OQE - PROBLEM SET 2 - SOLUTIONS

Exercise 1. We find infimum and supremum of the following sets

$X = \{\frac{n}{n+1}\}_{n\in\mathbb{N}}$;
$Y = \{a - b : a, b \in \mathbb{R}, 1 < a < 2, 3 < b < 4\}$.

We claim the following:

(a) $\inf X = \frac{1}{2}$ and $\sup X = 1$.
(b) $\inf Y = -3$ and $\sup Y = -1$.

(a) We observe that, if n, m are elements of $\mathbb{N}$, then $\frac{n}{n+1} \leq \frac{m}{m+1}$ if and only if $n \leq m$. It follows that, for each $n \in \mathbb{N}$, one has

$$\frac{1}{2} = \frac{1}{1+1} \leq \frac{n}{n+1}.$$

Since $\frac{1}{2}$ is an element of X, it follows that $\frac{1}{2} = \inf X$. We now prove $\sup X = 1$. For each $n \in \mathbb{N}$, one has $n < n+1$, and therefore $\sup X \leq 1$. Let now s be an upper bound of X and assume by contradiction that $s < 1$. Then the set X is contained in $[0, s]$. It follows that the sequence $(x_n)_n$, defined by $x_n = \frac{n}{n+1}$, is a sequence in $[0, s]$ which converges to 1 in $\mathbb{R}$. However, 1 does not belong to $[0, s]$, which is a contradiction to Theorem 1.3.8 from the notes.
(b) It is not difficult to show that $-3 \leq \inf Y \leq -2 \leq \sup Y \leq -1$. Let now $l = \sup Y$ and assume, by contradiction, that $l < -1$. Define $\delta = |-1-l|$, so that $0 < \delta \leq 1$. We define $a = 2 - \frac{\delta}{4}$ and $b = 3 + \frac{\delta}{4}$: it follows from their definitions that $1 < a < 2$ and $3 < b < 4$. However, one computes $a - b = -1 - \frac{\delta}{2} > -1 - \delta = l$, giving a contradiction to the minimality of l. We have proven thus that $\sup Y = -1$. To prove that $\inf Y = -3$, one uses a similar argument.

Exercise 2. We determine whether or not the following sequences

(a) $\bar{x} = (((-1)^n, 4, \frac{1}{n}))_{n\in\mathbb{N}}$
(b) $\bar{y} = ((\frac{n \sin n}{n^2+1}, \frac{(-1)^{n+1}}{n}))_{n\in\mathbb{N}}$

converge respectively in $\mathbb{R}^3$ and $\mathbb{R}^2$.
(a) We claim that $\bar{x}$ does not converge in $\mathbb{R}^3$. Assume by contradiction that $\bar{x}$ converges to a point $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$. Then, for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, for all $n > N_\epsilon$, one has

$$(x_1 - (-1)^n)^2 + (x_2 - 4)^2 + (x_3 - \frac{1}{n})^2 < \epsilon^2.$$

Fix $0 < \epsilon < \frac{1}{2}$ and let $n > N$ be odd. Then we have

$$(x_1 + 1)^2 + (x_2 - 4)^2 + (x_3 - \frac{1}{n})^2 = (x_1 - (-1)^n)^2 + (x_2 - 4)^2 + (x_3 - \frac{1}{n})^2 < \epsilon^2$$

and also

$$(x_1-1)^2+(x_2-4)^2+(x_3-\frac{1}{n})^2=(x_1-(-1)^{n+1})^2+(x_2-4)^2+(x_3-\frac{1}{n})^2<\epsilon^2.$$

However, one between x_1+1 and x_1-1 is larger than 1 and therefore $\epsilon>1$, which contradicts our choice of ϵ.

(b) We prove that $\bar{y}$ converges to $(0,0)$ in $\mathbb{R}^2$. We observe that, since sin is a function $\mathbb{R}\to[-1,1]$, one has, for each $n\in\mathbb{N}$, that

$$\frac{-n}{n^2+1}\leq\frac{n\sin n}{n^2+1}\leq\frac{n}{n^2+1}$$

and so, since both sequences $(\frac{-n}{n^2+1})_n$ and $(\frac{n}{n^2+1})_n$ tend to 0 as n goes to infinity, also $(\frac{n\sin n}{n^2+1})_n$ is convergent to 0. With a similar argument, one shows that $(\frac{(-1)^n}{n})_n$ is convergent to 0 and therefore $\bar{y}\to(0,0)$.

Exercise 3. We compute interior, closure, and boundary of $\mathbb{Q}$ in $\mathbb{R}$. We claim

- $\mathring{\mathbb{Q}}=\emptyset$.
- $\overline{\mathbb{Q}}=\partial\mathbb{Q}=\mathbb{R}$.

To prove that the interior of $\mathbb{Q}$ is empty, we work by contradiction. Assume that x is an element of $\mathring{\mathbb{Q}}$ and let $\epsilon>0$ be such that $\mathrm{B}_\epsilon(x)\subseteq\mathring{\mathbb{Q}}$. Since the sequence $(\frac{\sqrt{2}}{n})_{n>0}$ converges to 0, there exists $n\in\mathbb{N}$ such that $\frac{\sqrt{2}}{n}<\epsilon$. Fix such n. Then the element $x+\frac{\sqrt{2}}{n}$ belongs to $\mathrm{B}_\epsilon(x)\setminus\mathbb{Q}$. Contradiction. Hence $\mathring{\mathbb{Q}}=\emptyset$ and so $\overline{\mathbb{Q}}=\partial\mathbb{Q}$. Use a similar trick to prove that, for every element x of $\mathbb{R}$ and for every $\epsilon>0$, one has $\mathbb{Q}\cap\mathrm{B}_\epsilon(x)\neq\emptyset$.

Exercise 4. Let X be a non-empty set and let d be the discete metric on it. We show that the convergent sequences in (X,d) are exactly the stationary sequences, i.e. sequences $(x_n)_n$ such that there exists $N\in\mathbb{N}$ and $x\in X$ such that, for all $n>N$, one has $x_n=x$. Let indeed $(x_n)_n$ be a sequence in X. Then

$(x_n)_n$ converges to a point $x\in X$

$\Updownarrow$

for each $\epsilon>0$ there exists $N\in\mathbb{N}$ such that, for all $n>N$, one has $d(x_n,x)<\epsilon$

$\Updownarrow$

for each $1>\epsilon>0$ there exists $N\in\mathbb{N}$ such that, for all $n>N$, one has $d(x_n,x)<\epsilon<1$

$\Updownarrow$

for each $1>\epsilon>0$ there exists $N\in\mathbb{N}$ such that, for all $n>N$, one has $d(x_n,x)=0$

$\Updownarrow$

there exists $N\in\mathbb{N}$ such that, for all $n>N$, one has $x_n=x$.

Exercise 5. Let $C[0,1]$ be the collection of continuous maps $[0,1]\to\mathbb{R}$ and let $\|\cdot\|$ denote the max-norm on it, i.e. the norm associating to each $f\in C[0,1]$ the

element $\|f\| = \max_{t\in[0,1]} |f(t)|$ of $\mathbb{R}$.
(a) We first prove that, if $g \in C[0,1]$, then the set

$$A_g = \{f \in C[0,1] \mid \forall t \in [0,1] : f(t) < g(t)\}$$

is open in $C[0,1]$. Fix g and let $f \in A_g$. We prove that there exists $\epsilon > 0$ such that $\mathrm{B}_\epsilon(f)$ is contained in A_g. Define $\epsilon = \frac{1}{2}\min_{t\in[0,1]} |g(t) - f(t)|$. Since $f \in A_g$, the number ϵ is positive and so, for each $t \in [0,1]$, one has $f(t) + \epsilon < g(t)$. Let now $h \in \mathrm{B}_\epsilon(f)$. It follows that, for each $t \in [0,1]$, one has $h(t) < f(t) + \epsilon < g(t)$ and therefore $h \in A_g$. We have proven that $\mathrm{B}_\epsilon(f) \subseteq A_g$ and, the choice of f being arbitrary, A_g is open.
(b) Let f and g be respectively defined by $t \mapsto f(t) = 2t$ and $t \mapsto g(t) = 1 - t$. Then we compute

$$\|f - g\| = \max_{t\in[0,1]} |f(t) - g(t)| = \max_{t\in[0,1]} |3t - 1| = 2.$$

(c) We prove that the sequence $\bar{f} = (f_n(t))_n$, defined by $f_n(t) = t^n - t^{2n}$ is not convergent in $C[0,1]$. We will do so by contradiction. Assume that $\bar{f}$ has a limit f in $C[0,1]$. Then, for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that, for all $n > N_\epsilon$, one has $\|f - f_n\| < \epsilon$. Let now $\epsilon = \frac{1}{32}$ and choose $n > N_\epsilon$. Then one has

$$\|f_n - f_{2n}\| = \max_{t\in[0,1]} |(t^n - t^{2n}) - (t^{2n} - t^{4n})| = \max_{t\in[0,1]} |t^n - 2t^{2n} + t^{4n}|$$

and so, since $0 \leq \frac{1}{\sqrt[n]{2}} \leq 1$, we get

$$\|f_n - f_{2n}\| \geq \left|\Big(\frac{1}{\sqrt[n]{2}}\Big)^n - 2\Big(\frac{1}{\sqrt[n]{2}}\Big)^{2n} + \Big(\frac{1}{\sqrt[n]{2}}\Big)^{4n}\right| = \Big(\frac{1}{2} - 2\frac{1}{4} + \frac{1}{16}\Big) = \frac{1}{16}.$$

Then, as a consequence of the triangle inequality, one gets

$$\frac{1}{32} = \frac{1}{16} - \epsilon \leq \|f_{2n} - f_n\| - \|f - f_{2n}\| \leq \|f - f_n\| < \epsilon = \frac{1}{32}.$$

Contradiction.

Exercise 6. Let $(X, \|\cdot\|)$ be a normed space. Define, for all $x, y \in X$

(a) $\rho_1(x,y) = \min\{1, \|x - y\|\}$;
(b) $\rho_2(x,y) = \max\{1, \|x - y\|\}$.

(a) We claim that ρ_1 defines a metric on X, while ρ_2 does not. We start from ρ_1. Using the defining properties of a norm, one shows that

$$\rho_1(x,x) = \min\{1, \|x - x\|\} = \min\{1, \|0\|\} = 0$$

and also that

$$\rho_1(x,y) = \min\{1, \|x - y\|\} = \min\{1, \|y - x\|\} = \rho_1(y,x).$$

To prove the triangle inequality, one argues that, as a consequence of Lemma 1.1.7 from the notes, for all x, y, the following holds

$$\rho_1(x,z) = \min\{1, \|x - z\|\} \leq \min\{1, \|x - y\| + \|y - z\|\}$$

$$\leq \min\{1, \|x-y\|\} + \min\{1, \|y-z\|\} = \rho_1(x,y) + \rho_1(y,z).$$

We have proven that ρ_1 satisfies all requirements for being a metric on X and therefore so it is.

(b) To prove that ρ_2 is not a metric in general we fix $x \in X$. Then

$$\rho_2(x,x) = \max\{1, \|x-x\|\} = \max\{1, \|0\|\} = 1,$$

which contradicts the identity axiom for metrics.