Random Substitutions

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Easy dynamics

A topological dynamical system is a topological space X together with a self-action $\sigma\colon X \to X$.

Isomorphism: $(X_1, \sigma_1) \cong (X_2, \sigma_2)$ if there is $h\colon X_1 \to X_2$,

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma_1} & X_1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X_2 & \xrightarrow[\sigma_2]{} & X_2 \end{array}$$

Simple examples

$X = \{*\}, \sigma = \mathrm{id}_X$ too simple!

$$X = \{1, 2\}, \quad \sigma_1 = \mathrm{id}_X, \quad \sigma_2 \colon 1 \leftrightarrow 2$$

not the same because they have a different number of orbits

Subshifts

$\mathcal{A} = \{0, 1\}$ – Alphabet. $\mathcal{A}^{\mathbb{Z}}$ – full (two-sided) shift.
Metric d on $\mathcal{A}^{\mathbb{Z}}$:

$$d(\cdots x_{-1}.x_0x_1\cdots, \cdots y_{-1}.y_0y_1\cdots) = \inf\left\{1/2^{|i|} \mid x_i = y_i\right\} \cup \{1\}$$

"The more two sequences agree near their centers, the closer they are."

$$\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}} \qquad \text{shift map}$$

A (two-sided) *subshift* X is a closed, shift invariant subspace of the full shift $\mathcal{A}^{\mathbb{Z}}$.

Subshifts

Generally define X in one of a few ways:

- **Language** – $X_{\mathcal{L}} = \{w \in \mathcal{A}^{\mathbb{Z}} \mid u \triangleleft w \implies u \in \mathcal{L}\}$
  Example: $\mathcal{L} = \{0, 01, 010, 0101, \ldots\}$
- **Orbit closure** – $X_x = \overline{\{\sigma^n(x) \mid x \in \mathcal{A}^{\mathbb{Z}}, n \in \mathbb{Z}\}}$
- $\cdots$

Examples

$$(X_{\dots 000.000 \dots}, \sigma) \cong (\{*\}, \mathrm{id})$$

$$(X_{\dots 0101.0101 \dots}, \sigma) \cong (\{1, 2\}, 1 \leftrightarrow 2)$$

$$(X_{\dots 0000.1111 \dots}, \sigma) \cong \text{homework}$$

Invariants

- Cardinality and topology
- Number of closed invariant subspaces
- Number of n-periodic points (and the related zeta function)
- Topological entropy:

$$h_{\text{top}} := \lim_{n \to \infty} \frac{1}{n} \log \#\{\text{legal subwords of length } n\}$$

- "Dynamical spectrum", "Čech cohomology", "Ellis semigroup",...

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Substitutions

Substitutions (deterministic)

$\phi\colon \mathcal{A} \to \mathcal{A}^+$ – assign a word to the letters of $\mathcal{A}$.
Example: [*Period doubling substitution*]

$$\phi\colon 0 \mapsto 01, 1 \mapsto 00, \qquad M_\phi = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

Iterate

$$0 \mapsto 01 \mapsto 0100 \mapsto 01000101 \mapsto 0100010101000100 \mapsto \cdots$$

$$x = \cdots 01000101010001000100010101000101 \cdots$$

$X_\phi = X_x$ or
$X_\phi = X_\mathcal{L}$ where $\mathcal{L} = \{u \mid u \triangleleft \phi^n(0), n \geq 1\}$.

Primitivity

Definition

A substitution ϕ is *primitive* if there exists a $k \geq 1$ such that for all $i, j \in \mathcal{A}$ the letter j appears in the word $\phi^k(j)$.

Equivalently, if there exists a $k \geq 1$ such that the matrix M_ϕ^k has positive entries.

Proposition

Let ϕ be a primitive substitution. Then:

- *X_ϕ is non-empty*
- *Either every element of X_ϕ is periodic or every element is non-periodic*
- *X_ϕ is either finite or a Cantor set*
- *Every element of (X_ϕ, σ) has a dense orbit*
- $h_{top}(X_\phi, \sigma) = 0$

It would be nice to have a system which shares the nice hierarchical properties of substitutions, giving long range correlations, but also *positive entropy*!

Random substitutions

$\vartheta\colon \mathcal{A} \to \mathcal{P}(\mathcal{A}^+)$ – assign a set of words to the letters of $\mathcal{A}$.
For every $i \in \mathcal{A}$, $\boldsymbol{P}_i\colon \vartheta(i) \to [0,1]$ such that $\sum_{u\in\vartheta(i)} \boldsymbol{P}_i(u) = 1$.
Example: [*Random period doubling substitution*]

$$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$$

$$\boldsymbol{P}_0(01) = p, \quad \boldsymbol{P}_0(10) = 1-p, \quad \boldsymbol{P}_1(00) = 1$$

$$M_\vartheta = \begin{bmatrix} p + (1-p) & 2 \\ p + (1-p) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

When the matrix is independent of the probabilities we call ϑ *compatible*.

$$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$$

$$0 \mapsto \{01, 10\} \mapsto \{\overbrace{0100, 1000}^{01}, \overbrace{0001, 0010}^{10}\}$$

$$\mapsto \left\{ \begin{array}{l} \overbrace{01000101, 01000110, 01001001, 10000101}^{0100}, \\ 01001010, 10000110, 10001001, 10001010, \\ \overbrace{00010101, 00010110, 00011001, 00100101}^{1000}, \\ 00011010, 00100110, 00101001, 00101010, \\ \qquad\qquad \overbrace{\cdots}^{0001}, \overbrace{\cdots}^{0010} \end{array} \right\}$$

$$\mapsto \cdots$$

Long range correlations

Long range order is measured as non-trivial Bragg peaks in the pure point component of the diffraction measure of an associated 1-dimensional point-set.

Theorem (Baake, Spindeler, Strungaru, '18)

For the random period doubling substitution, the associated diffraction measure $\hat{\gamma}$ is given by

$$\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} = \sum_{k \in \mathbb{Z}[\frac{1}{2}]} I_p(k)\delta_k + \phi_p \lambda.$$

Where I_p and ϕ_p have explicit formulae.
Further, $\phi_p = 0$ if and only if $p = 0$ or $p = 1$. (This agrees with the deterministic setting)

The absolutely continuous component is measuring a kind of stochasticity (though it can be non-trivial even in the deterministic setting!).

Notation: If $u \triangleleft v \in \vartheta^k(i)$ for some $k \geq 1$, we write $u \blacktriangleleft \vartheta^k(i)$.
Example: [Random period doubling] $000100110 \in \vartheta^3(0)$ so $010011 \blacktriangleleft \vartheta^3(0)$.

Definition

A random substitution ϑ is *primitive* if $\exists k \geq 1$ such that for all $i, j \in \mathcal{A}$, $i \blacktriangleleft \vartheta^k(j)$.

If ϑ is compatible, then ϑ is primitive if and only if M_ϑ is primitive.

Language of ϑ is given by

$$\mathcal{L}_\vartheta = \{u \in \mathcal{A}^+ \mid u \blacktriangleleft \vartheta^k(i) \text{ for some } k \geq 1, i \in \mathcal{A}\}.$$

Subshift given by $X_\vartheta = X_{\mathcal{L}_\vartheta}$.

Example

$$\vartheta\colon 0 \mapsto \{00, 01, 10, 11\}, 1 \mapsto \{00, 01, 10, 11\}$$

Question:

What is X_ϑ?

Answer:

$$X_\vartheta = \mathcal{A}^\mathbb{Z}$$

Example

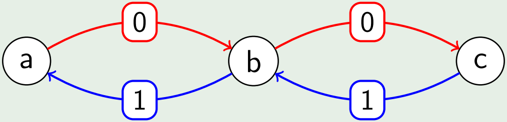
$$\vartheta\colon 0 \mapsto \{010, 0\}, 1 \mapsto \{01, 1\}$$

$$X_\vartheta = X_{\mathcal{F}=\{11\}} \quad \text{(Golden mean shift)}$$

Example

$$\vartheta\colon 0 \mapsto \{01, 10\}, b \mapsto \{01, 10\}$$

$$X_\vartheta = X_{\mathcal{G}} \quad \text{(Sofic shift)}$$



Results

Theorem (R., Spindeler, '18)

Let ϑ be primitive.
X_ϑ *is empty if and only if for all* $i \in \mathcal{A}$ *and* $u \in \vartheta(i)$, $|u| = 1$.
X_ϑ *is either finite or a Cantor set.*
(X_ϑ, σ) *is topologically transitive (there exists an element with dense orbit).*
Equivalently, there exists an element $x \in X_\vartheta$,

$$X_\vartheta = X_x = \overline{\{\sigma^n(x) \mid n \in \mathbb{Z}\}}.$$

Either $\mathrm{Per}(X_\vartheta) = \emptyset$ *or* $\mathrm{Per}(X_\vartheta) \overset{\text{dense}}{\subseteq} X_\vartheta$.

Periodic points

Example

The random period doubling subshift has periodic points.

$$001 \mapsto \overbrace{10}^{0}\overbrace{01}^{0}\overbrace{00}^{1} \mapsto 001001001001$$
$$\mapsto 100100100100100100100100 \mapsto \cdots$$

Definition

We say that ϑ has *disjoint images* if $\vartheta(i) \cap \vartheta(j) = \varnothing$ for all $i \neq j$.

The random period doubling substitution $0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$ has disjoint images.

Periodic Points

Franz Gähler computed some of the number of periodic points for the random period doubling susbtitution:

p	$\lvert \mathrm{Per}_p \rvert$	$\lvert \mathrm{Orb}_p \rvert$	p	$\lvert \mathrm{Per}_p \rvert$	$\lvert \mathrm{Orb}_p \rvert$
3	3	1	24	176,391	7,334
6	15	2	27	1,533	56
9	21	2	30	216,030	7,179
12	375	30	33	10,992	333
15	108	7	36	19,375,935	538,143
18	2,427	133	39	24,612	631
21	402	19	42	13,106,514	312,050

There are a few patterns but the data remains mostly mysterious.

Periodic Points

Question: What criteria ensure or exclude the existence of periodic points?

Theorem (R., '18)

If ϑ is primitive and $\mathrm{Per}(X_\vartheta) \neq \varnothing$ then the leading eigenvalue of M_ϑ is an integer.
Suppose ϑ is compatible with constant length ℓ and has disjoint images. If

$$\mathcal{L}_\ell \setminus \vartheta(\mathcal{A}) \neq \varnothing,$$

then $\mathrm{Per}(X_\vartheta) = \varnothing$.

Example

$\vartheta \colon 0 \mapsto \{00110, 01010\}, 1 \mapsto \{00000\}$, $M_\vartheta = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$, $\lambda = 5$.
$v = 00001 \in \mathcal{L}_5 \setminus \vartheta(\mathcal{A})$, so $\mathrm{Per}(X_\vartheta) = \varnothing$.

Entropy

Theorem (R., Spindeler, '18)

Let ϑ be primitive. Under mild conditions

$$h_{top}(X_\vartheta) > 0.$$

Theorem (Gohlke, '18)

Let ϑ be primitive and compatible. If $\mathrm{Per}(X_\vartheta) \neq \varnothing$, *then*

$$\limsup_{p\to\infty} \frac{1}{p} \log(\# \mathrm{Per}_p(X_\vartheta)) = h_{\mathrm{top}}.$$

Entropy

Theorem (Gohlke, '18)

Let ϑ be primitive and compatible. If ϑ has disjoint images, then

$$h_{\text{top}} = \frac{1}{\lambda - 1} \left\langle \begin{bmatrix} \log \#\vartheta(0) \\ \log \#\vartheta(1) \end{bmatrix}, \mathbf{r}_\lambda \right\rangle.$$

If $\vartheta(0) = \vartheta(1)$, then

$$h_{\text{top}} = \frac{1}{\lambda} \left\langle \begin{bmatrix} \log \#\vartheta(0) \\ \log \#\vartheta(1) \end{bmatrix}, \mathbf{r}_\lambda \right\rangle.$$

Otherwise, h_{top} is somewhere inbetween.

This result can be applied iteratively to ϑ^k to get better approximations in the intermediate cases (and in many cases calculate exact values as limits). **This is a big deal!**

Entropy

Philipp's result turns the following results into 1 page calculations.

Theorem (Godrèche, Luck, '89. Nilsson, '10)

$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{0\}$ *(random Fibonacci).*

$$h_{\text{top}} = \sum_{i=2}^{\infty} \frac{\log i}{\tau^{i+2}} \approx 0.444399.$$

Theorem (Baake, Spindeler, Strungaru, '18)

$\vartheta\colon 0 \mapsto \{01, 10\}, 1 \mapsto \{00\}$ *(random period doubling).* $h_{\text{top}} = \frac{2}{3} \log 2.$

Theorem (Nilsson, '13)

$\vartheta\colon 0 \mapsto \{100\}, 1 \mapsto \{01, 10\}$ *(random Fib^2).* $h_{\text{top}} = \frac{1}{\tau^3} \log 2.$