QE "Optimization", WS 2017/18

Problem Set No. 5

Submit your solutions by **16.10.2017**.

The problems will be discussed in the tutorials.

Questions marked with a star (*) are slightly more challenging and can be skipped if you get too stuck.

1. [**4 Points**] Under which conditions is a discrete metric space (X, d) **(a)** complete? **(b)** separable?

2. [**2 Points**] Let (X, d) and (Y, ρ) be metric spaces and $f\colon X \to Y$ be continuous. Prove that, for any set A which is dense in X, the image set $f(A)$ is dense in $f(X)$.

3. [**4 Points**] For which values of $r \in \mathbb{R}$ does the sequence $(r^n)_{n\in\mathbb{N}}$ belong to l_p, where $1 \leq p \leq +\infty$? Find its norm.

4. [**2 Points**] Which of the following sequences lie in l_1?

$$\textbf{(i)}\ \left(\frac{1}{n}\right)_{n\in\mathbb{N}}; \quad \textbf{(ii)}\ \left(\frac{\sin \pi n}{n^2}\right)_{n\in\mathbb{N}}$$

5. [**4 Points**] Show that the mapping $f \mapsto Tf$ given by

$$(Tf)(t) := \frac{1}{2}\int_0^t f(s)\, ds + 1,\ t \in [0, 1],\ f \in C[0, 1],$$

is a contraction in $C[0, 1]$. Determine the limit of the sequence defined by $f_1 \equiv 0, f_{n+1} = Tf_n,\ n \in \mathbb{N}$.

6. [**18 Points**] Which of the following sequences of functions converges to a limit in $C[0, 1]$? For those that converge, state what the limit is.

(i) $f_n(t) = t^{2n}$;

(ii) $f_n(t) = t\exp(-nt)$;

(iii) $f_n(t) = t\exp(-t/n)$;

(iv) $f_n(t) = n^{-1}\sin \pi nt$;

(v) $f_n(t) = (\sin \pi t)^n$;

(vi) $f_n(t) = \frac{1}{1+nt^2}$.

7.* [**4 Points**] Consider the set $C_0(\mathbb{R})$ of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$\exists \lim_{|t|\to+\infty} f(t) = 0.$$

Prove that $C_0(\mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} |f(t)|.$$

8. [**8 Points**] Check the continuity and find the norms of the following linear functionals $F\colon X \to \mathbb{R}$ (i.e., linear operators with values in $Y := \mathbb{R}$):

(i) $X := l_1$, $x = (x_i)_{i\geq 1}$, $\quad F(x) := \sum_{i=1}^{\infty} \left(1 - \frac{1}{i}\right) x_i$;

(ii) $X := l_2$, $x = (x_i)_{i\geq 1}$, $\quad F(x) := \sum_{i=1}^{\infty} \frac{1}{i}\, x_i$;

(iii) $X := l_2$, $x = (x_i)_{i\geq 1}$, $\quad F(x) := \sum_{i=1}^{\infty} [1 - (-1)^i]\, \frac{i-1}{i}\, x_i$;

(iv) $X := C([0,1])$, $F(f) = \int_0^1 f(t)\, \mathrm{sign}\left(t - \frac{1}{2}\right) dt$.

Here $\mathrm{sign}\, r := \begin{cases} 1, & \text{if } r > 0, \\ 0, & \text{if } r = 0, \\ -1, & \text{if } r < 0. \end{cases}$