QE "Optimization", WS 2017/18

Problem Set No. 3

Submit your solutions by **02.10.2017**.
The problems will be discussed in the tutorials.

Questions marked with a star (*) are slightly more challenging and can be skipped if you get too stuck.

1. (a) [**2 Points**] Prove that in any metric space (X, d) the *closed ball* defined by

$$\overline{B_r(x)} := \{y \in X \mid d(x, y) \leq r\}, \ x \in X, \ r \in \mathbb{R}_+,$$

is indeed a closed set.

(b) [**2 Points**] Prove that for each open ball

$$B_r(x) := \{y \in X \mid d(x, y) < r\}$$

its closure is always contained in $\overline{B_r(x)}$.

(c) [**2 Points**] Show that in general there is no identity in (b). **Hint:** consider the discrete metric space.

2.* (a) [**2 Points**] Prove that for any points $x, y \in X$ and any nonempty set $A \subseteq X$

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Here, $d(x, A) := \inf\{d(x, z) \mid z \in A\}$.

(b) [**1 Point**] Conclude from (a) that the mapping

$$(X, d) \ni x \to d(x, A) \in \mathbb{R}_+$$

is continuous.

3. [**2 Points**] Let $A \subseteq X$ be closed and $x \notin A$. Prove that $d(x, A) > 0$.

4. [**2 Points**] Let (X, d) and (Y, ρ) be metric spaces. Show that if d is the discrete metric, then any function $f: X \to Y$ is continuous.

5. [**2 Points**] For a metric space (X, d) and a given $x_0 \in X$, prove that the distance function

$$X \ni x \to f(x) := d(x, x_0) \in \mathbb{R}$$

is uniformly continuous.

6.* [**3 Points**] Prove that a function $f\colon \mathcal{I} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1-x}$$

is not uniformly continuous on the interval $\mathcal{I} = (0, 1)$.