OQE - PROBLEM SET 10 - SOLUTIONS

Exercise 1. Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^2$-function, where U is an open subset of $\mathbb{R}^2$. Let moreover (x_0, y_0) be a critical point of f satisfying

$$\det D^2 f(x_0, y_0) = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 < 0.$$

Then (x_0, y_0) is a saddle point (see Theorem 2.11.2).

Exercise 2. We want to find and classify the critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ that is defined by

$$(x, y) \mapsto f(x, y) = x^3 + y^3 - 3xy.$$

To do so, we compute the gradient of f

$$\nabla f(x, y) = (3x^2 - 3y, 3y^2 - 3x)$$

and the points at each it is equal to $(0, 0)$. We solve

$$\begin{cases} x^2 - y = 0 \\ y^2 - x = 0 \end{cases}$$

getting the points $P = (0, 0)$ and $Q = (1, 1)$. To determine the nature of the critical points P and Q we compute the Hessian of f:

$$D^2 f(x, y) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}.$$

We then compute

$$A = D^2 f(P) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \quad \text{and} \quad B = D^2 f(Q) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

In view of Proposition 2.11.4, the maxtrix B is positive definite and so, thanks to Theorem 2.11.2, the point Q is a strict local minimum; Q is however not a global minimum since, for example, one has $f(0, -5) = -25 < -1 = f(Q)$. The point P is a saddle point because A is indefinite: indeed one has

$$\begin{bmatrix} 1 & 1 \end{bmatrix} D^2 f(P) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -6 < 0$$

while

$$\begin{bmatrix} 1 & -1 \end{bmatrix} D^2 f(P) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 6 > 0.$$

Exercise 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $(x, y) \mapsto f(x, y) = xe^{-x}(y^2 - 4y)$.
(a) We want to find and classify all critical points of f. To this end, we compute

$$\nabla f(x, y) = (e^{-x}(1 - x)(y^2 - 4y), xe^{-x}(2y - 4)).$$

Since the image of the exponential function is $\mathbb{R}_{>0}$, the stationary points of f are exacly the pairs $(x, y) \in \mathbb{R}^2$ that are solutions to the following system

$$\begin{cases} (1 - x)(y^2 - 4y) = 0 \\ x(2y - 4)) = 0 \end{cases}.$$

With not much work, one shows that the stationary points of f are $(0, 0)$, $(0, 4)$, and $(1, 2)$. To decide the nature of the critical points, we compute

$$D^2 f(x, y) = \begin{bmatrix} (y^2 - 4y)e^{-x}(x - 2) & (2y - 4)e^{-x}(1 - x) \\ (2y - 4)e^{-x}(1 - x) & 2xe^{-x} \end{bmatrix}$$

and thus we have

$$D^2 f(0, 0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}, \; D^2 f(0, 4) = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \quad \text{and} \quad D^2 f(1, 2) = \begin{bmatrix} \frac{4}{e} & 0 \\ 0 & \frac{2}{e} \end{bmatrix}.$$

It follows from Proposition 2.11.4 and Theorem 2.11.2 that $(0, 0)$ and $(0, 4)$ are saddle points, while $(1, 2)$ is a strict local minimum.
(b) We show that f has neither a global maximum nor a global minimum. From (a), we know that there are no candidate points for local maxima (since the stationary points are either saddle or local minima). Moreover, since

$$f(-5, 5) = -20e^5 < -\frac{4}{e} = f(1, 2),$$

the local minimum $(1, 2)$ is not a global minimum and so f has no global minima either.
(c) Let the subset S of $\mathbb{R}^2$ be defined by $S = [0, 5] \times [0, 4]$. We claim that $f_{|S}$ has a global maximum and a global minimum. Indeed, the subset S is compact by the Heine-Borel theorem and so the claim follows from Weierstrass's theorem.
(d) We compute the global extrema of $f_{|S}$. Thanks to (a) and (b), we know that the global extrema of $f_{|S}$ belong to $\partial S \cup \{(1, 2)\}$. We compute

i. $f(0, y) = 0$;
ii. $f(5, y) = \frac{5}{e^5}(y^2 - 4y)$;
iii. $f(x, 0) = 0$;
iv. $f(x, 4) = 0$;
v. $f(1, 2) = -\frac{4}{e}$.

One can check that, for each $y \in [0, 4]$, one has $y^2 - 4y \leq 0$ with local minimum equal to -4 for $y = 2$. Since $f(5, 2) = -\frac{20}{e^5} > -\frac{4}{e} = f(1, 2)$, the global minimum of $f_{|S}$ corresponds to the point $(1, 2)$. On the other hand, the global maxima are given by all points in the set $\{(0, y), (x, 0), (x, 4) \in S\}$.

Exercise 4. Let $a, b, p, t \in \mathbb{R}_{>0}$ be such that $p > a + t$. Let moreover x be a variable standing for the number of units of a certain good. Let the functions $c, e, \tau, \pi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by

i. $c(x) = ax + bx^2$, the cost of production;
ii. $e(x) = px$, the amount earned;
iii. $\tau(x) = tx$, the tax;
iv. $\pi(x) = e(x) - c(x) - \tau(x)$, the profit.

(a) We want to find $x^* \in \mathbb{R}_{\geq 0}$ maximizing the profit. To do so, we first write

$$\pi(x) = px - tx - ax - bx^2$$

and therefore compute

$$\pi'(x) = p - t - a - 2bx = 0 \iff x = \frac{p - t - a}{2b}.$$

Since $\pi''(x) = -2b < 0$, we have that $x^* = \frac{p-t-a}{2b}$ is indeed a local maximum. The optimal profit is then $\pi^* = \pi(x^*) = \frac{(p-t-a)^2}{4b}$.

(b) We want to prove that $\partial \pi^* / \partial p = x^*$. Using the envelope theorem, we have

$$\frac{\partial \pi^*}{\partial p}(p, t, a, b) = \frac{\partial \pi}{\partial p}(x^*, p, t, a, b) = \frac{\partial (px - tx - ax - bx^2)}{\partial p}(x^*, p, t, a, b) = x^*.$$

We have thus that, whenever p is increasing, the optimal profit π^* increases proportionally to the amount of the good that is produced.