QE “Optimization”, WS 2017/18

Problem Set No. 2 (based on Lecture Notes, Items 1.1–1.4)

Submit your solutions by **25.09.2017**.
The problems will be discussed in the tutorials.

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1. [**4 Points**] Find the supremum and infimum of each of the following sets in $\mathbb{R}$. Justify your answers.

i $\left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\}$

ii $\{a - b \mid a, b \in \mathbb{R},\ 1 < a < 2,\ 3 < b < 4\}$.

2. [**4 Points**] Determine whether or not the following sequences converge. Find the limits, if they exist. Justify your work.

i $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^3$, $x_n := \left((-1)^n,\ 4,\ \frac{1}{n}\right)$

ii $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}^2$, $x_n := \left(\frac{n \sin n}{n^2+1}, \frac{(-1)^{n+1}}{n}\right)$.

3. [**3 Points**] Let $\mathbb{Q} \subset \mathbb{R}$ be the set of all rational numbers. Find its interior $\mathring{\mathbb{Q}}$, closure $\bar{\mathbb{Q}}$, and boundary $\partial\mathbb{Q}$.

4. [**2 Points**] Let X be a non-empty set and let d be the discrete metric;

$$d(x, y) := \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Describe all convergent sequences in the metric space (X, d).

5. Consider the space of continuous functions $C[0, 1]$ with the max-norm.

(a) [**2 Points**] Fix some function g from $C[0, 1]$. Prove that the set

$$\{f \in C[0, 1] \mid f(t) < g(t) \text{ for all } t \in [0, 1]\}$$

is open in $C[0, 1]$.

(b) [**2 Points**] Calculate the distance in $C[0, 1]$ between the functions $f(t) = 2t$ and $g(t) = 1 - t$.

(c) [**3 Points**] Prove that the sequence

$$f_n(t) := t^n - t^{2n},\ t \in [0,1]$$

is not convergent in $C[0,1]$.

6. Let X be a vector space equipped with a norm $\|x\|_X$.

(a) [**3 Points**] Prove that

$$\rho_1(x,y) := \min\{1, \|x-y\|_X\},\ x, y \in X$$

defines a metric on X (the so-called *Radar Screen* metric).

(b) [**2 Points**] Show that the function

$$\rho_2(x,y) := \max\{1, \|x-y\|_X\},\ x, y \in X$$

is not a metric on X.