OQE - PROBLEM SET 9 - SOLUTIONS

Exercise 1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x, y, z) \mapsto F(x, y, z) = x^2 - y^2 + z^3$.
(a) We want to determine the triples $(6, 3, z)$ for which $F(6, 3, z) = 0$. We impose

$$0 = F(6, 3, z) = 36 - 9 + z^3 = 27 + z^3$$

and therefore $F(6, 3, z) = 0$ if and only if $z^3 = -27$. It follows that the only element $(6, 3, z)$ of $\mathbb{R}^3$ satisfying $F(6, 3, z) = 0$ is $(6, 3, -3)$.
(b) We claim that F induces an implicit function in the indeterminate z. Indeed, the function $g : \mathbb{R} \to \mathbb{R}$, defined by $z \mapsto z^3$, is a bijection and thus, for any choice of $(x, y) \in \mathbb{R}^2$, there exists a unique $z \in \mathbb{R}$ such that $z = g^{-1}(-x^2 + y^2)$. In other words, for each $(x, y) \in \mathbb{R}$, there exists a unique z such that $F(x, y, z) = 0$, namely $z = g^{-1}(-x^2 + y^2)$.
(c) We compute partial derivatives of $z = z(x, y)$ at the point $(6, 3)$. We recall that

$$z = z(x, y) = (-x^2 + y^2)^{1/3}$$

and thus we have

$$\begin{aligned}\frac{\partial z}{\partial x}(6,3) &= \Big(\frac{1}{3}(-x^2+y^2)^{-2/3}(-2x)\Big)(6,3)\\ &= \Big(-\frac{2x}{3(-x^2+y^2)^{2/3}}\Big)(6,3)\\ &= -\frac{12}{3(-36+9)^{2/3}}\\ &= -\frac{4}{(-27)^{2/3}}\\ &= -\frac{4}{9}\end{aligned}$$

and also

$$\begin{aligned}\frac{\partial z}{\partial y}(6,3) &= \Big(\frac{1}{3}(-x^2+y^2)^{-2/3}(2y)\Big)(6,3)\\ &= \Big(\frac{2y}{3(-x^2+y^2)^{2/3}}\Big)(6,3)\\ &= \frac{6}{3(-36+9)^{2/3}}\\ &= \frac{2}{(-27)^{2/3}}\\ &= \frac{2}{9}.\end{aligned}$$

Exercise 2. Let $F : \mathbb{R}^4 \to \mathbb{R}$ be defined, for each $(x, y, z) \in \mathbb{R}^3$, by

$$F(x, y, z) = x^4 + 2x \cos y + \sin z.$$

We claim that $F(x, y, z) = 0$ defines z as an implicit function of x, y in a neighbourhood of $x = y = z = 0$. Indeed, the gradient of F is equal to

$$\nabla F = (4x^3 + 2 \cos y, -2x \sin y, \cos z)$$

and, since $\frac{\partial F}{\partial z}(0, 0, 0) = 1 \neq 0$, Theorem 2.7.4 yields the existence of open neighbourhoods U in $\mathbb{R}^2$ and V in $\mathbb{R}$, respectively of $(0, 0)$ and 0, and of a function $g : U \to V$ of x and y such that, for all $(x, y, z) \in U \times V$, if $F(x, y, z) = 0$, then $z = g(x, y)$. From Theorem 2.7.4 we know in addition that

$$\nabla g(0, 0) = -\frac{\partial F}{\partial z}(0, 0, 0)^{-1}\Big(\frac{\partial F}{\partial x}(0, 0, 0), \frac{\partial F}{\partial y}(0, 0, 0)\Big) = (-2, 0).$$

Exercise 3. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$(x, y, z) \mapsto F(x, y, z) = x^3 + 3y^2 + 4xz^2 - 3z^2y - 1.$$

We want to determine, for given $(x_0, y_0) \in \mathbb{R}^2$, if the equation $F(x, y, x) = 0$ defines z as an implicit function of x, y in a neighbourhood of (x_0, y_0). We look at the equation $0 = F(x_0, y_0, z) = x_0^3 + 3y_0^2 + 4x_0z^2 - 3z^2y_0 - 1$, which can be rewritten as

$$(4x_0 - 3y_0)z^2 = 1 - x_0^3 - 3y_0^2.$$

As a consequence, a necessary condition for (x_0, y_0, z_0) to be a zero of F is that

$$(4x_0 - 3y_0)(1 - x_0^3 - 3y_0^2) \geq 0.$$

We look at the following cases:

(a) $(x_0, y_0) = (1, 1)$;
(b) $(x_0, y_0) = (1, 0)$;
(c) $(x_0, y_0) = (1/2, 0)$.

(a) In this case $(4x_0 - 3y_0)(1 - x_0^3 - 3y_0^2) = -3 < 0$, so there is no $z_0 \in \mathbb{R}$ such that $F(x_0, y_0, z_0) = 0$. In particular, z is not an implicit function of x and y.
(b) In this case, the unique element $z_0 \in \mathbb{R}$ such that $F(x_0, y_0, z_0) = 0$ is $z_0 = 0$. We claim that there is however no triple (U, V, g), with U an open neighbourhood of (x_0, y_0) in $\mathbb{R}^2$, with V an open neighbourhood of z_0 in $\mathbb{R}$, and $g : U \to V$ a function satisfying

$$(x, y, z) \in U \times V \quad \text{with} \quad F(x, y, z) = 0 \Longrightarrow z = g(x, y).$$

Assume by contradiction that such a triple (U, V, g) exists. Then, for each $(x, y) \in U$ and $z \in V$ satisfying $F(x, y, z) = 0$, one has

$$z^2 = g(x, y)^2 = \frac{1 - x^3 - 3y^2}{4x - 3y}.$$

However, if $(x,y) \in U$ is such that $\frac{1-x^3-3y^2}{4x-3y} \neq 0$, the last equation gives rise to $z_1, z_2 \in \mathbb{R} \setminus \{0\}$, with $z_1 = -z_2$, such that $0 = F(x,y,z_1) = F(x,y,z_2)$. Contradiction.

(c) Set $z_0 = \sqrt{7}/4$. Then we have

$$F(x_0, y_0, z_0) = \frac{1}{8} + \frac{7}{8} - 1 = 0.$$

In view of Theorem 2.7.4, we compute

$$\nabla F(x,y,z) = (3x^2 + 4z^2, 6y - 3z^2, 8xz - 6zy)$$

and therefore, since $\frac{\partial F}{\partial z}(1/2, 0, \sqrt{7}/4) = \sqrt{7} \neq 0$, there exist open neighbourhoods U and V, respectively of (x_0, y_0) and z_0, in $\mathbb{R}^2$ and $\mathbb{R}$ and a continuously differentiable function $g : U \to V$ such that, if $(x,y,z) \in U \times V$ is such that $F(x,y,z) = 0$, then $z = g(x,y)$. To compute $\nabla g(1/2, 0)$ we rely on Theorem 2.7.4 and compute

$$\begin{aligned}\nabla g(1/2,0) &= -\frac{\partial F}{\partial z}(1/2,0,\sqrt{7}/4)^{-1}\Big(\frac{\partial F}{\partial x}(1/2,0,\sqrt{7}/4), \frac{\partial F}{\partial y}(1/2,0,\sqrt{7}/4\Big) \\ &= -\sqrt{7}\Big(\frac{10}{4}, -\frac{21}{16}\Big) \\ &= \Big(-\frac{10\sqrt{7}}{4}, \frac{21\sqrt{7}}{16}\Big).\end{aligned}$$

Exercise 4. Let $x, y, u, v \in \mathbb{R}$ and consider the system

$$\begin{cases} u + xe^y + v = e - 1 \\ x + e^{u+v^2} - y = e^{-1} \end{cases}.$$

We prove that the given system defines u and v in terms of x, y around the point $(1, 1, -1, 0)$. To do so, we define additional functions $F_1, F_2 : \mathbb{R}^4 \to \mathbb{R}$ by means of

$$(x,y,u,v) \mapsto F_1(x,y,u,v) = u + xe^y + v - e + 1$$

and also

$$(x,y,u,v) \mapsto F_2(x,y,u,v) = x + e^{u+v^2} - y - e^{-1}.$$

We then write $F = (F_1, F_2)$. As we want to apply Theorem 2.7.4, we compute

$$DF(x,y,u,v) = \begin{bmatrix} e^y & xe^y & 1 & 1 \\ 1 & -1 & e^{u+v^2} & 2ve^{u+v^2} \end{bmatrix}$$

and so, if we restrict to partial derivatives with respect to u and v, we get

$$\det \frac{\partial F}{\partial(u,v)}(1,1,-1,0) = \det \begin{bmatrix} 1 & 1 \\ e^{-1} & 0 \end{bmatrix} = -e^{-1} \neq 0.$$

Thanks to Theorem 2.7.4 there exist U and V open neighbourhoods in $\mathbb{R}^2$, respectively of $(1,1)$ and $(-1,0)$, and a function $g : U \to V$ with the property that

$$(x,y,u,v) \in U \times V \ \text{ with } \ F(x,y,u,v) = 0 \Longrightarrow (u,v) = g(x,y).$$

Moreover, thanks to Theorem 2.7.4, we can also compute

$$\begin{aligned}\frac{\partial g}{\partial(x,y)}(1,1) &= -\frac{\partial F}{\partial(u,v)}(1,1,-1,0)^{-1}\frac{\partial F}{\partial(x,y)}(1,1,-1,0)\\ &= -\begin{bmatrix}1 & 1\\ e^{-1} & 0\end{bmatrix}^{-1}\begin{bmatrix}e & e\\ 1 & -1\end{bmatrix}\\ &= e\begin{bmatrix}0 & -1\\ -e^{-1} & 1\end{bmatrix}\begin{bmatrix}e & e\\ 1 & -1\end{bmatrix}\\ &= e\begin{bmatrix}-1 & 1\\ 0 & -2\end{bmatrix}\end{aligned}$$

Exercise 5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x, y) \mapsto f(x, y) = (x + e^y, y + e^{-x})$. We claim that f is everywhere locally invertible. In view of Theorem 2.8.1, it suffices to show that, for any choice of $(x, y) \in \mathbb{R}^2$ the determinant of the matrix

$$Df(x,y) = \begin{bmatrix}1 & e^y\\ -e^{-x} & 1\end{bmatrix}$$

is different from 0. For each $x, y \in \mathbb{R}$, we compute $\det Df(x, y) = 1 + e^{-x}e^y$ and so, since the image of the exponential is contained in $\mathbb{R}_{>0}$, we have $\det Df(x, y) > 0$. This proves the claim. Let now U and V be open neighbourhoods in $\mathbb{R}^2$, respectively of $(1, -1)$ and $f(1, -1) = (1 + e^{-1}, -1 + e^{-1})$, and let $g : V \to U$ be such that $g \circ f_{|U} = \mathrm{id}_U$. Thanks to Theorem 2.8.1, we can compute

$$\begin{aligned}Dg(1+e^{-1}, -1+e^{-1}) &= Df(1,-1)^{-1}\\ &= \begin{bmatrix}1 & e^{-1}\\ -e^{-1} & 1\end{bmatrix}^{-1}\\ &= \frac{1}{1+e^{-2}}\begin{bmatrix}1 & -e^{-1}\\ e^{-1} & 1\end{bmatrix}.\end{aligned}$$

Exercise 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x, y) \mapsto f(x, y) = (e^x \cos y, e^x \sin y)$. We claim that f is locally invertible everywhere, but it is not globally invertible. We start by showing that f is not globally invertible. A necessary requirement for f to be invertible is that f is injective: this is not the case as, for example $f(1, 0) = f(1, 2\pi)$. We now show that f is locally invertible everywhere. To do so, for each $(x, y) \in \mathbb{R}^2$, we compute

$$\det Df(x,y) = \det\begin{bmatrix}e^x\cos y & -e^x\sin y\\ e^x\sin y & e^x\cos y\end{bmatrix} = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} > 0$$

and so, thanks to Theorem 2.8.1, the function f is locally invertible everywhere. In particular, there exist open neighbourhoods U and V in $\mathbb{R}^2$, respectively of $(0, 0)$

and $f(0,0) = (1,0)$, and $g : V \to U$ be such that $g \circ f_{|U} = \mathrm{id}_U$. Fix such a triple (U, V, g). Theorem 2.8.1 yields

$$Dg(1,0) = Df(0,0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$