QE "Optimization", WS 2017/18

Problem Set No. 6

Submit your solutions by **23.10.2017**, 12 noon.

The problems will be discussed in the tutorials.

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1. [**3 Points**] Let A, B be two compact sets in a metric space (X, d). Prove that $A \cap B$ and $A \cup B$ are also compact.

2. [**2 Points**] Under which conditions is a *discrete* metric space (X, d) compact?

3. [**2 Points**] Show that if $f\colon X \to Y$ is continuous and X is compact, then $f(X)$ is compact in Y.

4. [**8 Points**] Prove or disprove compactness of the following sets in $\mathbb{R}^2$:

(i) $A := (\mathbb{Q} \cap [0,1]) \times [0,1]$;

(ii) $B := \{(x, y) \in \mathbb{R}^2 \mid x = 0\}$;

(ii) $C = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\} \times [0,1]$;

(iv) $D := \left\{\left(\frac{1}{n}, \frac{n-1}{n}\right) \mid n \in \mathbb{N}\right\}$.

5*. [**5 Points**] Prove that $\overline{B_1(0)}$ is not (sequentially) compact in l_∞.

6*. [**5 Points**] Apply the Weierstrass theorem to prove the following statement:

Let $K \subset R^2$ be a compact set of the plane with the Euclidean distance $d(x, y) := |x - y|$, $x, y \in \mathbb{R}^2$. Then there exists a point $x^* \in K$ which is the furthest point from the origin in K,i.e.,

$$d(x^*, 0) := \max_{x \in K} d(x, 0).$$

7. [**3 Points**] Let $X = C[0,1]$ with the supremum norm and let A be the subset defined by

$$A := \{f \in C[0,1] \mid f(0) = 1\}.$$

Show that A is closed.