# OQE - PROBLEM SET 7 - SOLUTIONS

**Exercise 1.** We show that the following functions $\mathbb{R}^2 \to \mathbb{R}$

(a) $f(x, y) = 2x + y$.
(b) $g(x, y) = x^2 + y^2$.

are partially differentiable with respect to both $x$ and $y$ everywhere in $\mathbb{R}^2$.
(a) We compute

$$\begin{aligned} D_1 f(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h) + y - (2x+y)}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} D_2 f(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{2x + y + h - (2x+y)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= 1. \end{aligned}$$

The choice of $(x, y)$ being arbitrary in $\mathbb{R}^2$, the function $f$ is partially differentiable.
(b) We compute

$$\begin{aligned} D_1 g(x,y) &= \lim_{h\to 0} \frac{g(x+h,y) - g(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + y^2 - (x^2+y^2)}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} 2x + h = 2x \end{aligned}$$

and

$$\begin{aligned} D_1 g(x,y) &= \lim_{h\to 0} \frac{g(x,y+h)-g(x,y)}{h} \\ &= \lim_{h\to 0} \frac{x^2+(y+h)^2-(x^2+y^2)}{h} \\ &= \lim_{h\to 0} \frac{2hy+h^2}{h} \\ &= \lim_{h\to 0} 2y+h = 2y. \end{aligned}$$

The choice of $(x,y)$ being arbitrary, $g$ is everywhere partially differentiable in $\mathbb{R}^2$.

**Exercise 2.** We compute the gradient of the following functions $\mathbb{R}^3 \to \mathbb{R}$

(a) $f(x,y,z) = x^2 + ze^{2y}$.
(b) $g(x,y) = e^{xyz}$.

According to Definition 2.1.4, the gradient of a function is the vector of all partial derivatives. We get

(a) $\nabla f(x,y,z) = (2x, 2ze^{2y}, e^{2y})$.
(b) $\nabla g(x,y,z) = (yze^{xyz}, xze^{xyz}, xye^{xyz})$.

**Exercise 3.** We calculate the directional derivative of the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $(x,y) \mapsto f(x,y) = \sin(xy)$, along $v = (1/2, \sqrt{3}/2)$ and at $(1,0)$. We get

$$\begin{aligned} \partial_v f(1,0) &= \lim_{h\to 0} \frac{f(1+h/2, 0+(\sqrt{3}h)/2) - f(1,0)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\Big( \sin\Big(\frac{(2+h)(\sqrt{3}h)}{4}\Big) - \sin(0)\Big) \\ &= \lim_{h\to 0} \frac{(2+h)(\sqrt{3}h)}{4h} \\ &= \lim_{h\to 0} \frac{\sqrt{3}(2+h)}{4} = \frac{\sqrt{3}}{2}. \end{aligned}$$

Alternatively, one can use the relationship between $\nabla f$ and $D_v f$.

**Exercise 4.** We calculate the derivative of $g : \mathbb{R}^3 \to \mathbb{R}$, defined by $(x,y,z) \mapsto g(x,y,z) = e^{xyz}$ along $v = (1/\sqrt{6}, \sqrt{2/3}, -1/\sqrt{6})$ and at $(1,1,1)$. We know from Exercise 2$b$ that $\nabla g(x,y,x) = (yze^{xyz}, xze^{xyz}, xye^{xyz})$ and so we compute

$$\begin{aligned} D_v g(1,1,1) &= \langle \nabla g(1,1,1), (1/\sqrt{6}, \sqrt{2/3}, -1/\sqrt{6})\rangle \\ &= \langle (e,e,e), (1/\sqrt{6}, \sqrt{2/3}, -1/\sqrt{6})\rangle \\ &= \frac{e}{\sqrt{6}} + \frac{\sqrt{2}e}{\sqrt{3}} - \frac{e}{\sqrt{6}} \\ &= \sqrt{\frac{2}{3}}e. \end{aligned}$$

**Exercise 5.** We compute the directional derivative of $f : \mathbb{R}^3 \to \mathbb{R}$, defined by $(x, y, z) \mapsto f(x, y, z) = x^3 + 2ze^{3y}$ along $v = (1/2, -1/2, 1/\sqrt{2})$ and at $(-1, 0, 1)$. We first compute, for each $(x, y, z) \in \mathbb{R}^3$, the gradient

$$\nabla f(x, y, z) = (3x^2, 6ze^{3y}, 2e^{3y})$$

and so, as a consequence, we get

$$\begin{aligned} D_v f(-1, 0, 1) &= \langle \nabla f(-1, 0, 1), (1/2, -1/2, 1/\sqrt{2}) \rangle \\ &= \langle (3, 6, 2), (1/2, -1/2, 1/\sqrt{2}) \rangle \\ &= \frac{3}{2} - 3 + \sqrt{2} \\ &= \frac{-3 + 2\sqrt{2}}{2}. \end{aligned}$$

**Exercise 6.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{y^3}{(x^2+y^2)^{1/2}} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We claim that $f$ is totally differentiable at $(0, 0)$. To prove so, we will rely on Theorem 2.4.5 and prove that all partial derivatives are continuous at $(0, 0)$. For $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the gradient of $f$ is equal to

$$\nabla f(x, y) = \Big( \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \Big) = \Big( \frac{-y^3 x}{(x^2 + y^2)^{3/2}}, \frac{3x^2y^2 + 2y^4}{(x^2 + y^2)^{3/2}} \Big)$$

while $\nabla f(0, 0) = (0, 0)$. To prove continuity of the partial derivatives, we will show that $\lim_{(x,y)\to(0,0)} \nabla f(x, y) = (0, 0)$. We first show that $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = 0$ and, to do so, we will show that $\lim_{(x,y)\to(0,0)} \Big| \frac{-y^3 x}{(x^2+y^2)^{3/2}} \Big| = 0$. We remark that, for each $(x, y) \in \mathbb{R}^2$, we have $(x - y)^2 \geq 0$ and also $(x + y)^2 \geq 0$: it follows from those that $|xy| \leq \frac{x^2+y^2}{2} \leq x^2 + y^2$. We then have

$$\Big| \frac{-y^3 x}{(x^2 + y^2)^{3/2}} \Big| = \frac{y^2 |yx|}{(x^2 + y^2)^{3/2}} \leq \frac{y^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}} \leq \frac{(x^2 + y^2)^2}{(x^2 + y^2)^{3/2}} = (x^2 + y^2)^{1/2}$$

and, since $\lim_{(x,y)\to(0,0)} x^2 + y^2 = 0$, we have that $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = 0$. We now move to $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x, y)$, for which we adopt the same technique. We have indeed that

$$\begin{aligned} \frac{3x^2y^2 + 2y^4}{(x^2 + y^2)^{3/2}} &= \frac{3y^2}{(x^2 + y^2)^{3/2}} \Big( x^2 + \frac{2}{3} y^2 \Big) \leq \frac{3y^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}} \\ &\leq \frac{3(x^2 + y^2)^2}{(x^2 + y^2)^{3/2}} = 3(x^2 + y^2)^{1/2}. \end{aligned}$$

from which it follows that $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x, y) = 0$. As claimed, we have proven that $\lim_{(x,y)\to(0,0)} \nabla f(x, y) = (0, 0)$ and so the function $f$ is differentiable at $(0, 0)$.