Ordered cohomology and co-dimension one cut-and-project sets

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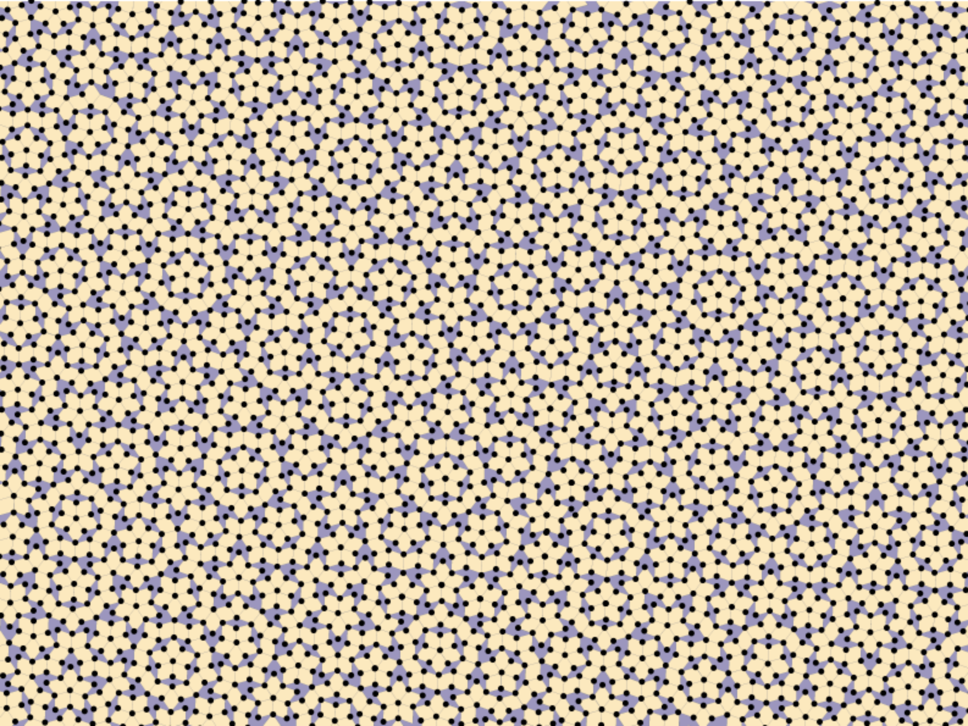
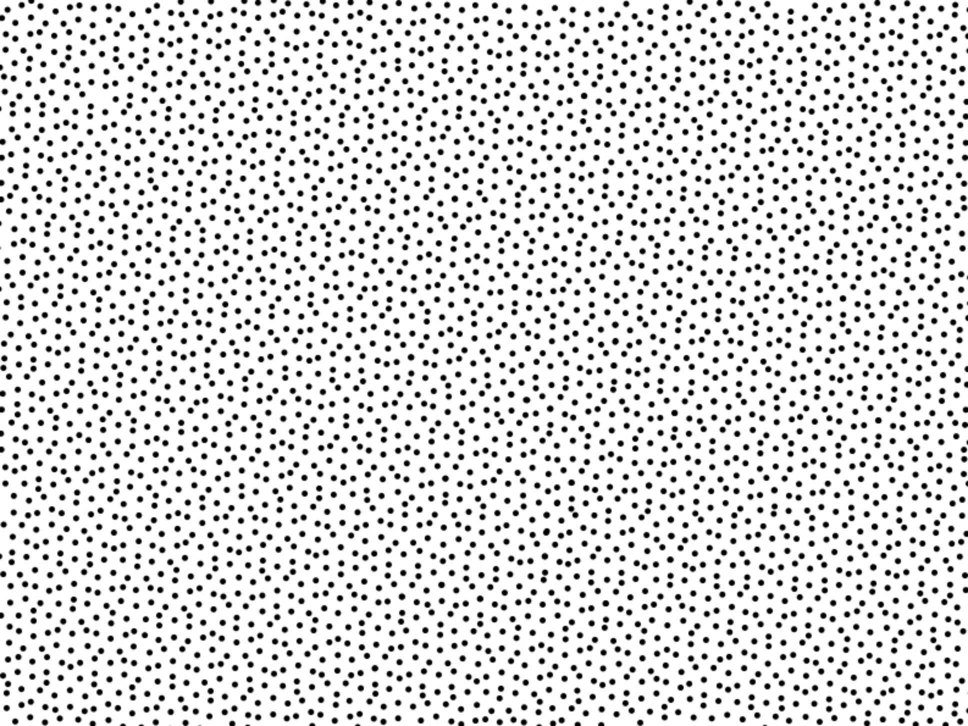
Quasicrystals

- Dan Shechtman won the 2011 Nobel Prize in Chemistry
- Not a periodic lattice – Has no global translational symmetries (may have rotational symmetry)
- Does have 'long range order' and 'local symmetries' on arbitrarily large scales
- Quotient of Euclidean space by a lattice gives a torus.
- Observation: $H^1(\mathbb{R}^n/\Gamma) = \Gamma$
- We want to go the other way
- Form the moduli space which parametrises 'locally isomorphic' quasicrystals and then use cohomology to measure the 'symmetry' of the space



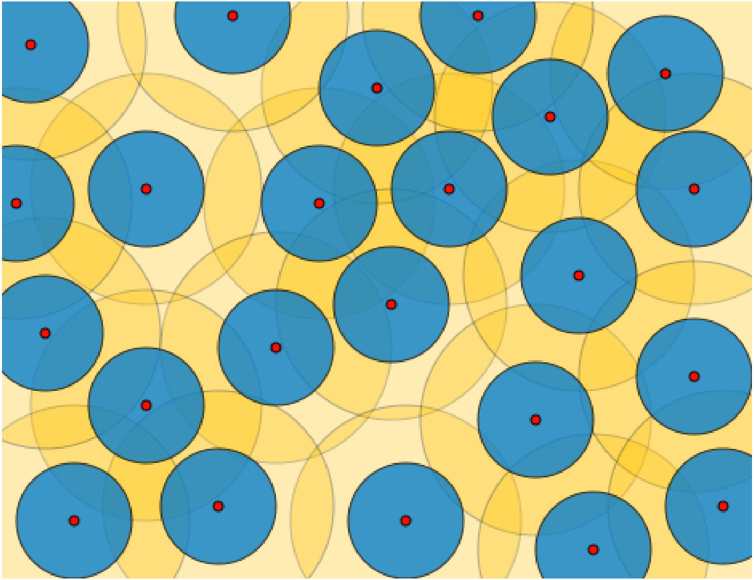


Set of points - $\mathcal{T} \subset \mathbb{R}^n$.

Definition

Uniformly discrete - There exists an $r > 0$ such that for all $x \in \mathcal{T}$, $B_r(x) = \{x\}$.
Relatively dense - There exists an $R > 0$ such that for all $x \in \mathbb{R}^n$, $B_R(x) \cap \mathcal{T} \neq \emptyset$.
Delone set = uniformly discrete + relatively dense.

Roughly - points in a Delone set never get too far apart or too close together.



Delone set - $\mathcal{T} \subset \mathbb{R}^n$

Definition

Finite Local Complexity (FLC) - For every $R > 0$ there are only finitely many kinds of 'patches' appearing in balls of radius R.
Repetitive - For every $r > 0$, there exists an $R > 0$ such that every patch of radius R contains every patch of radius r as a sub-patch.

We only care about repetitive, FLC Delone sets.
Pattern metric - Delone sets $\mathcal{T}_1$ and $\mathcal{T}_2$ are close if there exists a small $\epsilon > 0$ and vectors $|v_1|, |v_2| \leq \epsilon$ such that

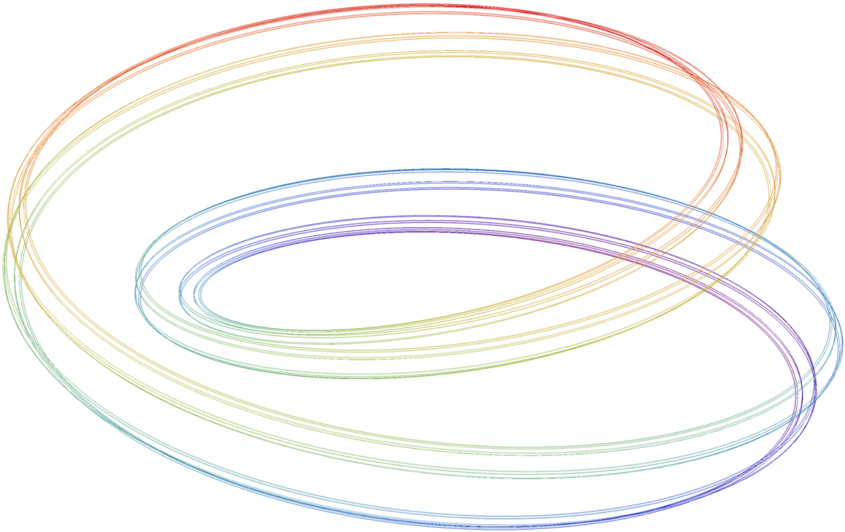
$$B_{\frac{1}{\epsilon}}(0) \cap (\mathcal{T}_1 + v_1) = B_{\frac{1}{\epsilon}}(0) \cap (\mathcal{T}_2 + v_2)$$

"After a small translation, the sets look the same around the origin out to a large radius."

Definition

Tiling space -

$$\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} + v \mid v \in \mathbb{R}^n\}}$$



Definition

Tiling space -

$$\Omega = \overline{\{\mathcal{T} + v \mid v \in \mathbb{R}^n\}}$$

Proposition (Topological properties - "why tiling spaces are weird")

- *$\Omega_{\mathcal{T}}$ is compact if and only if $\mathcal{T}$ is FLC.*
- *$(\Omega_{\mathcal{T}}, \mathbb{R}^n)$ is minimal (every orbit is dense) if and only if $\mathcal{T}$ is repetitive.*
- *$\Omega_{\mathcal{T}} \cong T^n$ if and only if $\mathcal{T}$ is periodic under the action of a lattice.*
- *$\Omega_{\mathcal{T}}$ is connected. Never path-connected, unless $\mathcal{T}$ is periodic.*
- *Fiber bundle*

$$\text{Cantor} \hookrightarrow \Omega_{\mathcal{T}} \twoheadrightarrow T^n$$

How do we compare Delone sets? What does it mean for $\mathcal{T}_1$ to be 'equivalent' to $\mathcal{T}_2$?

- **Local derivation** - Local rule in terms of patches which transforms $\mathcal{T}_1 \overset{\text{LD}}{\to} \mathcal{T}_2$.
- **Mutual local derivation** (MLD) - Local derivations in both directions $\mathcal{T}_1 \overset{\text{LD}_1}{\to} \mathcal{T}_2 \overset{\text{LD}_2}{\to} \mathcal{T}_1$. We write $\mathcal{T}_1 \overset{\text{MLD}}{\longleftrightarrow} \mathcal{T}_2$.
- **Linearly MLD** (L-MLD) - exists a linear isomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\mathcal{T}_1) \overset{\text{MLD}}{\longleftrightarrow} \mathcal{T}_2$. We write $\mathcal{T}_1 \overset{\text{L-MLD}}{\longleftrightarrow} \mathcal{T}_2$.
- **Homeomorphism** - continuous map $f\colon \Omega_{\mathcal{T}_1} \to \Omega_{\mathcal{T}_2}$ with continuous inverse.

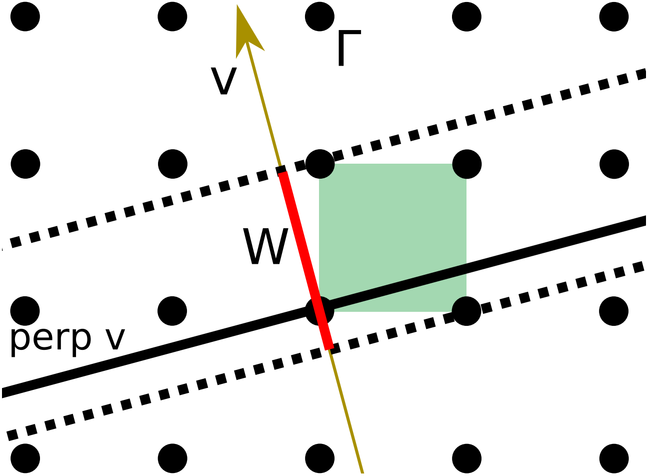
$$\text{MLD} \implies \text{L-MLD} \implies \text{Homeomorphic}$$

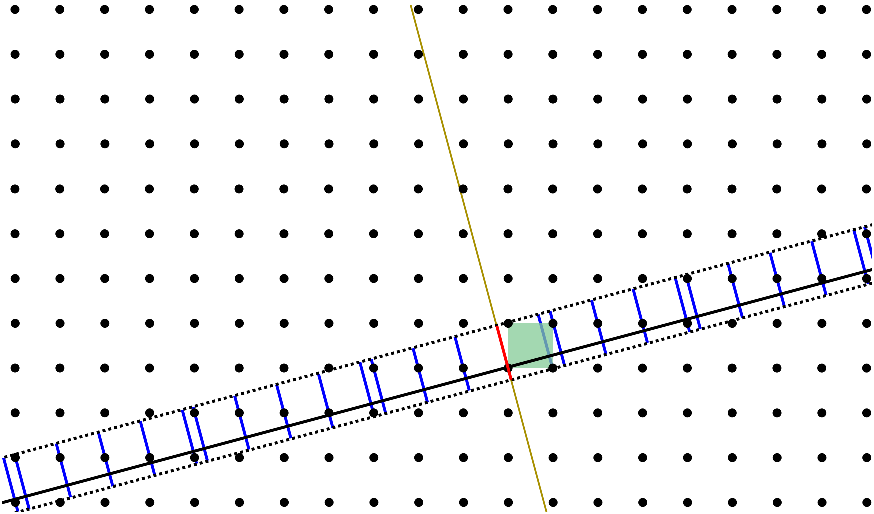
Cut-and-project sets

Definition

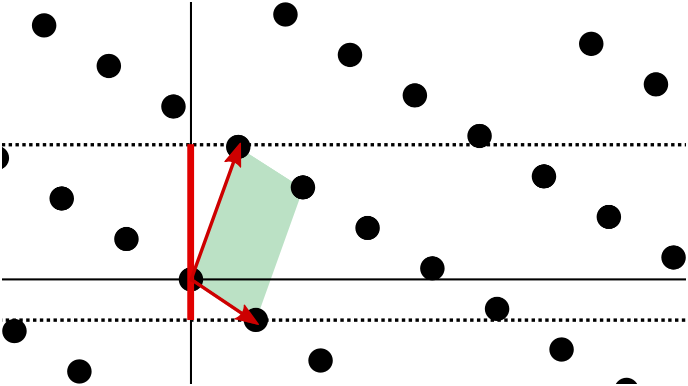
We say $v, v' \in \mathbb{R}^n \setminus \{0\}$ are *equivalent* if there exists $M \in GL(n, \mathbb{Z})$ such that $\mathrm{span}(Mv) = \mathrm{span}(v')$. We write $v \simeq v'$.

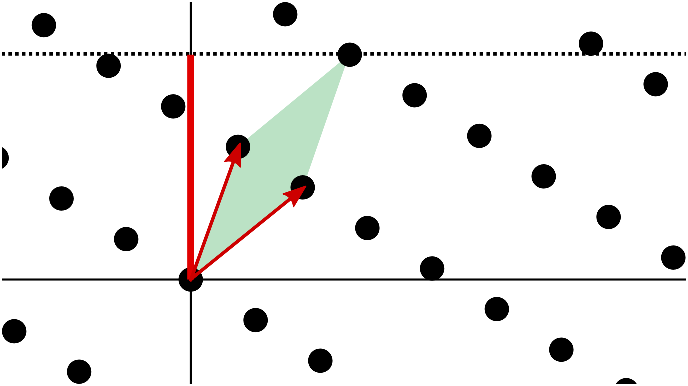
$\mathbb{R}^{n+1} = \text{span}\, v \oplus \text{perp}\, v,\ W = \pi([0,1]^{n+1}),\ S = W + \text{perp}\, v$





$$
\begin{array}{ccccc}
(\cong \mathbb{R}^n) & & & & (\cong \mathbb{R}) \\
\text{perp } v & \xleftarrow{\ \pi\ } & \mathbb{R}^{n+1} & \xrightarrow{\ \pi_{\text{int}}\ } & \text{span } v \\
\uparrow & & \uparrow & & \uparrow \text{ dense} \\
\pi(\Gamma) & \xleftarrow{\ 1-1\ } & \Gamma & \longrightarrow & \pi_{\text{int}}(\Gamma) \\
\uparrow & & \uparrow & & \\
\mathcal{T}_v & \longleftarrow & S \cap \Gamma & &
\end{array}
$$





Theorem (Baake, Schlottmann, Jarvis - '91)

If β_1 and β_2 generate the same lattice $\Gamma = \langle \beta_1 \rangle = \langle \beta_2 \rangle$ as $\mathbb{Z}$-submodules of $\mathbb{R}^{n+1}$, then their cut-and-project sets are MLD.

$$\langle \beta_1 \rangle = \langle \beta_2 \rangle \implies \mathcal{T}_{\beta_1} \overset{\text{MLD}}{\longleftrightarrow} \mathcal{T}_{\beta_2}$$

Corollary

If $v \simeq v'$ then $\mathcal{T}_v \overset{\text{L-MLD}}{\longleftrightarrow} \mathcal{T}_{v'}$. In particular $\Omega_{\mathcal{T}_v} \cong \Omega_{\mathcal{T}_{v'}}$.

Cohomology

Slogan:

"Count the holes"

Cohomology of nice spaces

$$\begin{array}{llcll} \text{cohomology}: & \textbf{Top} & \to & \textbf{Ab} & (\text{really } \textbf{GrRng}) \\ & X & \mapsto & H^n(X) & (\bigoplus H^n(X)) \end{array}$$

$$X \cong Y \implies H^n(X) \cong H^n(Y)$$

$H^n(X) = \mathbb{Z}^k$
$k =$ no. of n-dimensional holes in X.

Example

$H^n(\mathbb{R}^d) = 0$ for all $n \geq 1$
$H^1(S^1) = \mathbb{Z}, \quad H^2(S^1) = 0, \quad H^3(S^1) = 0 \dots$
$H^n(S^n) = \mathbb{Z}, \quad H^k(S^n) = 0$ otherwise
$H^1(T^2) = \mathbb{Z}^2, \quad H^2(T^2) = \mathbb{Z}, \quad H^k(T^2) = 0$ otherwise
$H^1(T^2_{\text{punc}}) = \mathbb{Z}^2$
$H^n(T^{n+1}_{\text{punc}}) = \mathbb{Z}^{n+1}$

Ordered cohomology

Definition (Ordered abelian group)

$(A, \leq)$ — Abelian group A with partial order $\leq$.

- $a \leq b \implies a + c \leq b + c$

Positive cone $A^+ = \{a \in A \mid 0 \leq a\}$

Definition (positive cone)

$H \subset A$ satisfies

i $H + H \subseteq H$

ii $H \cap -H = \{0\}$

iii $H - H = A$

Define $a \leq_H b$ if and only if $b - a \in H$. Write

$$(A, H) = (A, \leq_H) = (A, A^+)$$

$h\colon (A, A+) \to (B, B^+)$ is an isomorphism if $h\colon A \overset{\cong}{\to} B$ and $h(A^+) = B^+$.

Example

$$(-1)\colon (\mathbb{Z}, \mathbb{Z}^+) \to (\mathbb{Z}, -\mathbb{Z}^+)$$

Let $H_1 = \{(m, n) \mid m + n > 0\} \cup \{0\}$
and $H_2 = \{(m, n) \mid m > 0\} \cup \{0\}$

Example

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} : (\mathbb{Z}^2, H_1) \to (\mathbb{Z}^2, H_2)$$

For every vector $v \in \mathbb{R}^n \setminus \{0\}$ the set $H_v = \{x \in \mathbb{Z}^n \mid \langle x, v\rangle > 0\} \cup \{0\}$ is a positive cone for $\mathbb{Z}^n$. Note: $Aut(\mathbb{Z}^n) = GL(n, \mathbb{Z})$ is countable. Implies there are uncountably many isomorphism classes of order structures on $\mathbb{Z}^n$ for $n \geq 2$.

Proposition (Gähler, R. - '16)

$$(\mathbb{Z}^{n+1}, H_v) \cong (\mathbb{Z}^{n+1}, H_{v'}) \iff v \simeq v'$$

Key facts for proof.

- any isomorphism is realised by a matrix $M \in GL(n+1, Z)$
- tensoring with $\mathbb{R}$ preserves isomorphisms
- $0 = \langle M^{-1}x, v\rangle \iff 0 = \langle x, (M^{-1})^T v\rangle$
- $x \in H_{v'} \setminus \{0\} \iff 0 < \langle v', x\rangle$
- $\langle -, -\rangle$ is positive definite

□

$\mathcal{T} \subset \mathbb{R}^n$ - Delone set

Theorem (Kellendonk, Putnam - '05)

There is a natural graded ring homomorphism $\tau_{\mathrm{rs}} \colon H^*(\Omega_{\mathcal{T}}) \to \Lambda \mathbb{R}^{n*}$ *that is stable under homeomorphism of* $\Omega_{\mathcal{T}}$ *(Ruelle-Sullivan map).*

Theorem (Barge, Kellendonk, Schmieding - '12)

- *The restriction* $\tau_{\mathrm{rs}} \colon H^n(\Omega_{\mathcal{T}}) \to \Lambda^n \mathbb{R}^{n*} \cong \mathbb{R}$ *induces an order structure on* $H^n(\Omega_{\mathcal{T}})$ *by*

$$H^{n+} = \tau_{\mathrm{rs}}^{-1}(\mathbb{R}^{>0}) \cup \{0\}$$

- $\Omega_{\mathcal{T}} \cong \Omega_{\mathcal{T}'} \implies (H^n(\Omega_{\mathcal{T}}), H^{n+}) \cong (H^n(\Omega_{\mathcal{T}'}), H^{n+})$
- $\tau_{\mathrm{rs}}(H^n(\Omega_{\mathcal{T}})) = \mathrm{freq}(\mathcal{T})$ — *Frequency module of* $\mathcal{T}$

Theorem (Gähler, R. - '16)

$$(H^n(\Omega_{\mathcal{T}_v}), H^{n+}) \cong (\mathbb{Z}^{n+1}, H_v)$$

Sketch of proof.

- $H^n(\Omega_{\mathcal{T}_v}) \cong H^n(T^{n+1}_{\text{punc}}) = \mathbb{Z}^{n+1}$ - [Forrest, Hunton, Kellendonk '02]
- The Ruelle-Sullivan map $\tau_{\text{rs}} \colon H^{n+} \to \text{freq}(\mathcal{T}_v)$ is injective
- $\text{freq}(\mathcal{T}_v)$ is generated by the frequency of each tile type
- The frequency of a tile t_i is given by the length of the associated dual basis vector β_i^* after projecting by π — freq $t_i = \pi(\beta_i^*)$
- Pullback to $\Gamma = \mathbb{Z}^{n+1} \cong H^n(\Omega_{\mathcal{T}_v})$ along τ_{rs} using bijectivity - $\pi(\beta_i^*) \mapsto \beta_i^*$

□

Main Theorem

We can now give a topological classification of co-dimension one canonical projection tilings

Theorem (Gähler, R. -'16)

Let v and v' be totally irrational vectors in $\mathbb{R}^n$. The following are equivalent:

1. $\mathcal{T}_v \overset{\text{L-MLD}}{\longleftrightarrow} \mathcal{T}_{v'}$
2. $\Omega_{\mathcal{T}_v} \cong \Omega_{\mathcal{T}_{v'}}$
3. $(H^n(\Omega_{\mathcal{T}_v}), H^{n+}) \cong (H^n(\Omega_{\mathcal{T}_{v'}}), H^{n+})$
4. $(\mathbb{Z}^n, H_v) \cong (\mathbb{Z}^n, H_{v'})$
5. $v \simeq v'$

Question: Can this be extended to other co-dimensions? Can this be extended to non-canonical window?

Thank you!