Spatial decay of rotating waves in parabolic systems

Dynamics of Patterns, MFO, Oberwolfach, December 16-22, 2012

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December 17, 2012

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CRC 701

Outline

1. Introduction: Rotating pattern in $\mathbb{R}^d$
2. Main result: Exponential decay of $v_\star$
3. Outline of proof: Exponential decay of $v_\star$
4. Multisolitons: Interaction of spinning solitons

Rotating Patterns in $\mathbb{R}^d$

Consider a **reaction diffusion system**

$$(1)\quad \begin{aligned} u_t(x,t) &= A\triangle u(x,t) + f(u(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ u(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

where $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{R}^N$, $A\in\mathbb{R}^{N,N}$, $f\in C^2(\mathbb{R}^N,\mathbb{R}^N)$.
Assume a **rotating wave** solution $u_\star : \mathbb{R}^d \times [0,\infty[\to \mathbb{R}^N$ of (1)

$$u_\star(x,t) = v_\star(e^{tS}x)$$

$v_\star : \mathbb{R}^d \to \mathbb{R}^N$ profile (pattern), $0\neq S\in\mathbb{R}^{d,d}$ skew-symmetric.
Transformation (into a rotating frame): $v(x,t) = u(e^{tS}x,t)$ solves

$$(2)\quad \begin{aligned} v_t(x,t) &= A\triangle v(x,t) + \langle Sx, \nabla v(x,t)\rangle + f(v(x,t)),\ t>0,\ x\in\mathbb{R}^d,\ d\geqslant 2,\\ v(x,0) &= u_0(x) \qquad\qquad ,\ t=0,\ x\in\mathbb{R}^d. \end{aligned}$$

$$\langle Sx, \nabla v(x)\rangle := \sum_{i=1}^{d}\sum_{j=1}^{d} S_{ij}x_j D_i v(x)\ \textbf{(rotational term)}.$$

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$$\langle Sx,\nabla v(x)\rangle := \sum_{i=1}^{d-1}\sum_{j=i+1}^{d} S_{ij}\left(x_j D_i - x_i D_j\right)v(x)\ \textbf{(rotational term)}.$$

Rotating Patterns in $\mathbb{R}^d$

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$v_\star$ is a stationary solution of (2).

Question: How to show exponential decay of $v_\star$ at $|x|=\infty$?
Consequence: Exponentially small error by restriction to bounded domain.

Rotating Patterns in $\mathbb{R}^d$

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$v_\star$ is a stationary solution of (2).
$d=2$: Spectral stability implies nonlinear stability.

[BL] W.-J. Beyn, J. Lorenz. 2008.

Example

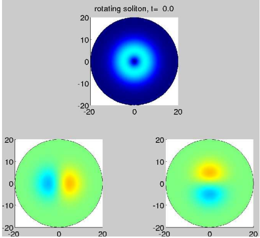
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

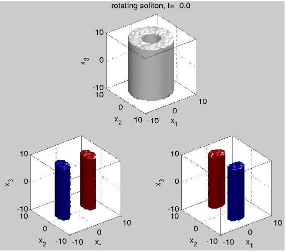
$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$. For the parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}$$

this equation exhibits so called **spinning soliton** solutions.





Applications: superconductivity, superfluidity, nonlinear optical systems.

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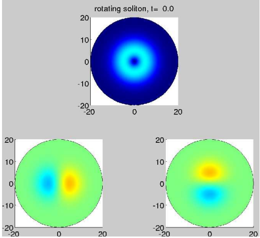

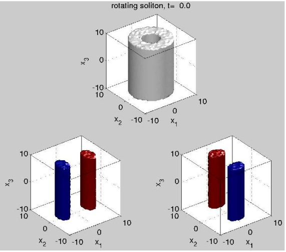

[CMM] L.-C. Crasovan, B.A. Malomed, D. Mihalache. 2001

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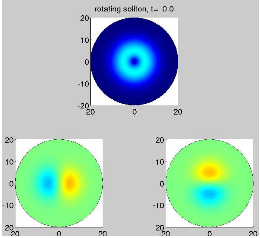

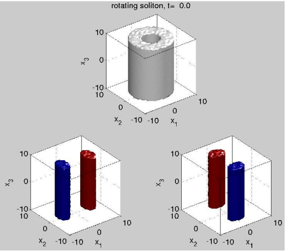

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Main result: Exponential decay of $v_\star$

Theorem: (Exponential Decay of $v_\star$)

Let $f(v_\infty) = 0$ and $\mathrm{Re}\,\sigma(Df(v_\infty)) < 0$. Under further assumptions holds:
For every $1 < p < \infty$, $0 < \vartheta < 1$ and for every radially nondecreasing weight function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ of exponential growth rate $\eta \geqslant 0$ with

$$0 \leqslant \eta^2 \leqslant \vartheta \frac{2}{3} \frac{a_0 b_0}{a_{\max}^2 p^2}$$

there exists $K_1 = K_1(A, f, v_\infty, d, p, \theta, \vartheta) > 0$ with the following property:
Every classical solution $v_\star$ of

$$A \triangle v(x) + \langle Sx, \nabla v(x) \rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,$$

such that $v_\star - v_\infty \in L^p(\mathbb{R}^d, \mathbb{R}^N)$ and

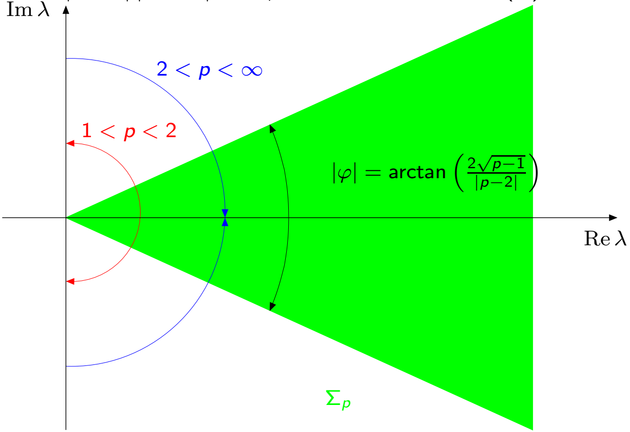
$$\sup_{|x| \geqslant R_0} |v_\star(x) - v_\infty| \leqslant K_1 \text{ for some } R_0 > 0$$

satisfies

$$v_\star - v_\infty \in W_\theta^{1,p}(\mathbb{R}^d, \mathbb{R}^N)$$ (**weighted Sobolev space**).

Main result: The assumptions

- $\mathrm{Re}\,\lambda > 0$ and $|\mathrm{Im}\,\lambda|\,|p-2| \leqslant 2\sqrt{p-1}\mathrm{Re}\,\lambda \quad \forall\,\lambda \in \sigma(A)$



$A, Df(v_\infty) \in \mathbb{R}^{N,N}$ simultaneously diagonalizable over $\mathbb{C}$

- $a_0 \leqslant \mathrm{Re}\,\lambda, \quad |\lambda| \leqslant a_{\max} \quad \forall\,\lambda \in \sigma(A)$
  $\mathrm{Re}\,\mu \leqslant -b_0 < 0 \quad \forall\,\mu \in \sigma(Df(v_\infty))$

Main result: The assumptions

- A positive function $\theta \in C(\mathbb{R}^d, \mathbb{R})$ is called a **weight function of exponential growth rate** $\eta \geqslant 0$ provided that

$$\exists\, C_\theta > 0 : \ \theta(x+y) \leqslant C_\theta \theta(x) e^{\eta|y|} \quad \forall\, x, y \in \mathbb{R}^d.$$

[ZM] S. Zelik, A. Mielke. 2009.

Examples: $\mu \in \mathbb{R}$, $x \in \mathbb{R}^d$

$$\theta_1(x) = \exp\left(-\mu|x|\right), \quad \theta_3(x) = \exp\left(-\mu\sqrt{|x|^2+1}\right),$$
$$\theta_2(x) = \cosh\left(\mu|x|\right), \quad \theta_4(x) = \cosh\left(\mu\sqrt{|x|^2+1}\right).$$

- **Exponentially weighted Sobolev spaces**: $1 \leqslant p \leqslant \infty$, $k \in \mathbb{N}_0$

$$L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^N) \mid \|\theta v\|_{L^p} < \infty \right\},$$
$$W^{k,p}_\theta(\mathbb{R}^d, \mathbb{R}^N) := \left\{ v \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N) \mid D^\beta u \in L^p_\theta(\mathbb{R}^d, \mathbb{R}^N)\ \forall\, |\beta| \leqslant k \right\}.$$

Outline of proof: Exponential Decay of $v_\star$

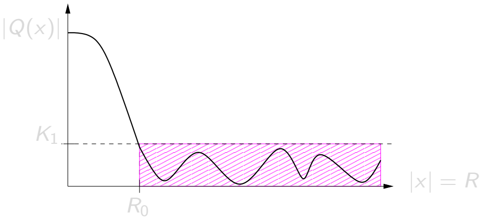
Consider the nonlinear problem

$$A\triangle v_\star(x) + \langle Sx, \nabla v_\star(x)\rangle + f(v_\star(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Far-Field Linearization: $f \in C^1$, Taylor's theorem, $f(v_\infty) = 0$

$$a(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + a(x)w(x) = 0,\ x \in \mathbb{R}^d.$$



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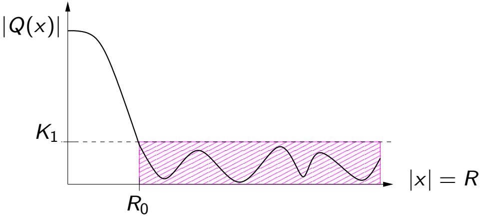
Decomposition of a:

$$Df(v_\infty) + Q(x) = \int_0^1 Df(v_\infty + t(v_\star(x) - v_\infty))dt, \quad w(x) := v_\star(x) - v_\infty$$

$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q(x)\right) w(x) = 0,\ x \in \mathbb{R}^d.$$



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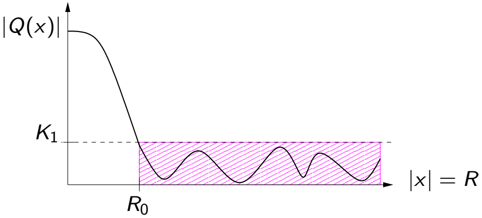
$$A\triangle w(x) + \langle Sx, \nabla w(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) w(x) = 0,\ x \in \mathbb{R}^d.$$



Decomposition of Q**:**

$Q(x) = Q_\varepsilon(x) + Q_c(x)$,

$Q, Q_\varepsilon, Q_c \in L^\infty(\mathbb{R}^d, \mathbb{R}^{N,N})$,

Q_ε small, i.e. $\|Q_\varepsilon\|_{L^\infty} < K_1$,

Q_c compactly supported.

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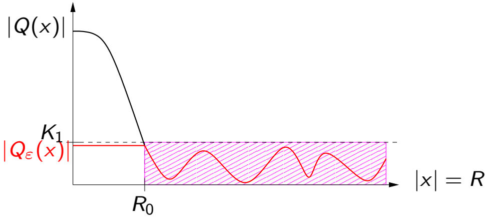

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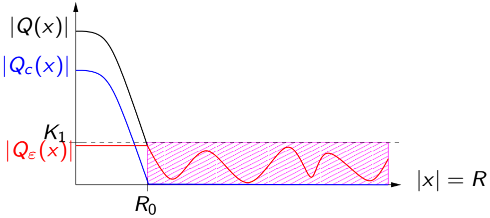

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Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) v(x) = 0,\ x \in \mathbb{R}^d,\ d \geqslant 2.$$

Operators: Study the following operators

$$\begin{aligned}
\mathcal{L}_Q v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v + Q_c v,\\
\mathcal{L}_{Q_\varepsilon} v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v,\\
\mathcal{L}_\infty v :=& A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v,\\
\mathcal{L}_0 v :=& A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator).} \quad \text{(max. domain)}
\end{aligned}$$

Ornstein-Uhlenbeck Operator

Let $P, B \in \mathbb{R}^{d,d}$, $P = P^T$, $P > 0$ and $B \neq 0$.

$$\nabla^T P \nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij} D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x \in \mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

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$$\nabla^T P \nabla v(x) + \langle Bx, \nabla v(x)\rangle = \sum_{i=1}^{d}\sum_{j=1}^{d} D_i\left(P_{ij} D_j v(x)\right) + \sum_{i=1}^{d}\sum_{j=1}^{d} D_i v(x) B_{ij} x_j,\ x \in \mathbb{R}^d$$

Here: $P = I_d$ and $B = S$.

Exponential Decay: To show exponential decay for the solution $v_\star$ of

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + f(v(x)) = 0,\; x\in\mathbb{R}^d,\; d\geqslant 2$$

investigate the far-field linearization (w.l.o.g. $v_\infty = 0$)

$$A\triangle v(x) + \langle Sx, \nabla v(x)\rangle + \left(Df(v_\infty) + Q_\varepsilon(x) + Q_c(x)\right) v(x) = 0,\; x\in\mathbb{R}^d,\; d\geqslant 2.$$

Operators: Study the following operators

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\mathcal{L}_Q v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v + Q_\varepsilon v + Q_c v,\\
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\mathcal{L}_\infty v &:= A\triangle v + \langle S\cdot, \nabla v\rangle + Df(v_\infty)v, && \text{(exp. decay)}\\
\mathcal{L}_0 v &:= A\triangle v + \langle S\cdot, \nabla v\rangle \quad \text{(Ornstein-Uhlenbeck operator).} && \text{(max. domain)}
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[MPV] G. Metafune, D. Pallara, V. Vespri. 2005.

[MPRS] G. Metafune. 2001.

Work in progress

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- rotating waves in **bounded domains**
- **approximation theorem** for rotating waves
- asymptotic boundary conditions
- **numerical computations** (interaction of multisolitons)

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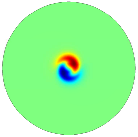
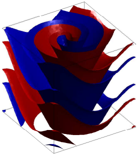
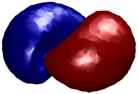
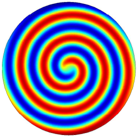








Multisolitons: Interaction of spinning solitons

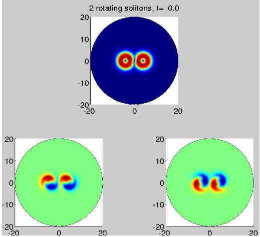
Consider the **quintic complex Ginzburg-Landau equation** (QCGL):

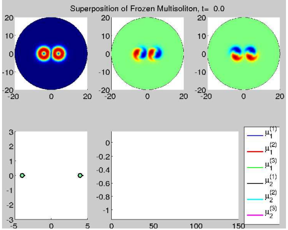
$$u_t = \alpha \triangle u + u\left(\mu + \beta |u|^2 + \gamma |u|^4\right), \quad u = u(x,t) \in \mathbb{C}$$

with $u : \mathbb{R}^d \times [0,\infty[\to \mathbb{C}$, $d \in \{2,3\}$ and parameters

$$\alpha = \frac{1}{2} + \frac{1}{2}i,\ \beta = \frac{5}{2} + i,\ \gamma = -1 - \frac{1}{10}i,\ \mu = -\frac{1}{2}.$$

- **Weak interaction** of 2 spinning solitons:





without freezing (left), with decompose and freeze (right)

Center of solitons at $\pm(4,0)$.

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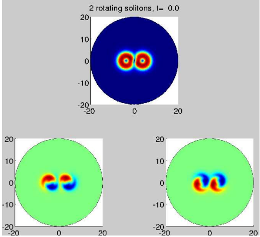

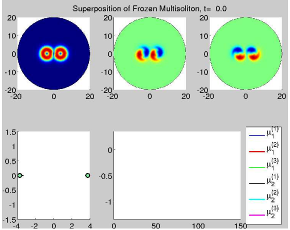

without freezing (left), with decompose and freeze (right)

Center of solitons at $\pm(3.75, 0)$.

Multisolitons: Interaction of spinning solitons

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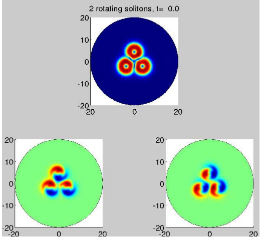
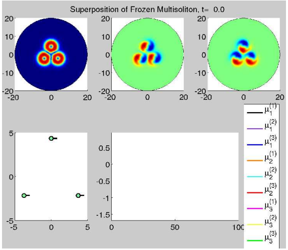
- **Strong interaction** of 3 spinning solitons:





without freezing (left), with decompose and freeze (right)

Centers on a equilateral triangle with radius of circumcircle 3.75.