Exponential decay of two-dimensional rotating waves

GAMM 83rd Annual Scientific Conference, TU Darmstadt, March 26-30, 2012

Denny Otten
Department of Mathematics
Bielefeld University
Germany

March 29, 2012

Universität Bielefeld

CRC 701

Outline

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion equation**

$$(1)\qquad \begin{aligned} u_t(x,t) &= \alpha\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u:\mathbb{R}^d\times[0,\infty[\to\mathbb{C}$, $\alpha\in\mathbb{C}$ with $\mathrm{Re}\,\alpha>0$, $f:\mathbb{C}\to\mathbb{C}$ smooth nonlinearity.
Assume a **rotating wave** solution $u_\star:\mathbb{R}^d\times[0,\infty[\to\mathbb{C}$ of (1)

$$u_\star(x,t) = v_\star(e^{tS}x)$$

$v_\star:\mathbb{R}^d\to\mathbb{C}$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.
Transformation (into a rotating frame): $u(x,t)$ solves (1) if and only if $v(x,t)=u(e^{tS}x,t)$ solves

$$(2)\qquad \begin{aligned} v_t(x,t) &= \alpha\triangle v(x,t) + \langle Sx,\nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

Note: $v_\star$ is a stationary solution of (2).

Question: How to show exponential decay of $v_\star$ at $|x|=\infty$?
Consequence: Exponentially small error by restriction to bounded domain.

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Note: $v_\star$ is a stationary solution of (2) and stable under spectral conditions.

[BL] W.-J. Beyn, J. Lorenz.
Nonlinear stability of rotating patterns, 2008.

Example

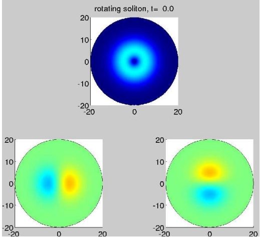
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

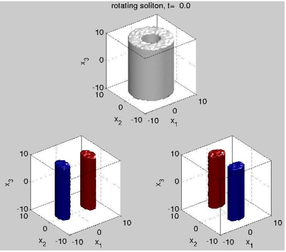
$$u_t = \alpha \triangle u + u \left(\mu + \beta \left| u \right|^2 + \gamma \left| u \right|^4 \right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0, \infty[\to \mathbb{C}$, $d \in \{2, 3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2} i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10} i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





Applications: superconductivity, superfluidity, nonlinear optical systems.

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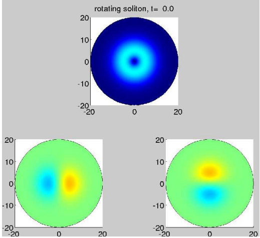

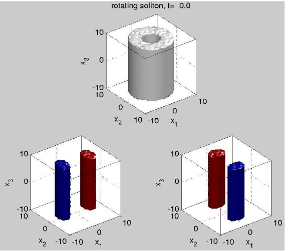

[CMM] L.-C. Crasovan, B.A. Malomed, D. Mihalache.
Spinning solitons in cubic-quintic nonlinear media, 2001.

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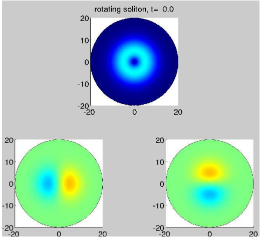

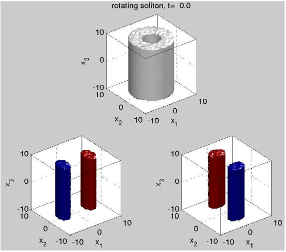

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Main Assumptions (and Theorem)

Problem: Consider the steady state problem

$$(3)\qquad \alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2,$$

where $v:\mathbb{R}^d\to\mathbb{C}$, $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$, $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric,

$$\langle Sx, \nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) s_{ij} x_j \text{ (\textbf{rotational term})}.$$

Assumptions: There exists a classical solution $v_\star\in C^2(\mathbb{R}^d,\mathbb{C})\cap C_b(\mathbb{R}^d,\mathbb{C})$ of (3) and a constant asymptotic state $v_\infty\in\mathbb{C}$, i.e. $f(v_\infty)=0$, such that

$$(A1)\qquad \exists R_0>0:\ \sup_{|x|\geqslant R_0}|v_\star(x)-v_\infty|<K_1,$$

$$(A2)\qquad f\in C^2(\mathbb{C},\mathbb{C}) \text{ real differentiable},\quad -\delta_1:=\operatorname{Re}Df(v_\infty)<0$$

Aim:

$v_\star - v_\infty \in W^{2,p}_{eucl,\eta}(\mathbb{R}^d,\mathbb{C})$, $1<p<\infty$ (weighted Sobolev space) (for L^p-theory)

$\left|D^\beta(v_\star(x)-v_\infty)\right|\leqslant Ce^{-|\eta||x|}$, $x\in\mathbb{R}^d$, $0\leqslant|\beta|\leqslant 2$. (for C_b-theory)

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Theorem: (Exponential Decay of $v_\star$)

Let $\alpha \in \mathbb{C}$, $\mathrm{Re}\,\alpha > 0$, $S \in \mathbb{R}^{d,d}$ skew-symmetric and $1 < p < \infty$. Then for every $0 < \varepsilon \leqslant \frac{\mathrm{Re}\,\alpha\delta_1}{|\alpha|^2 p^2}$ there exists a constant $K_1 = K_1(\alpha,\delta_1,p,d,\varepsilon) > 0$ such that every classical solution $v_\star$ of (3) with (A1), (A2), $v_\star - v_\infty \in L^p(\mathbb{R}^d,\mathbb{C})$ satisfies

$$v_\star - v_\infty \in W^{1,p}_{eucl,\eta}(\mathbb{R}^d,\mathbb{C}) \quad \forall\, 0 \leqslant \eta^2 \leqslant \frac{\mathrm{Re}\,\alpha\delta_1}{|\alpha|^2 p^2} - \varepsilon.$$

Motivation (Exponential Decay of $v_\star$)

Far-Field Linearization: Consider the nonlinear problem

$$\alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\; x \in \mathbb{R}^d,\; d \geqslant 2.$$

$v_\infty \in \mathbb{C}$ asymptotic state, $f \in C^1(\mathbb{C},\mathbb{C})$. By Taylor's theorem

$$f(v(x)) = \underbrace{f(v_\infty)}_{=0} + \underbrace{\int_0^1 Df(v_\infty + t(v(x)-v_\infty))dt}_{=:a(x),\, a\in C^1(\mathbb{R}^d,\mathbb{C})}(v(x)-v_\infty),\; x \in \mathbb{R}^d.$$

We arrive at

$$\alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + a(x)(v(x)-v_\infty) = 0,\; x \in \mathbb{R}^d,\; d \geqslant 2.$$

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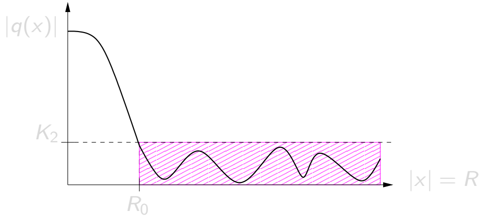
and since $f \in C^2(\mathbb{C}, \mathbb{C})$

$$\sup_{|x| \geqslant R_0} |q(x)| < \frac{K_1}{2} \cdot \sup_{w \in B_{K_1}(v_\infty)} |D^2 f(w)| =: K_2.$$

Decomposition of q:



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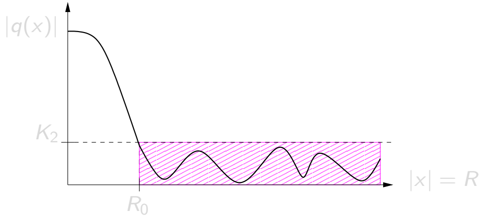

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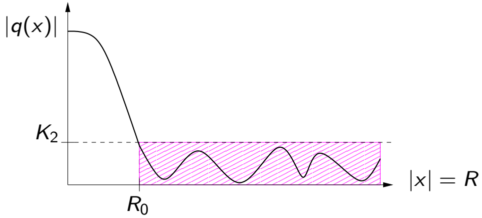

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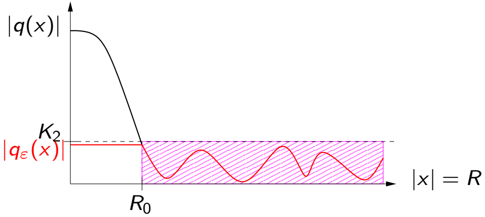

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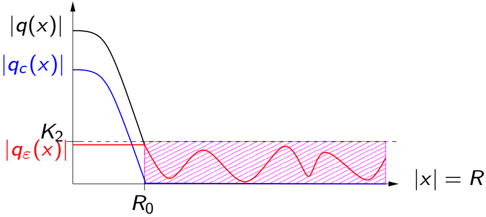

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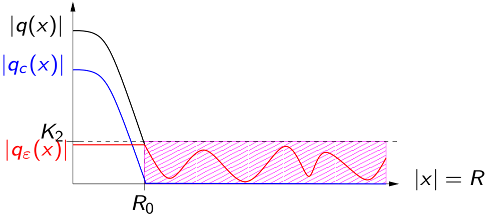
$$\sup_{|x| \geqslant R_0} |q(x)| < \frac{K_1}{2} \cdot \sup_{w \in B_{K_1}(v_\infty)} |D^2 f(w)| =: K_2.$$

Decomposition of q:



$q(x) = q_\varepsilon(x) + q_c(x)$,

$q, q_\varepsilon, q_c : \mathbb{R}^d \to \mathbb{C}$,

q_ε small, i.e. $\|q_\varepsilon\|_{L^\infty} < K_2$,

q_c compactly supported.

$$\alpha \triangle v(x) + \langle Sx, \nabla v(x) \rangle + (Df(v_\infty) + q_\varepsilon(x) + q_c(x))(v(x) - v_\infty) = 0,\ x \in \mathbb{R}^d.$$

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$\alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

$$\alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + (Df(0) + q_\varepsilon(x) + q_c(x))\, v(x) = 0,\ x\in\mathbb{R}^d,\ d\geqslant 2.$$

Operators: Study the following operators ($\delta := -Df(0)\in\mathbb{C}$, $\mathrm{Re}\,\delta > 0$)

$$\begin{aligned}
\mathcal{L}_q v &:= \alpha\triangle v + \langle S\cdot, \nabla v\rangle - \delta v + q_\varepsilon v + q_c v,\\
\mathcal{L}_\varepsilon v &:= \alpha\triangle v + \langle S\cdot, \nabla v\rangle - \delta v + q_\varepsilon v,\\
\mathcal{L}_\infty v &:= \alpha\triangle v + \langle S\cdot, \nabla v\rangle - \delta v,\\
\mathcal{L}_0 v &:= \alpha\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator)}.
\end{aligned}$$

Ornstein-Uhlenbeck Operator

Let $P, B\in\mathbb{R}^{d,d}$, P be positive definite and $B\neq 0$.

$$\nabla^T P\nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij}D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x\in\mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

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[MPRS] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt.
The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure, 2002.

[MPV] G. Metafune, D. Pallara, V. Vespri.
L^p-estimates for a class of elliptic operators with unbounded coefficients in $\mathbb{R}^N$, 2005.

Exponentially Weighted Sobolev Spaces

Group action: $v : \mathbb{R}^d \to \mathbb{C}$, $\gamma = (\tau, R) \in \mathrm{SE}(d) = \mathbb{R}^d \ltimes \mathrm{SO}(d)$

$$[a(\gamma)v](x) = v(R^{-1}(x - \tau)),\ x \in \mathbb{R}^d$$

Weight function: $\eta \in \mathbb{R}$ exponential growth rate

$$\omega_\eta(x) = \cosh(\eta|x|),\ x \in \mathbb{R}^d$$

Exponentially weighted Sobolev spaces:

$$L^p_\eta(\mathbb{R}^d, \mathbb{C}) = \{v \in L^1_{loc}(\mathbb{R}^d, \mathbb{C}) \mid \|v\|_{L^p_\eta} < \infty\},\ \|v\|^p_{L^p_\eta} = \int_{\mathbb{R}^d} (\omega_\eta(x)\,|v(x)|)^p\,dx$$

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[M] A. Mielke.
The Ginzburg-Landau equation in its role as a modulation equation, 2002.

[BT] W.-J. Beyn, V. Thümmler.
Freezing solutions of equivariant evolution equations, 2004.

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[MZ] A. Mielke, S. Zelik.
Multi-pulse evolution and space-time chaos in dissipative systems, 2009.

$[\mathcal{L}_\infty v](x) = \alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle - \delta v(x)$

Consider the inhomogeneous problem

$$[\mathcal{L}_\infty v](x) = g(x),\ x \in \mathbb{R}^d,\ d \geqslant 2. \quad (4)$$

whose solution is given by

$$v(x) = \int_{\mathbb{R}^d} G(x,\xi)g(\xi)d\xi,\ x \in \mathbb{R}^d \quad (5)$$

with Green's function $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ defined by

$$G(x,\xi) = -\int_0^\infty H(x,\xi,t)dt,\ x,\xi \in \mathbb{R}^d, \quad H : \text{Heat kernel}.$$

Theorem: (Regularity)

Let $\alpha,\delta \in \mathbb{C}$, $\mathrm{Re}\,\alpha, \mathrm{Re}\,\delta > 0$, $S \in \mathbb{R}^{d,d}$ skew-symmetric, $g \in L^p_{eucl,\eta}(\mathbb{R}^d,\mathbb{C})$, $1 < p < \infty$, $0 \leqslant \eta^2 < \frac{\mathrm{Re}\,\alpha \mathrm{Re}\,\delta}{|\alpha|^2 p^2}$, v be the solution of (4) given by (5).
Then $v \in W^{1,p}_{eucl,\eta}(\mathbb{R}^d,\mathbb{C})$ with

$$\exists\, C > 0 : \ \left\|D^\beta v\right\|_{L^p_{eucl,\eta}} \leqslant C\,\|g\|_{L^p_{eucl,\eta}},\ |\beta| \leqslant 1.$$

Heat Kernel

Recall the definition of the heat operator

$$[\mathcal{L}_\infty v](x) = \alpha \triangle v(x) + \langle Sx, \nabla v(x) \rangle - \delta v(x), \quad x \in \mathbb{R}^d.$$

Theorem: (Heat Kernel)

Let $\alpha, \delta \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ and $S \in \mathbb{R}^{d,d}$ skew-symmetric, then the function $H : \mathbb{R}^d \times \mathbb{R}^d \times]0,\infty[\to \mathbb{C}$ given by

$$H(x, \xi, t) = (4\pi\alpha t)^{-\frac{d}{2}} \exp\left(-\delta t - (4\alpha t)^{-1} \left|e^{tS}x - \xi\right|^2\right)$$

is a heat kernel of $\mathcal{L}_\infty$.

For the scalar real case see:

[B] R. Beals.
A note on fundamental solutions, 1999.

[A] J. Aarão.
Fundamental solutions for some partial differential operators from fluid dynamics and statistical physics, 2007.

Semigroup Properties

Consider the heat operator

$$[\mathcal{L}_\infty v](x) = \alpha \triangle v(x) + \langle Sx, \nabla v(x) \rangle - \delta v(x), \quad x \in \mathbb{R}^d$$

on $L^p_{eucl,\eta}(\mathbb{R}^d, \mathbb{C})$ and the associated family of operators

$$[T(t)v_0](x) = \begin{cases} \int_{\mathbb{R}^d} H(x,\xi,t) v_0(\xi) d\xi & , t > 0, \\ v_0(x) & , t = 0. \end{cases}$$

Then

- $(T(t))_{t \geqslant 0}$ is a semigroup on $L^p_{eucl,\eta}$ $\forall \eta \in \mathbb{R}$, $1 \leqslant p \leqslant \infty$,
- $(T(t))_{t \geqslant 0}$ is strongly continuous on $L^p_{eucl,\eta}$ $\forall \eta \in \mathbb{R}$, $1 \leqslant p < \infty$,
- $(T(t))_{t \geqslant 0}$ is <u>not</u> analytic on $L^p_{eucl,\eta}$ $\forall \eta \in \mathbb{R}$, $1 < p < \infty$,

The maximal domain is given by $(1 < p < \infty)$

$$\mathcal{D}^p_{eucl,\eta} = \{ v \in W^{2,p}_{eucl,\eta} \mid \langle S \cdot, \nabla v(\cdot) \rangle \in L^p_{eucl,\eta} \}.$$

$$[\mathcal{L}_\varepsilon v](x) = \alpha \triangle v(x) + \langle Sx, \nabla v(x)\rangle - \delta v(x) + q_\varepsilon(x)v(x)$$

Consider the inhomogeneous problem

$$[\mathcal{L}_\varepsilon v](x) = g(x),\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Unique solvability of $\mathcal{L}_\infty v = g$ implies

$$v(x) = \int_{\mathbb{R}^d} G(x,\xi)\left(g(\xi) - q_\varepsilon(\xi)v(\xi)\right) d\xi =: [Sv](x).$$

By contraction mapping principle:

Theorem: (Solvability, Uniqueness, Regularity)

Let $\alpha, \delta \in \mathbb{C}$, $\mathrm{Re}\,\alpha, \mathrm{Re}\,\delta > 0$, $S \in \mathbb{R}^{d,d}$ skew-symmetric, $g \in L^p_{eucl,\eta}(\mathbb{R}^d, \mathbb{C})$, $1 < p < \infty$, $0 \leqslant \eta^2 < \frac{\mathrm{Re}\,\alpha \mathrm{Re}\,\delta}{|\alpha|^2 p^2}$, $q_\varepsilon \in L^\infty(\mathbb{R}^d, \mathbb{C})$ with $\|q_\varepsilon\|_{L^\infty} < K_2$. Then

$$\exists_1\, v \in L^p_{eucl,\eta}(\mathbb{R}^d, \mathbb{C}) :\ Sv = v \text{ in } L^p_{eucl,\eta}.$$

Moreover, we have $v \in W^{1,p}_{eucl,\eta}$ with

$$\exists\, C > 0 :\ \left\|D^\beta v\right\|_{L^p_{eucl,\eta}} \leqslant C \|g\|_{L^p_{eucl,\eta}},\ |\beta| \leqslant 1.$$

$$[\mathcal{L}_q v](x) = \alpha \triangle v(x) + \langle Sx, \nabla v(x) \rangle - \delta v(x) + q(x)v(x)$$

Finally, consider the homogeneous problem

$$[\mathcal{L}_q v](x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2. \tag{6}$$

Theorem: (Exponential Decay)

Let $\alpha, \delta \in \mathbb{C}$, $\mathrm{Re}\,\alpha, \mathrm{Re}\,\delta > 0$, $S \in \mathbb{R}^{d,d}$ skew-symmetric, $q \in L^\infty(\mathbb{R}^d, \mathbb{C})$ with

$$\sup_{|x| \geqslant R_0} |q(x)| < K_2, \text{ for some } R_0 > 0.$$

Moreover, let $v \in L^p(\mathbb{R}^d, \mathbb{C})$ be a solution of (6), $1 < p < \infty$. Then

$$v \in W^{1,p}_{eucl,\eta}(\mathbb{R}^d, \mathbb{C})\ \forall\, 0 \leqslant \eta^2 < \frac{\mathrm{Re}\,\alpha \mathrm{Re}\,\delta}{|\alpha|^2 p^2}.$$

Note: $\mathcal{D}(\mathcal{L}_q) = \mathcal{D}(\mathcal{L}_\varepsilon) = \mathcal{D}(\mathcal{L}_\infty)$.

Exponential Decay

Now consider the nonlinear problem

(7) $$\alpha\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Theorem: (Exponential Decay of $v_\star$)

Let $\alpha \in \mathbb{C}$, $\mathrm{Re}\,\alpha > 0$, $S \in \mathbb{R}^{d,d}$ skew-symmetric and $1 < p < \infty$. Then for every $0 < \varepsilon \leqslant \frac{\mathrm{Re}\,\alpha\delta_1}{|\alpha|^2 p^2}$ there exists a constant $K_1 = K_1(\alpha, \delta_1, p, d, \varepsilon) > 0$ such that every classical solution $v_\star$ of (7) with (A1), (A2), $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{C})$ satisfies

$$v_\star - v_\infty \in W^{1,p}_{eucl,\eta}(\mathbb{R}^d, \mathbb{C}) \quad \forall\, 0 \leqslant \eta^2 \leqslant \frac{\mathrm{Re}\,\alpha\delta_1}{|\alpha|^2 p^2} - \varepsilon.$$