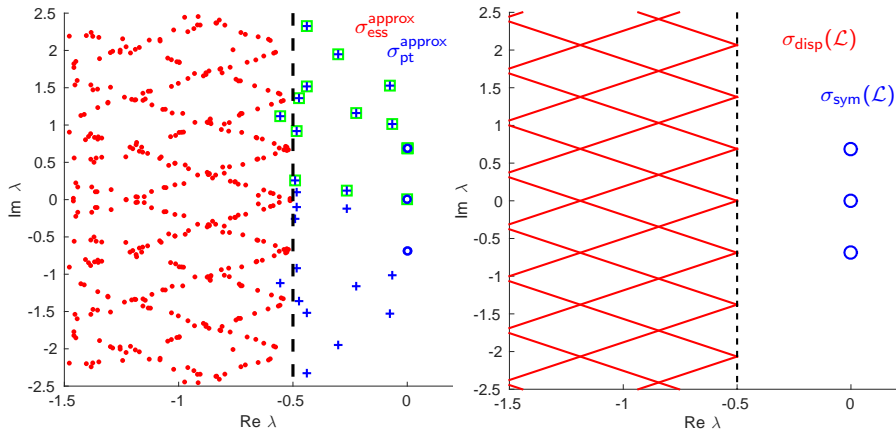
$$\mu = -\frac{1}{2}, \quad v_{\infty} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_0 = \operatorname{Re}\alpha,$$

$$a_{\max} = |\alpha|, \quad b_0 = \beta_{\infty} = -\operatorname{Re}\delta = -\frac{1}{2},$$

Numerical vs. theoretical decay rate: ($\rho = 2$)

$$\text{NDR} \approx 0.5387, \quad \text{TDR} = \mu_{\max}^{\text{pro}} = \frac{\sqrt{2}}{4} \approx 0.4714.$$

Spectrum of QCGL for a spinning soliton with $d = 3$: (numerical vs. analytical)



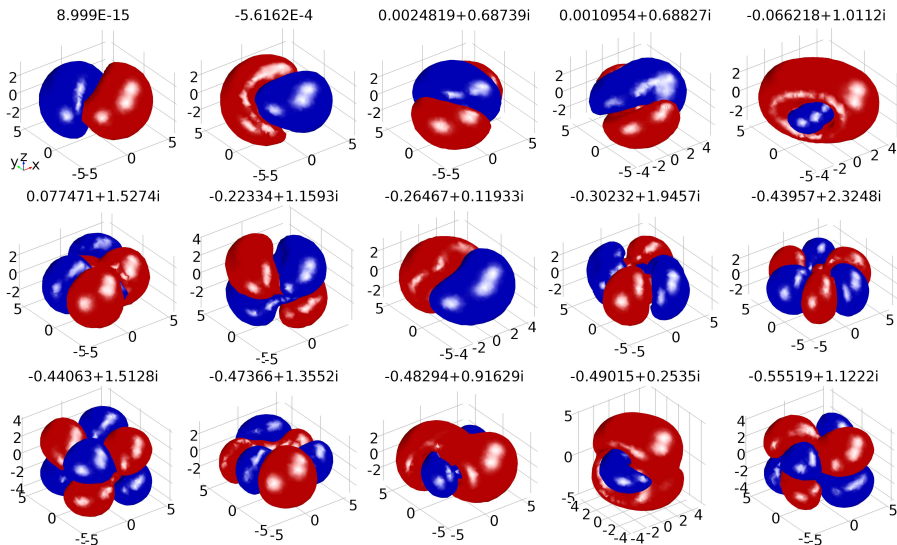
Point spectrum on $i\mathbb{R}$ and essential spectrum by dispersion relation:

$$\sigma_{\text{disp}}(\mathcal{L}) = \{\lambda = -\omega^2 \alpha_1 + \delta_1 + i(\mp \omega^2 \alpha_2 \pm \delta_2 - n \sigma_1) : \omega \in \mathbb{R}, n \in \mathbb{Z}\},$$

$$\sigma_{\text{sym}}(\mathcal{L}) = \{0, \pm i \sigma_1\}, \quad \sigma_1 = 0.6888$$

for parameters $\alpha = \frac{1}{2} + \frac{1}{2}i$, $\beta = \frac{5}{2} + i$, $\gamma = -1 - \frac{1}{10}i$, $\mu = -\frac{1}{2}$.

Eigenfunctions of QCGL for a spinning soliton with $d = 3$: $\text{Re}v(x) = \pm 0.8$

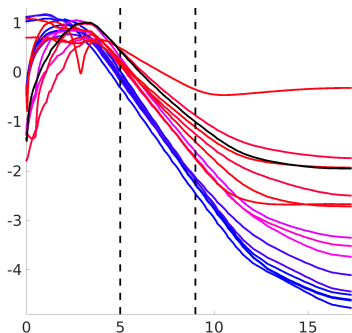


Spatial decay of eigenfunctions of QCGL at a spinning soliton for $d = 3$: Note

$$\operatorname{Re}\lambda \geq -(1 - \varepsilon)\beta_\infty = -(1 - \varepsilon)(-\operatorname{Re}\delta) \Leftrightarrow \varepsilon \leq \frac{\operatorname{Re}\lambda - \operatorname{Re}\delta}{-\operatorname{Re}\delta} =: \varepsilon(\lambda).$$

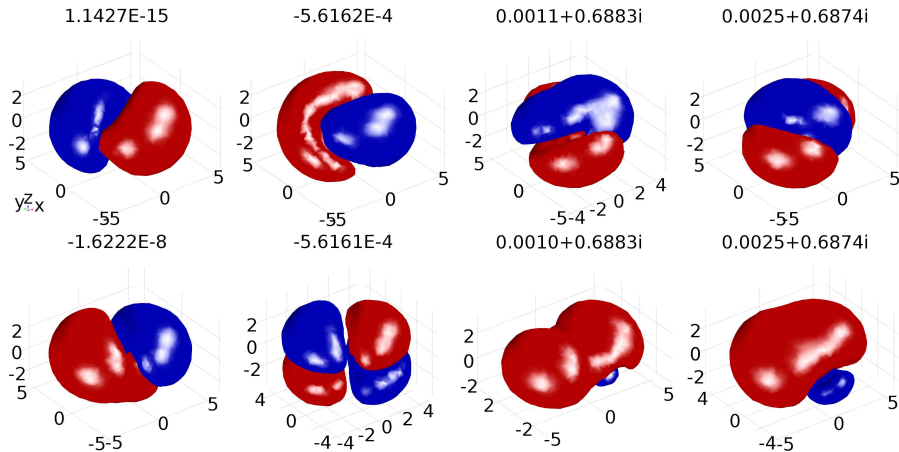
Decay rate of eigenfunctions:

$$0 \leq \mu \leq \frac{\varepsilon(\lambda)\sqrt{-\operatorname{Re}\alpha\operatorname{Re}\delta}}{|\alpha|\rho} =: \mu^{\operatorname{eig}}(\rho, \lambda) < \frac{\varepsilon(\lambda)\sqrt{-\operatorname{Re}\alpha\operatorname{Re}\delta}}{|\alpha|\max\{\rho_{\min}, \frac{d}{2}\}} =: \mu_{\max}^{\operatorname{eig}}(\lambda).$$



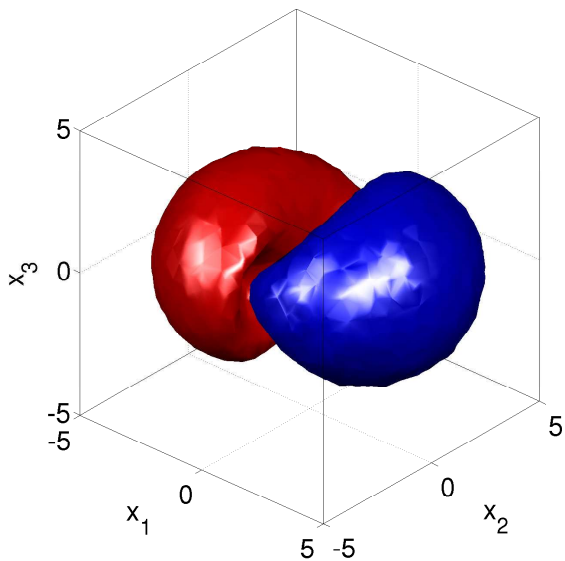
eigenvalue	NDR	TDR
$8.999 \cdot 10^{-15}$	0.5387	0.4714
$-5.6162 \cdot 10^{-4}$	0.5478	0.4714
$0.00110 \pm 0.68827i$	0.5507	0.4714
$0.00248 \pm 0.6874i$	0.5398	0.4714
$-0.06622 \pm 1.0112i$	0.4899	0.4090
$-0.07747 \pm 1.5274i$	0.5355	0.3984
$-0.22334 \pm 1.1593i$	0.4756	0.2608
$-0.26467 \pm 0.1193i$	0.4785	0.2219
$-0.30232 \pm 1.9457i$	0.4649	0.1864
$-0.43957 \pm 2.3248i$	0.3595	0.0570
$-0.44063 \pm 1.5128i$	0.3310	0.0560
$-0.47366 \pm 1.3552i$	0.4781	0.0248
$-0.48294 \pm 0.9163i$	0.4145	0.0161
$-0.48506 \pm 0.0991i$	0.2126	0.0141
$-0.49015 \pm 0.2535i$	0.3307	0.0093
$-0.55519 \pm 1.1222i$	0.3581	—

Eigenfunctions vs. adjoint eigenfunctions of QCGL for a spinning soliton with $d = 3$:



Eigenfunctions (above) and adjoint eigenfunctions (butoff) for $\lambda \in \sigma_{\text{sym}}(\mathcal{L})$

Eigenfunction $\langle Sx, \nabla v_*(x) \rangle$ of QCGL for a spinning soliton with $d = 3$:



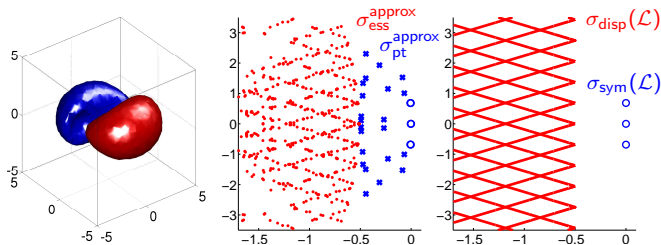
Conclusion:

Theoretical results:

- 1 spatial decay of rotating waves
- 2 spectral properties of linearization at localized rotating waves
 - ▶ Fredholm properties in L^p
 - ▶ symmetry set, point L^p -spectrum, shape of eigenfunctions and spatial decay of eigenfunctions and adjoint eigenfunctions
 - ▶ dispersion set, essential L^p -spectrum

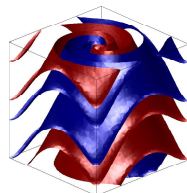
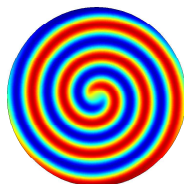
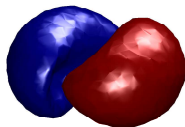
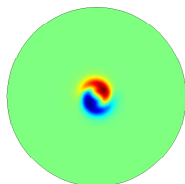
Numerical results:

- 3 approximation of rotating waves, spectra, eigenfunctions and adjoint eigenfunctions of QCGL (computation: COMSOL, postprocessing: MATLAB)



Open problems and work in progress

- **Fredholm properties and L^p -spectra of localized rotating waves**
(joint work with: W.-J. Beyn)
- **Fourier-Bessel method on \mathbb{R}^d and on circular domains**
(joint work with: W.-J. Beyn, C. Döding)
- **Nonlinear stability of relative equilibria in evolution equations**
(joint work with: W.-J. Beyn, C. Döding)
- **Freezing traveling waves in incompressible Navier-Stokes equations**
(joint work with: W.-J. Beyn, C. Döding)
- **Nonlinear stability of rotating waves for $d \geq 3$**
(joint work with: W.-J. Beyn)
- **Approximation theorem for rotating waves**



Outline

8 Outline of proof: Theorem 1

9 Outline of proof: Theorem 2

10 Outline of proof: Theorem 7

Outline of proof: Theorem 1 (Exponential decay of v_\star)

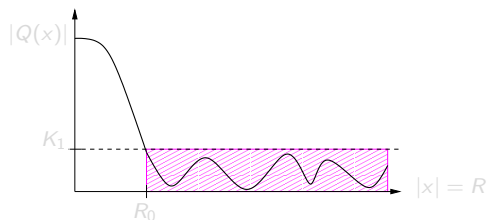
Consider the nonlinear problem

$$A\Delta v_\star(x) + \langle Sx, \nabla v_\star(x) \rangle + f(v_\star(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

1. Far-Field Linearization: $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) := \int_0^1 Df(v_\infty + tw_\star(x)) dt, \quad w_\star(x) := v_\star(x) - v_\infty$$

$$A\Delta w_\star(x) + \langle Sx, \nabla w_\star(x) \rangle + a(x)w_\star(x) = 0, \quad x \in \mathbb{R}^d.$$



Outline of proof: Theorem 1 (Exponential decay of v_*)

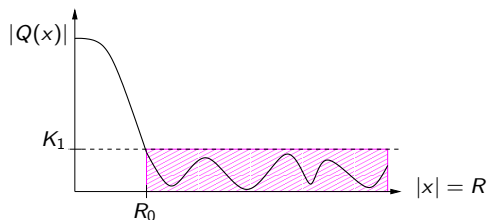
Consider the nonlinear problem

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

2. Decomposition of a : Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A\Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$



Outline of proof: Theorem 1 (Exponential decay of v_*)

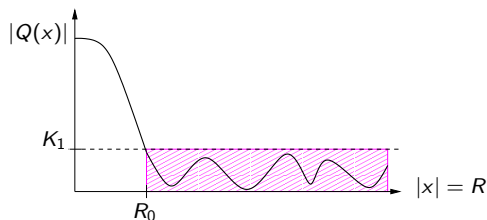
Consider the nonlinear problem

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

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$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A\Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$



3. Decomposition of Q :

$$\begin{aligned} Q(x) &= Q_s(x) + Q_c(x), \\ Q, Q_s, Q_c &\in L^\infty(\mathbb{R}^d, \mathbb{R}^{m,m}), \\ Q_s \text{ small, i.e. } &\|Q_s\|_{L^\infty} < K_1, \\ Q_c &\text{ compactly supported.} \end{aligned}$$

Outline of proof: Theorem 1 (Exponential decay of v_*)

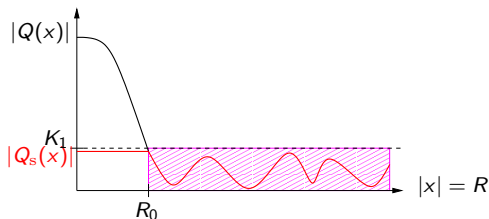
Consider the nonlinear problem

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

2. Decomposition of a : Let $a(x) = Df(v_\infty) + Q(x)$ with

$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A\Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$



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Outline of proof: Theorem 1 (Exponential decay of v_*)

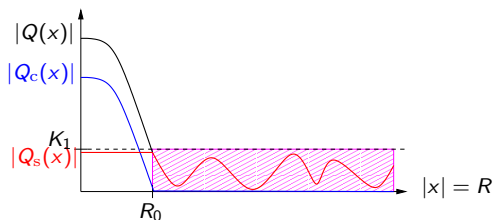
Consider the nonlinear problem

$$A\Delta v_*(x) + \langle Sx, \nabla v_*(x) \rangle + f(v_*(x)) = 0, \quad x \in \mathbb{R}^d, \quad d \geq 2.$$

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$$Q(x) := \int_0^1 Df(v_\infty + tw_*(x)) - Df(v_\infty) dt, \quad w_*(x) := v_*(x) - v_\infty$$

$$A\Delta w_*(x) + \langle Sx, \nabla w_*(x) \rangle + (Df(v_\infty) + Q_s(x) + Q_c(x)) w_*(x) = 0, \quad x \in \mathbb{R}^d.$$



3. Decomposition of Q :

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Outline

8 Outline of proof: Theorem 1

9 Outline of proof: Theorem 2

10 Outline of proof: Theorem 7

Outline of proof: Theorem 2 (Decay of eigenfunctions)

Consider

$$A\Delta v(x) + \langle Sx, \nabla v(x) \rangle + Df(v_*(x))v(x) = \lambda v(x), \quad x \in \mathbb{R}^d.$$

1. Splitting off the stable part:

$$Df(v_*(x)) = Df(v_\infty) + (Df(v_*(x)) - Df(v_\infty)) =: Df(v_\infty) + Q(x), \quad x \in \mathbb{R}^d,$$

leads to

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q(x))v(x) = \lambda v(x), \quad x \in \mathbb{R}^d.$$

2. Decomposition of (the variable coefficient) Q :

$$Q(x) = Q_\varepsilon(x) + Q_c(x), \quad Q_\varepsilon \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ small w.r.t. } \|\cdot\|_{C_b}, \\ Q_c \in C_b(\mathbb{R}^d, \mathbb{R}^{N,N}) \text{ compactly supported on } \mathbb{R}^d,$$

leads to

$$[\mathcal{L}_0 v](x) + (Df(v_\infty) + Q_\varepsilon(x) + Q_c(x))v(x) = \lambda v(x), \quad x \in \mathbb{R}^d.$$

(\rightarrow inhomogeneous Cauchy problem for \mathcal{L}_c)

Outline

8 Outline of proof: Theorem 1

9 Outline of proof: Theorem 2

10 Outline of proof: Theorem 7

Outline of proof: Theorem 7 (Essential L^p -spectrum of \mathcal{L})

Choose $R \geq 2$ large and cut-off function $\chi_R \in C_b^2$ (bounded indep. on R)

$$\chi_R : [0, \infty) \rightarrow [0, 1], \chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0, 1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases}$$

$I_1 = [0, R - 1]$, $I_2 = [R - 1, R]$, $I_3 = [R, 2R]$, $I_4 = [2R, 2R + 1]$, $I_5 = [2R + 1, \infty)$.

Introducing

$$v_R(\xi) := \left[\prod_{l=1}^k \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}},$$

we want show that $w_R \in \mathcal{D}_{loc}^p(\mathcal{L}_0)$ and

$$\|(\lambda I - \mathcal{L})w_R\|_{L^p}^p = \frac{\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d} = \frac{C}{R} + \eta_R \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Then, $\lambda \notin \rho(\mathcal{L})$ (by continuity of resolvent), i.e. $\lambda \in \sigma(\mathcal{L})$. But $\lambda \notin \sigma_{pt}(\mathcal{L})$ (since varying ω or ρ shows that λ is not isolated), hence $\lambda \in \sigma_{ess}(\mathcal{L})$.

Outline of proof: Theorem 7 (Essential L^p -spectrum of \mathcal{L})

$$\chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0, 1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases} \quad v_R(\xi) := \left[\prod_{l=1}^k \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

$$I_1 = [0, R-1], \quad I_2 = [R-1, R], \quad I_3 = [R, 2R], \quad I_4 = [2R, 2R+1], \quad I_5 = [2R+1, \infty).$$

Aim: $\frac{\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d}$ and $w_R \in \mathcal{D}_{loc}^p(\mathcal{L}_0)$

Show:

- 1 $\|v_R\|_{L^p}^p \geq CR^d$
- 2 $\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p \leq CR^{d-1} + CR^d \eta_R$
- 3 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| = 0$, if $|\tilde{y}| \in I_1 \cup I_5$ or $r_l \in I_1 \cup I_5$ for some $1 \leq l \leq k$,
 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leq C \forall |\tilde{y}|, r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leq l \leq k$,
 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leq (\sum_{l=1}^k \frac{C_l}{r_l} + \eta_R)^{\frac{1}{p}} \forall |\tilde{y}|, r_l \in I_3$ for all $1 \leq l \leq k$,
- 4 $\|(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R\|_{L^p}^p \leq CR^{d-1}$
- 5 $(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R(\xi) = 0$

Outline of proof: Theorem 7 (Essential L^p -spectrum of \mathcal{L})

$$\chi_R(r) = \begin{cases} 0 & , r \in I_1 \cup I_5, \\ \in [0, 1] & , r \in I_2 \cup I_4, \\ 1 & , r \in I_3, \end{cases} \quad v_R(\xi) := \left[\prod_{l=1}^k \chi_R(r_l) \right] \chi_R(|\tilde{y}|) \hat{v}(\xi), \quad w_R := \frac{v_R}{\|v_R\|_{L^p}}$$

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Outline of proof: Theorem 7 (Essential L^p -spectrum of \mathcal{L})

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 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leq (\sum_{l=1}^k \frac{C_l}{r_l} + \eta_R)^{\frac{1}{p}} \forall |\tilde{y}|, r_l \in I_3$ for all $1 \leq l \leq k$,
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- 5 $(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R(\xi) = 0$

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$$I_1 = [0, R-1], \quad I_2 = [R-1, R], \quad I_3 = [R, 2R], \quad I_4 = [2R, 2R+1], \quad I_5 = [2R+1, \infty).$$

Aim: $\frac{\|(\lambda I - \mathcal{L})v_R\|_{L^p}^p}{\|v_R\|_{L^p}^p} \leq \frac{CR^{d-1} + CR^d \eta_R}{CR^d}$ and $w_R \in \mathcal{D}_{\text{loc}}^p(\mathcal{L}_0)$

Show:

- ① $\|v_R\|_{L^p}^p \geq CR^d$
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- ③ $|(\lambda I - \mathcal{L}_2)v_R(\xi)| = 0$, if $|\tilde{y}| \in I_1 \cup I_5$ or $r_l \in I_1 \cup I_5$ for some $1 \leq l \leq k$,
 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leq C \forall |\tilde{y}|, r_l \in I_2 \cup I_3 \cup I_4$ for some $1 \leq l \leq k$,
 $|(\lambda I - \mathcal{L}_2)v_R(\xi)| \leq \left(\sum_{l=1}^k \frac{C_l}{r_l} + \eta_R \right)^{\frac{1}{p}} \forall |\tilde{y}|, r_l \in I_3$ for all $1 \leq l \leq k$,
- ④ $\|(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R\|_{L^p}^p \leq CR^{d-1}$
- ⑤ $(\lambda I - \mathcal{L}_\infty^{\text{sim}})v_R(\xi) = 0$