OQE - PROBLEM SET 12 - SOLUTIONS

Exercise 1. We consider the constrained optimization problem in $\mathbb{R}^2$

$$\min / \max \; f(x,y) = xy$$
$$\text{subject to the constraint } x^2 + y^2 = 1.$$

To this end, we define the function $g : \mathbb{R}^2 \to \mathbb{R}$ by $(x,y) \mapsto g(x,y) = x^2 + y^2 - 1$.
(a) We expect the problem to be solvable for the following reasons. The subset $D = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$ is closed and bounded in $\mathbb{R}^2$, and thus compact. Since the function $f : D \to \mathbb{R}$ is continuous, Weierstrass's theorem guarantees the existence of both a maximum and a minimum in D.
(b) The constrained qualification fails only at the point $(0,0)$. Indeed, for each $(x,y) \in \mathbb{R}^2$, one computes $\nabla g(x,y) = (2x, 2y)$, which is equal to $(0,0)$ if and only if $(x,y) = (0,0)$.
(c) We form the Lagrangean associated to the given EC problem by taking $\lambda \in \mathbb{R}$ and defining

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = xy - \lambda(x^2 + y^2 - 1).$$

The first order conditions are then the following

$$\text{(e1)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = y - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = x - 2\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) = -x^2 - y^2 + 1 = 0 \end{cases}.$$

(d) We derive from (e1) that the four candidate solutions for our EC problem are $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$.
(e) To compute which points from (d) are maxima or minima, we calculate

(1) $f(1/\sqrt{2}, 1/\sqrt{2}) = 1/2$ (maximum);
(2) $f(-1/\sqrt{2}, 1/\sqrt{2}) = -1/2$ (minimum);
(3) $f(1/\sqrt{2}, -1/\sqrt{2}) = -1/2$ (minimum);
(4) $f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/2$ (maximum).

Exercise 2. We consider the constrained optimization problem in $\mathbb{R}^3$

$$\max / \min \; f(x,y,z) = x^2 + y^2 + z$$
$$\text{subject to } (x-1)^2 + y^2 = 5,\; y = z.$$

(a) The reason why we expect the given EC problem to be solvable is the following. Define $C = \{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + y^2 = 5\}$ and let $\pi = \{(x,y,y) \in \mathbb{R}^3\}$. Both

C and π are closed subsets of $\mathbb{R}^3$ and therefore also

$$D = C \cap \pi = \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + y^2 = 5, y = z\}$$

is closed in $\mathbb{R}^3$. Moreover, each $(x, y, z) \in D$ satisfies $(x-1)^2, y^2 \leq 5$, from which it follows that

$$\|(x, y, z)\|^2 = x^2 + y^2 + z^2 = x^2 + 2y^2 \leq (1 + \sqrt{5})^2 + 10$$

and therefore D is bounded. It follows that D is compact and, $f : D \to \mathbb{R}$ being continuous, f admits both a global maximum and a global minimum on D.
(b) Though the given EC problem involves three variables, we can reduce it to the case of two variables. Indeed, since each $(x, y, z) \in D$ satisfies $y = z$, we can reduce to studying $\tilde{f} : C \to \mathbb{R}$, where $\tilde{f}(x, y) = f(x, y, y) = x^2 + y(y+1)$. *Note: here we are slightly abusing notation, since C is a subset of $\mathbb{R}^3$ and not $\mathbb{R}^2$.*
(c) To check that the constrained qualification holds at each point of C, define $g : \mathbb{R}^2 \to \mathbb{R}$ by $(x, y) \mapsto g(x, y) = (x-1)^2 + y^2 - 5$. We compute, at each point of the domain, $\nabla g(x, y) = (2x - 2, 2y)$, which is equal to $(0, 0)$ if and only if $(x, y) = (1, 0) \notin C$.
(d) We form the Lagrangean associated to the given EC problem by taking $\lambda \in \mathbb{R}$ and defining

$$\mathcal{L}(x, y, \lambda) = \tilde{f}(x, y) - \lambda g(x, y) = x^2 + y^2 + y - \lambda((x-1)^2 + y^2 - 5).$$

The first order conditions are then the following

$$\text{(e2)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x, y, \lambda) = 2x - 2\lambda x + 2\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x, y, \lambda) = 2y - 2\lambda y + 1 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x, y, \lambda) = -(x-1)^2 - y^2 + 5 = 0 \end{cases}.$$

(e) We derive the candidate solutions from (e2). Indeed, from (e2.2), we know that $y \neq 0$ and, from the combination of (e2.1),(e2.2), and (e2.3), we also have $x \neq 0$. Computing y(e2.1) $- x$(e2.2) $= 0$, we can rewrite (e.2) as

$$\begin{cases} 2\lambda y - x = 0 \\ 2y - x + 1 = 0 \\ (x-1)^2 + y^2 = 5 \end{cases}.$$

Solving the new system leads to finding the points $(3, 1, 1)$ and $(-1, -1, -1)$. We conclude by computing

(1) $f(3, 1, 1) = 11$ (maximum);
(2) $f(-1, -1, -1) = 1$ (minimum).

Exercise 3. We consider the EC problem in $U = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$

$$\max f(x,y) = 2x + 3y$$
$$\text{subject to } \sqrt{x} + \sqrt{y} = 5.$$

For each $x, y \in U$, write $g(x,y) = \sqrt{x} + \sqrt{y} - 5$ and, given $\lambda \in \mathbb{R}$, write

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = 2x + 3y - \lambda(\sqrt{x} + \sqrt{y} - 5).$$

Following the Lagrange multiplier method, we have to solve the system associated to the first order conditions, i.e.

$$\text{(e3)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = 2 - \lambda/(2\sqrt{x}) = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = 3 - \lambda/(2\sqrt{y}) = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) = -\sqrt{x} - \sqrt{y} + 5 = 0 \end{cases}.$$

The unique triple (λ, x, y) solving (e3) is $(16, 9, 4)$, corresponding to $f(9,4) = 30$. However, if we compute $f(25,0) = 50$, we have $f(9,4) < f(25,0)$ and so the Lagrangean method does not return a maximum point. The reason for this, is that f is not partially differentiable (neither in x nor y) for $x = 0$ or $y = 0$.

Exercise 4. We find the local extrema in $\mathbb{R}^2$ of

$$f(x,y) = x + 2y$$
$$\text{subject to } x^2 + y^2 = 5.$$

To this end, we define $g : \mathbb{R}^2 \to \mathbb{R}$ by $(x,y) \mapsto g(x,y) = x^2 + y^2 - 5$ and, for $\lambda \in \mathbb{R}$, we define

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y) = x + 2y - \lambda(x^2 + y^2 - 5).$$

Following the Lagrange multiplier method, we solve the system associated to the first order conditions, i.e.

$$\text{(e4)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = 1 - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = 2 - 2\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) = -x^2 - y^2 + 5 = 0 \end{cases}.$$

Solving (e4), we find the two candidate local extrema $(1,2)$, with $\lambda = 1/2$, and $(-1,-2)$, with $\lambda = -1/2$. Define $D = \{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}$. The subset D being compact, f achieves both a maximum and a minimum on D. Moreover, since $\nabla g(x,y) = (-2x, -2y)$, both $(1,2)$ and $(-1,-2)$ will be local extrema of D. We calculate

(1) $f(1,2) = 5$ (maximum);
(2) $f(-1,-2) = -5$ (minimum).

Relying on Theorem 4.3.1, one could have looked at the functions

i. $\mathcal{L}^+(x,y) = x + 2y - (x^2 + y^2 - 5)/2$ (for $\lambda = 1/2$)
ii. $\mathcal{L}^-(x,y) = x + 2y + (x^2 + y^2 - 5)/2$ (for $\lambda = -1/2$)

which have Hessians respectively equal to

$$D^2\mathcal{L}^+(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad D^2\mathcal{L}^-(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $D^2\mathcal{L}^+(x,y)$ is negative definite and $D^2\mathcal{L}^-(x,y)$ is positive definite, Theorem 4.3.1 would give that $(1,2)$ is a global maximum and $(-1,-2)$ a global minimum.

Exercise 5. We will use the Lagrange multiplier method to solve the constrained optimization problem in $\mathbb{R}^3$

$$\begin{gathered} \max / \min\ f(x,y,z) = x+y+z \\ \text{subject to } x^2+y^2+z^2 = 12. \end{gathered}$$

We start by defining $g : \mathbb{R}^3 \to \mathbb{R}$ by $(x,y,z) \mapsto x^2+y^2+z^2-12$ and the associated set of zeroes $D = \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0\}$. For $\lambda \in \mathbb{R}$, we define moreover

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z) = x+y+z - \lambda(x^2+y^2+z^2-12)$$

and compute, for each $(x,y,z,\lambda) \in \mathbb{R}^4$

$$\text{(e5)} \qquad \begin{cases} \frac{\partial \mathcal{L}}{\partial x}(x,y,z,\lambda) = 1 - 2\lambda x = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,z,\lambda) = 1 - 2\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial y}(x,y,z,\lambda) = 1 - 2\lambda z = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x,y,z,\lambda) = -x^2-y^2-z^2+12 = 0 \end{cases}.$$

Moreover, $\nabla g(x,y,z) = (1,1,1)$ and so no point in D fails the constraint qualification. Solving (e5), we find points $(2,2,2)$, with $\lambda = 1/4$, and $(-2,-2,-2)$, with $\lambda = -1/4$. By looking at the functions

i. $\mathcal{L}^+(x,y,z) = x+y+z-(x^2+y^2-5)/4$ (for $\lambda = 1/4$)
ii. $\mathcal{L}^-(x,y,z) = x+y+z+(x^2+y^2-5)/4$ (for $\lambda = -1/4$)

and computing

$$D^2\mathcal{L}^+(x,y,z) = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \quad \text{and} \quad D^2\mathcal{L}^-(x,y,z) = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

we derive from Theorem 4.3.1 that $(2,2,2)$ is a global maximum and $(-2,-2,-2)$ is a global minimum.