QE "Optimization", WS 2017/18

Problem Set No. 4

Submit your solutions by **09.10.2017**.

The problems will be discussed in the tutorials.

Questions marked with a star (*) are slightly more challenging and can be skipped if you get too stuck.

A geometric progression is a sequence of the form $(a, ar, ar^2, ar^3, ar^4, \ldots)$. The sum of a geometric progression will appear several times in this problem sheet. Hence, recall that the following summation formulae hold for all $|\beta| < 1$,

$$\sum_{i=1}^{n} \beta^i = \frac{\beta - \beta^n}{1-\beta}, \quad \sum_{i=1}^{\infty} \beta^i = \frac{\beta}{1-\beta}.$$

1. [**14 Points**] Recall that the space of p-summable sequences in $\mathbb{R}$ with the p-norm $\|\cdot\|_p$ is denoted by l_p. The p-norm is given by

$$\|(x_1, x_2, \ldots)\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

Check whether the following sequences are convergent in the corresponding l_p space or not. Find the limits if they exist. Prove that the sequence does not converge if not. [Be careful. We're looking at sequences *of sequences*.]

Example:

$X = l_2$, $y_n := (1, 2, 3, \ldots, n, 0, 0, 0, \ldots)$;

So $y_1 = (1, 0, 0, 0, \ldots)$, $y_2 = (1, 2, 0, 0, \ldots)$, $y_3 = (1, 2, 3, 0, 0 \ldots)$, etc., and we have norms $\|y_1\|_2 = 1$, $\|y_2\|_2 = \sqrt{5}$, $\|y_3\|_2 = \sqrt{14}$, etc.

We see that the sequence $(y_n)_{n\geq 1}$ cannot converge because $\|y_n - y_{n-1}\|_2 = n \to \infty$ and if y_n were to converge, then $\|y_n - y_{n-1}\|_2$ would converge to 0.

(**a**) $X = l_1$, $y_n := \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, 0, 0, \ldots\right)$;

(**b**) $X = l_1$, $y_n := \left(\frac{n+1}{n^2}, \frac{n+2}{n^2}, \ldots, \frac{2n}{n^2}, 0, 0, \ldots\right)$;

(**c**) $X = l_1$, $y_n := \Big(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n \text{ times}}, 0, 0, \dots\Big)$;

(**d**) $X = l_1$, $y_n := \Big(\underbrace{0, \dots, 0}_{n-1 \text{ times}}, \frac{1}{n^\sigma}, \frac{1}{(n+1)^\sigma}, \dots\Big)$, $\sigma > 1$;

(**e**) $X = l_2$, $y_n := \Big(\underbrace{\frac{1}{n}, 0, \dots, 0, 1}_{n}, 0, 0, \dots\Big)$;

(**f**) $X = l_2$, $y_n := \Big(\underbrace{\frac{1}{n}, \dots, \frac{1}{n}}_{n^2 \text{ times}}, 0, 0, \dots\Big)$;

(**g**) $X = l_3$, $y_n := \big(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\big)$.

2. [**4 Points**] Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in a metric space (X, d). Suppose that $\exists \lim_{k\to\infty} x_{n_k} =: x \in X$ for some subsequence $(x_{n_k})_{k\geq 1}$. Prove that $x_n \to x$ as $n \to \infty$.

3. [**3 Points**] Check the following inequality

$$d(x_n, x^*) \leq \frac{\beta^n}{1-\beta} d(x_1, x_0),\ n \in \mathbb{N},$$

describing the speed of convergence in the Banach fixed point theorem.

Hint: Work by induction on n.

4. [**5 Points**] Let (X, d) be a complete metric space and let $(x_n)_{n\geq 1}$ be a sequence in X such that there is $0 < \beta < 1$ with

$$d(x_{n+2}, x_{n+1}) \leq \beta d(x_{n+1}, x_n) \text{ for all } n \in \mathbb{N}.$$

Show that $(x_n)_{n\geq 1}$ is convergent.

Hint: What is the consequent relationship between $d(x_{n+1}, x_{n+1})$ and $d(x_2, x_1)$? Use the triangle inequality to show that $(x_n)_{n\geq 1}$ is a Cauchy sequence. You might want to use geometric series.

5. [**5 Points**] Let (X, d) be a complete metric space, and $T\colon X \to X$ be such that the operator T^n is a contraction for some $n \in \mathbb{N}$. Show that T has a unique fixed point.

Hint: (i) Prove that T^n has a unique fixed point, say x^*.
(ii) Check that $d(Tx^*, x^*) = 0$, i.e., x^* is the unique fixed point for T.
6. [**5 Points**] Show that the map F defined by

$$f \mapsto F(f),\ [F(f)](t) = \frac{1}{2}\int_0^1 tsf(s)\,ds + \frac{5}{6}t,\ t \in [0,1],$$

is a contraction in $C([0,1])$. Use the second part of Banach's fixed point theorem concerning convergence of iterates to find the unique fixed point f^*.

Hint: Is it true that $F^n(f_0)(t) = f_n(t) = c_n t$ for some $c_n \in \mathbb{R}$? If yes, do we have $\lim_{n\to\infty} c_n = 1$? Choosing $f_0(t)$ wisely eases the computations!!
7. [**4 Points**] Check that the mapping

$$F(x) := \frac{x^2+2}{2x}$$

is a contraction on the closed interval $[1,2]$. Using the above, apply the Banach fixed point theorem to show that the expression

$$\frac{1}{x} - \frac{x}{2}$$

has exactly one root in the interval $[1,2]$.
Hint: To show that F is a contraction on $[1,2]$, rewrite $|F(x) - F(y)|$ as $c|x-y||f(x,y)|$, where c is a constant in $(0,1)$ and $f(x,y)$ is a function such that, for all x, y, one has $|f(x,y)| \leq 1$.
8*. [**4 Points**] Prove that the space $C([-1,1])$ is not complete with respect to the metric

$$d(f,g) := \left(\int_0^1 |f(t) - g(t)|^2\,dt\right)^{1/2}.$$

Hint: Consider the sequence

$$f_n(t) := \begin{cases} -1 & \text{if } -1 \leq t \leq -\frac{1}{n}, \\ nt & \text{if } -\frac{1}{n} \leq t \leq \frac{1}{n}, \\ 1 & \text{if } \frac{1}{n} \leq t \leq 1. \end{cases}$$

9*. [**5 Points**] Consider the functional sequence $f_n\colon \mathbb{R}_+ \to \mathbb{R}$, $n \in \mathbb{N}$,

$$f_n(t) := \cos\left(\frac{t}{n}\right) e^{-t},\ t \in \mathbb{R}_+ := [0,\infty).$$

(**a**) Find the pointwise limit of this sequence.

(**b**) Show that this sequence converges even uniformly on $\mathbb{R}_+$ (equipped with the usual distance $|\cdot|$ from $\mathbb{R}$). That is, show that $\|f - f_n\| \to 0$ as $n \to \infty$, where f is the pointwise limit of f_n and $\|g\| := \sup\{|g(t)| \, : \, t \in \mathbb{R}_+\}$.

Hint: Use the elementary inequality $\cos x \geq 1 - \frac{x^2}{2}$, $x \in \mathbb{R}_+$.