1. Traveling Front in the Nagumo equation

Consider the Nagumo equation

\[ u_t = u_{xx} + u(1 - u)(u - b), \quad x \in \mathbb{R}, \quad t \geq 0 \]

for \( 0 < b < 1 \), where \( u = u(x, t) \in \mathbb{R} \). We want to solve this equation numerically for a suitable initial value function \( u_0 \) with Comsol Multiphysics. Therefore, we have to restrict the equation on a sufficiently large bounded domain \( \Omega \subset \mathbb{R} \) with homogeneous Neumann boundary conditions, i.e. we solve the initial boundary value problem

\[
\begin{align*}
    u_t &= u_{xx} + u(1 - u)(u - b), \quad x \in \Omega, \quad t \in (0, T], \\
    u(\cdot, 0) &= u_0, \quad x \in \Omega, \quad t = 0, \\
    u_x &= 0, \quad x \in \partial \Omega, \quad t \in [0, T],
\end{align*}
\]

(1.1)

on the spatial domain \( \Omega = (-50, 50) \) for end time \( T = 100 \), initial data

\[ u_0(x) = \frac{1}{1 + \exp\left(-\frac{x}{\sqrt{2}}\right)}, \quad x \in \mathbb{R} \]

and parameter \( b = \frac{1}{4} \). For the space discretization we use linear Lagrange elements with maximal element size \( \Delta x = 0.1 \). For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize \( \Delta t = 0.1 \), relative tolerance \( rtol = 10^{-3} \) and absolute tolerance \( atol = 10^{-4} \) with global method set to be unscaled. The nonlinear equations should be solved by the Newton method, i.e. automatic (Newton).

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2. Model Wizard

Start Comsol Multiphysics.
To start Comsol Multiphysics 5.2 open the Terminal and enter
• comsol -ckl

Model Wizard.
Space dimension
• In the New window, click Model Wizard.
• In the Model Wizard window, click 1D in the Select Space Dimension menu.

Equation
• In the Select Physics tree, select Mathematics>PDE Interfaces>Coefficient Form PDE (c).
• Click Add.
• Next, locate the Dependent Variables section.
• In the Field name text field, type u.
• In the Dependent variables text field, type also u.

Study settings
• Click Study and choose Preset Studies>Time Dependent.
• Click Done.

Some Advanced Settings.
Hint: In the Model Builder window you should click on the Show icon and enable everything that is possible from the menu: Equation Sections (Equation View, Override and Contribution, Discretization, Stabilization, Advanced Physics Options, Advanced Study Options and Advanced Results Options). Done this, click Expand All icon.

3. Geometry
• In the Model Builder tree, expand the Component 1 (comp1) node, right-click Geometry 1 and select Interval.
• In the Settings window for Interval, locate the Interval section.
• In the Left endpoint text field, type -50.
• In the Right endpoint text field, type 50.
• In the Model Builder tree, right-click on the Component 1 (comp1)→Geometry 1 node and select Build all. (Alternatively, press the short cut F8.)

4. Partial differential equation

General Settings.
• Click on Component 1 (comp1)→Coefficient Form PDE (c).
• Locate the Settings window for Coefficient Form PDE.
• In the Label text field, type Nagumo Equation.
• In the Discretization section choose
  – Shape function type: Lagrange,
  – Element order: Linear.
Partial differential equation. We define the PDE:

- Switch to Component 1 (comp1)→Nagumo Equation (c)→Coefficient Form PDE 1

\[ e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} + \nabla \cdot \left( -c \nabla u - \alpha u + \gamma \right) + \beta \cdot \nabla u + au = f \]

with \( \nabla = \frac{\partial}{\partial x} \), and enter the following values

- Mass Coefficient \( e_a \): 0,
- Damping or Mass Coefficient \( d_a \): 1,
- Diffusion coefficient \( c \): 1,
- Conservative Flux Convection Coefficient \( \alpha \): 0,
- Conservative Flux Source \( \gamma \): 0,
- Convection Coefficient \( \beta \): 0,
- Absorption Coefficient \( a \): 0,
- Source Term \( f \): \( fu \).

Boundary Conditions. Since the PDE requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. Hint: By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

Initial Values. We define the initial value \( u(\cdot, 0) = u_0 \) for the partial differential equation:

- Click on Component 1 (comp1)→Nagumo Equation (c)→Initial Values 1.
- In the Initial Values section enter
  - Initial value for \( u \): \( u_0 \),
  - Initial time derivative of \( u \): 0.

The quantity \( u_0 \) will be defined below in Section 5. This completes the implementation of the initial boundary value problem.

5. Parameters and Variables

Parameters. We first define the parameters and constants arising in our model as 'global parameters':

- In the Model Builder tree, right-click on the Global Definitions node and select Parameters. (Alternatively: On the Model toolbar, click Parameters.)
- In the Settings window for Parameters, locate the Parameters section.
- In the table add the following entry:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>1/4</td>
<td>0.25</td>
<td>constant of Nagumo equation</td>
</tr>
<tr>
<td>T</td>
<td>100</td>
<td>100</td>
<td>end time</td>
</tr>
</tbody>
</table>

Variables 1. We now define all functions which appear in our model as 'local variables':

- In the Model Builder tree, right-click on the Component 1 (comp1)→Definitions node and select Variables.
- In the Settings window for Variables, locate the Variables section.
- In the table add the following entries:

<table>
<thead>
<tr>
<th>Name</th>
<th>Expression</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>u0</td>
<td>1/(1+\exp(-x/sqrt(2)))</td>
<td>unit</td>
<td>initial value</td>
</tr>
<tr>
<td>fu</td>
<td>( u^<em>(1-u)^</em>(u-b) )</td>
<td>unit</td>
<td>nonlinearity</td>
</tr>
</tbody>
</table>
6. Mesh

- In the Model Builder tree, click on Component 1 (comp1) → Mesh 1.
- In the Settings window for Mesh, locate the Mesh Settings section.
- Set the Sequence type on User-controlled mesh.
- In the Model Builder tree, switch to Component 1 (comp1) → Mesh 1 → Size.
- In the Settings window for Size, locate the Element Size Parameters section.
- In the Maximum element size text field, type 0.1.
- In the Model Builder tree, right-click on Component 1 (comp1) → Mesh and select Build All.

7. Studies and Computation

Study 1. Study 1

- Click on Study 1.
- Locate the Settings window for Study.
- In the Label text field, type Study 1: Nagumo Equation.

Step 1

- Click on Study 1: Nagumo Equation → Step 1: Time Dependent.
- Locate the Settings window for Time Dependent.
- In the Study Settings section enter
  - Time unit: s,
  - Times: range(0,0.1,T),
  - Relative tolerance: 0.001.
  The last input requires to enable the corresponding checkbox.

Solver Configurations

- Right-click on Study 1: Nagumo Equation → Solver Configurations and select Show Default Solver.
- Click on Study 1: Nagumo Equation → Solver Configurations → Solution 1 (sol1) → Time-Dependent Solver 1.
- Locate the Settings window for Time Dependent Solver.
- In the Absolute Tolerance section enter
  - Global method: Unscaled,
  - Tolerance: 0.0001.
- In the Time Stepping section enter
  - Method: BDF,
  - Steps taken by solver: intermediate,
  - Maximum BDF order: 2.
- Click on Study 1: Nagumo Equation → Solver Configurations → Solution 1 (sol1) → Time-Dependent Solver 1 → Fully Coupled 1.
- Locate the Settings window for Fully Coupled.
- In the Method and Termination section, choose
  - Nonlinear Method: Automatic (Newton),
Solution Store

- Right-click on **Study 1: Nagumo Equation**\rightarrow**Solver Configurations**\rightarrow**Solution 1 (sol1)** and select **Other>Solution Store** from the list.
- Click on **Study 1: Nagumo Equation**\rightarrow**Solver Configurations**\rightarrow**Solution 1 (sol1)**\rightarrow**Solution Store 1 (sol2)**.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Nagumo Equation Solution**.


- Right-click on **Study 1: Nagumo Equation** and select **Compute** from the list.

8. Postprocessing and graphical output

In this section we generate 2 Plot groups and a movie for visualizing our results.

8.1. Results for the Nagumo equation.

Plot Group 1: Traveling Front, View 1

- Click on **Results**\rightarrow**1D Plot Group 1**. Hint: If **1D Plot Group 1** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 1**.
- In the **Data** section select **Data set** Study 1: Nagumo Equation/Nagumo Equation Solution (sol2). **Time selection** Interpolated and **Times (s)** 0 20 40 60 80 100.
- In the **Title** section select **Title type** None.
- In the **Plot Settings** section select **x-axis label** x and **y-axis label** u(x,t).
- Click on **Results**\rightarrow**Traveling Front, View 1**\rightarrow**Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results**\rightarrow**Traveling Front, View 1** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph 1.
- In the **Data** section select **Data set** From parent.
- In the **Selection** section select **Selection** All domains.
- In the **y-Axis Data** section select **Expression** u.
- In the **x-Axis Data** section select **Parameters** Expression and **Expression** x.
- In the **Coloring and Style** section select **Line Solid, Color Cycle** and **Width 2** in the **Line style** subsection.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends** Manual and enter the legends t=0, t=20, t=40, t=60, t=80 and t=100.

Plot Group 2: Traveling Front, View 2

- Click on **Results**\rightarrow**1D Plot Group 2**. Hint: If **1D Plot Group 2** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 2**.
- In the **Data** section select **Data set** Study 1: Nagumo Equation/Nagumo Equation Solution (sol2) and **Time selection** All.
• In the **Title** section select **Title type None**.
• In the **Plot Settings** section select **x-axis label x** and **y-axis label t**.
• Click on **Results → Traveling Front, View 2 → Line Graph 1**. Hint: If Line Graph 1 does not exists, right-click on **Results → Traveling Front, View 2** and select **Line Graph** from the list.
• Locate the **Settings** window for Line Graph 1.
• In the **Data** section select **Data set From parent**.
• In the **Selection** section select **Selection All domains**.
• In the **y-Axis Data** section select **Expression t**.
• In the **x-Axis Data** section select **Parameter Expression** and **Expression x**.
• Right-click on **Results → Traveling Front, View 2 → Line Graph 1** and select **Color Expression**.
• Click on **Results → Traveling Front, View 2 → Line Graph 1 → Color Expression 1**.
• Locate the **Settings** window for Color Expression.
• In the **Expression** section select **Expression u**.

### Plot Group 3: Plot for Animation

• Right-click on **Results** and select **1D Plot Group** from the list.
• Locate the **Settings** window for 1D Plot Group 3.
• In the **Label** text field, type **Plot Group for Animation**.
• In the **Data** section select **Data set Study 1: Nagumo Equation/Nagumo Equation Solution (sol2), Time selection First**.
• In the **Title** section select **Title type None**.
• In the **Plot Settings** section select **x-axis label x** and **y-axis label u(x,t)**.
• Click on **Results → Plot Group for Animation → Line Graph 1**. Hint: If Line Graph 1 does not exists, right-click on **Results → Plot Group for Animation** and select **Line Graph** from the list.
• Locate the **Settings** window for Line Graph 1.
• In the **Data** section select **Data set From parent**.
• In the **Selection** section select **Selection All domains**.
• In the **y-Axis Data** section select **Expression u**.
• In the **x-Axis Data** section select **Parameters Expression** and **Expression x**.
• In the **Coloring and Style** section select **Line Solid, Color Cycle and Width 2** in the **Line style** subsection.
• In the **Legends** section enable the **Show legends checkbox**, select **Legends Automatic**.

### Animation 1: Traveling Front, Animation

• Click on the **Animation** icon on top of the **Settings** window and select **Player**.
• Locate the **Setting** window for Animation 1.
• In the **Label** text field, type **Traveling Front, Animation**.
• In the **Target** section select **File**.
• In the **Output** section select **Format GIF** and in the **Filename** text field type **NagumoEquation.gif**.
• In the Frames section enter **Number of Frames** 100.

• In the Layout section enable the checkbox for Include and then the checkboxes for Title, Legend and Axes and enter **Font size** 10.

• In the Advanced section disable the checkbox Synchronize scales between frames.

• Right-click Results→Export→Traveling Front, Animation and select Export.