Random substitutions, mixing and Rauzy fractals

Dan Rust

History

- AKA Stochastic substitutions, multi-valued substitutions, $0L$-systems, expansion-modification systems, …
- Peyrière '81. Seems to have been first to consider random substitutions (in context of percolation theory).
- Godrèche–Luck '89. Concrete examples (random Fibonacci, Penrose) in context of entropy and diffraction (heuristics).
- Dekking–Meester '91. Rediscovered (percolation theory).
- Baake, Moll, Nilsson '10–'13. Improved rigour and studied hull rather than single tilings (entropy, diffraction, ergodic th.).
- Gähler–Miro '14. Cohomology (specific examples).
- Some sporadic work in physics, genetics, computer science, …

More recently

- R.–Spindeler '18. Groundwork for rigorous study of subshifts in terms of dynamics, ergodic theory, topology.
- Baake–Spindeler–Strungaru '18. Much better understanding of diffraction, covering model sets.
- Gohlke–Spindeler '19. Rigorous foundation for ergodic theory and probability theory—tools to study *frequency measures*.
- R. '19. Beginning to understand periodicity/recognisability.
- Gohlke '20. Mostly understand topological entropy in the case of *compatible* random susbtitutions.
- and more coming...

Today

Discuss new work with

- Miro, Sadun, Tadeo C-balancedness, topological mixing.
- Samuel, Mitchell Rauzy fractals.

Random substitutions

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Locally mix tribonacci with twisted tribonacci (Random tribonacci)

$$\tau\colon \begin{cases} a & \mapsto & \{ab, ba\} \\ b & \mapsto & \{ac\} \\ c & \mapsto & \{a\} \end{cases} \qquad \text{with probabilities } (p, 1-p)$$

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Choices are independent for each letter:

$$a \mapsto ab \mapsto \overbrace{ba}^{\tau(a)}\, ac \mapsto ac\, \overbrace{ab}^{\tau(a)}\, \overbrace{ba}^{\tau(a)}\, a \mapsto baabaacacabba \mapsto \cdots$$

$$w = \cdots baabaacacabba.acababacbaababaaabacacab \cdots$$

$$X_\tau := \overline{\{\sigma^n(w) \mid n \in \mathbb{Z}, w \text{ generic}\}}.$$

Notation: $[u] = (|u|_{a_1}, \ldots, |u|_{a_d})$ is the abelianisation of the word u.
Let ϑ be a random substitution.

Two assumptions

1. Mainly consider **compatible** random substitutions. That is,

$$u, v \in \vartheta(a) \implies [u] = [v].$$

   Ensures we have well-defined **constant** substitution matrix M_ϑ.
2. ϑ is **primitive**, i.e., $M_\vartheta^k > 0$ for some $k \geq 1$.

- Note that primitivity can be defined for non-compatible random substitutions.

Results

Theorem (R., Spindeler '18)

***Properties of X_ϑ for primitive ϑ** (don't need compatible):*

- *Cantor set*
- *Either no periodic points or periodic points are dense*
- *Uncountably many minimal components*
- *Uncountably many invariant prob. measure*
- *Canonical measure μ_p induced by probabilities*
- *Almost all orbits are dense (in particular topologically transitive)*
- *Positive topological entropy (needs mild conditions)*

Even though X_τ has positive topological entropy, we can still get long-range correlations. Allows us to form **quasicrystals with entropy**.

Theorem (Baake, Spindeler, Strungaru, '18. Godrèche, Luck, '89)

For random Fibonacci ϑ_{Fib}: $a \mapsto \{ab, ba\}, b \mapsto \{a\}$, and random period doubling ϑ_{pd}: $a \mapsto \{ab, ba\}, b \mapsto \{aa\}$,

- *For every $x \in \Omega_\vartheta$, the diffraction spectrum has a non-trivial pp component and for μ_p-a.e. x, non-trivial ac component (pos. entropy).*
- *Eigenfunctions are continuous on a set of full measure (very close to having discrete 'topological' dynamical spectrum).*

They used the fact that $x \in X_\vartheta$ is a relatively dense subset of a regular cut-and-project set. The C+P scheme is the same as for the deterministic case, but the windows are larger (unions of the windows for the marginals). Potentially very powerful method for studying Pisot substitutions.

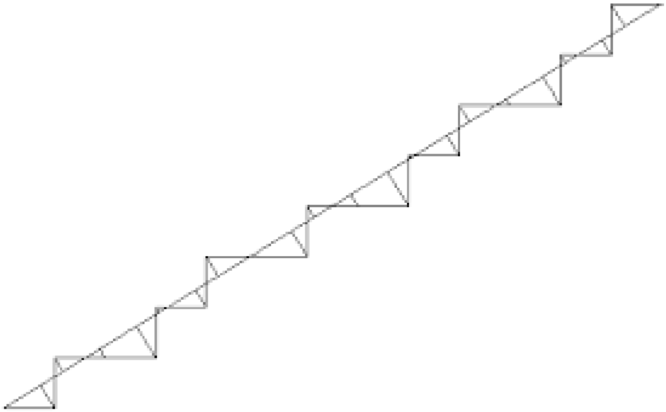
C-balancedness and Rauzy fractals

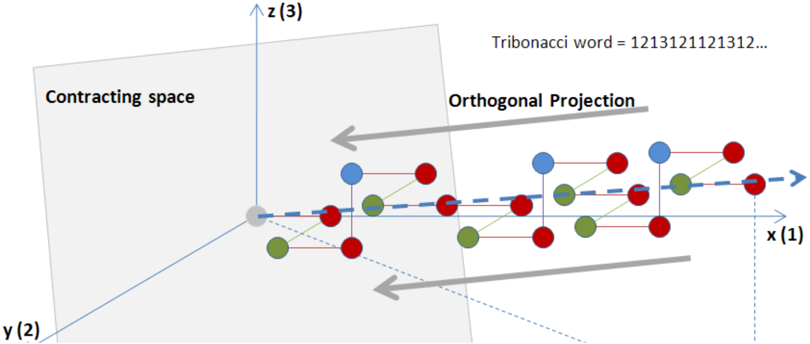
Let w be an (bi-)infinite word with well-defined letter-frequencies $\mathbf{R}_i$. Then w is **C-balanced** or has **bounded discrepancy** if there is a $C > 0$ such that for all subwords u,

$$\left| |u|_{a_i} - \mathbf{R}_i |u| \right| \leq C.$$

That is, the number of a_is in u never deviates more than C from the expected number.

> If w is C-balanced, then the lattice vectors
>
> $$[w_1], [w_1 w_2], [w_1 w_2 w_3], \ldots$$
>
> all remain within a small neighbourhood of the ray spanned by the frequency vector $\mathbf{R}$.

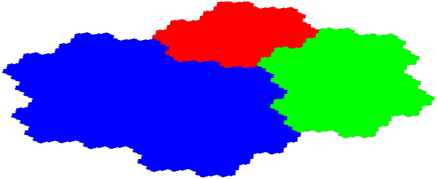




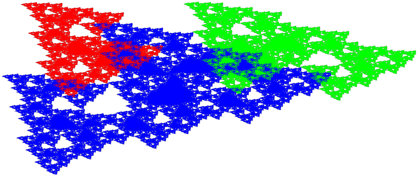
[*Shamelessly stolen from wikipedia*]

Define $\mathcal{R}(\theta) = \mathcal{R}(w) := \overline{\{\text{proj}_{\mathbf{R}}(w_{[0,n]}) \mid n \geq 0\}}$, the **Rauzy fractal** of w.

$$\tau : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases} \qquad \tilde{\tau} : \begin{cases} a & \mapsto & ba \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$



Tribonacci



Twisted Tribonaci

Rauzy fractals for deterministic substitutions

- $\mathcal{R}(\theta)$ is a compact subset of $\mathbb{R}^{d-1}$ equal to closure of its interior. Not always connected.
- For $w, w' \in X_\theta$, $\mathcal{R}(w) = \mathcal{R}(w') + \mathbf{x}$ so $\mathcal{R}(\theta)$ makes sense.
- $\mathcal{R}(\theta)$ is the unique attractor of an associated GIFS.
- Properties of $\mathcal{R}(\theta)$ correspond to properties of τ and X_θ.

$\mathcal{R}(\theta)$	X_θ
Tiles the plane	Discrete spectrum, equiv. Measure iso. with $\mathbb{T}$ translation

- Useful tools for studying *Pisot substitutions*.
- If Pisot conj. true, then $\mathcal{R}(\theta)$ is *window* for a cut-and-project scheme a.a. of whose model sets are the $x \in \Omega_\theta$.

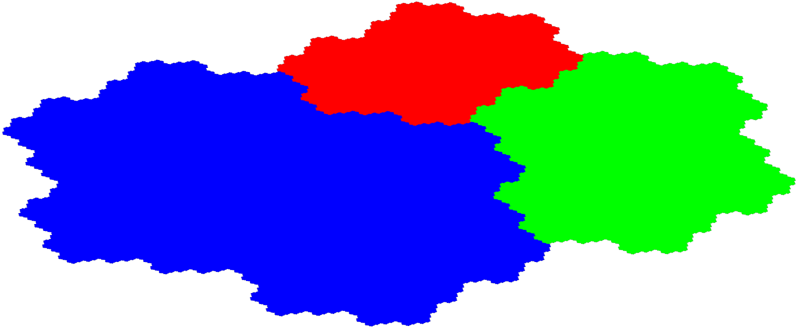
Let ϑ be a primitive, compatible random substitution.
Write $\lambda_{PF} > |\lambda_2| \geq \cdots \geq |\lambda_d|$.

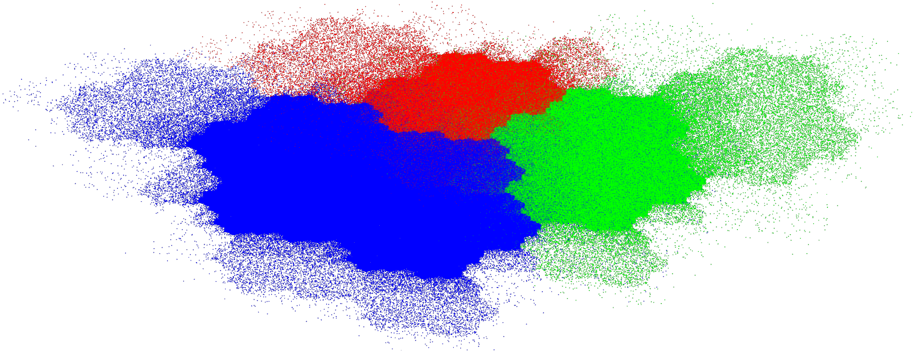
Theorem (Miro, R., Sadun, Tadeo '20)

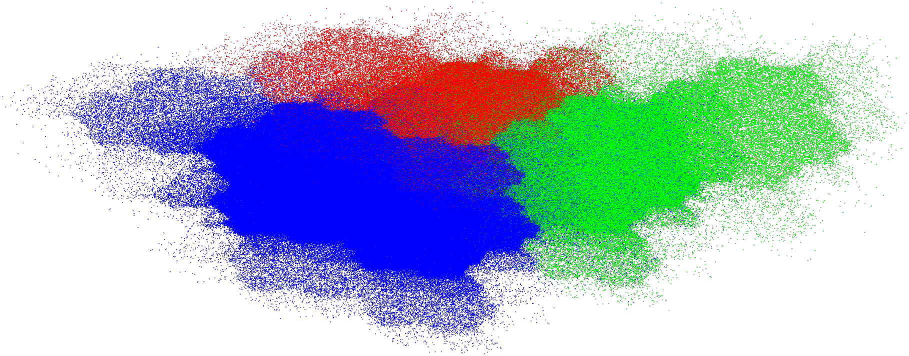
- *If* $|\lambda_2| < 1$, *then* X_ϑ *is C-balanced.*
- *If* $|\lambda_2| > 1$, *then* X_ϑ *is not C-balanced.*
- *If* $|\lambda_2| = 1$, *then both can happen (i.e., "M is not enough").*

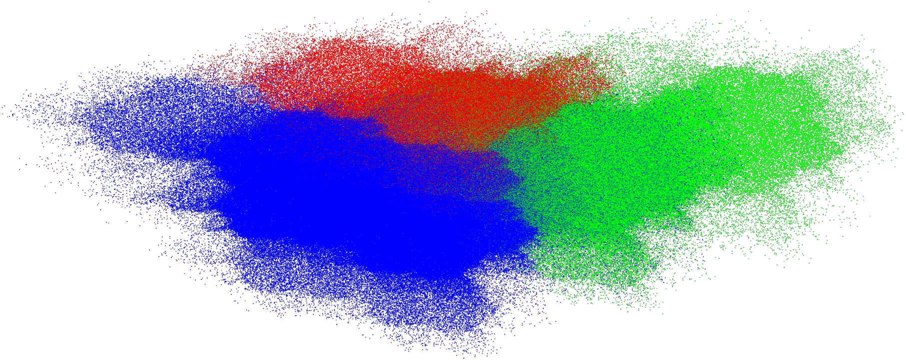
Open: Classify C-balancedness when $|\lambda_2| = 1$.
In deterministic setting, this is solved, but **extremely** complicated [Adamczewski '03]

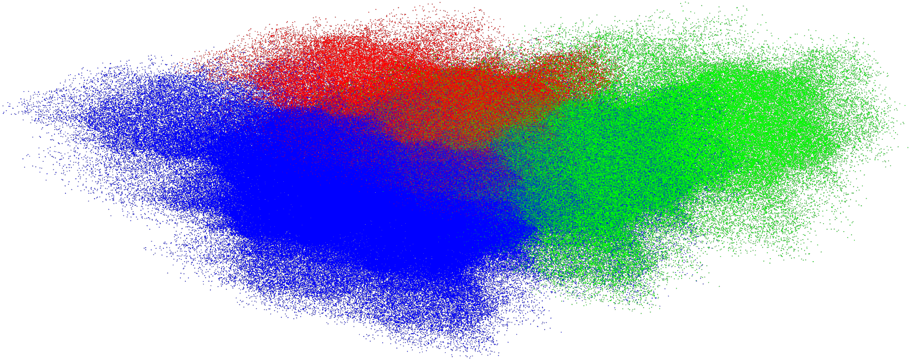
(With T. Samuel in Birmingham)
We can therefore generate approximate Rauzy fractals $\mathcal{R}(\vartheta, p)$ when ϑ is Pisot.

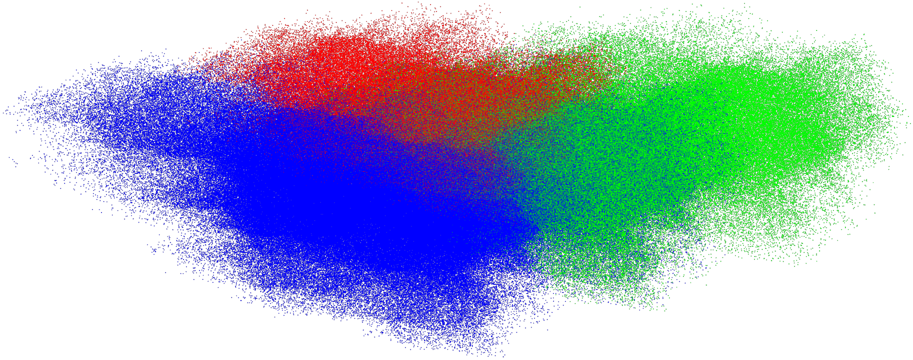


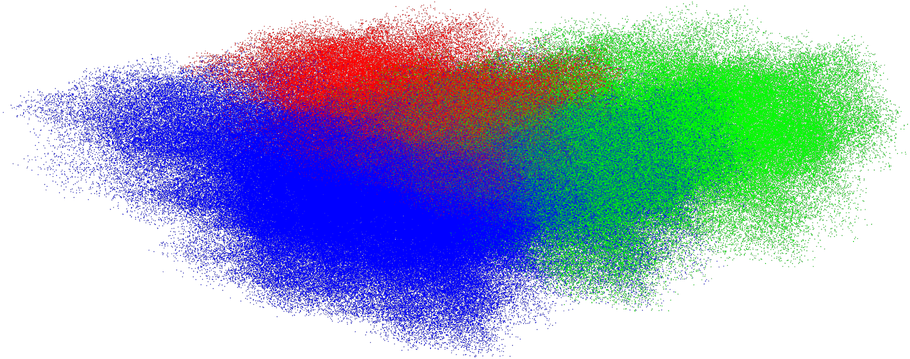


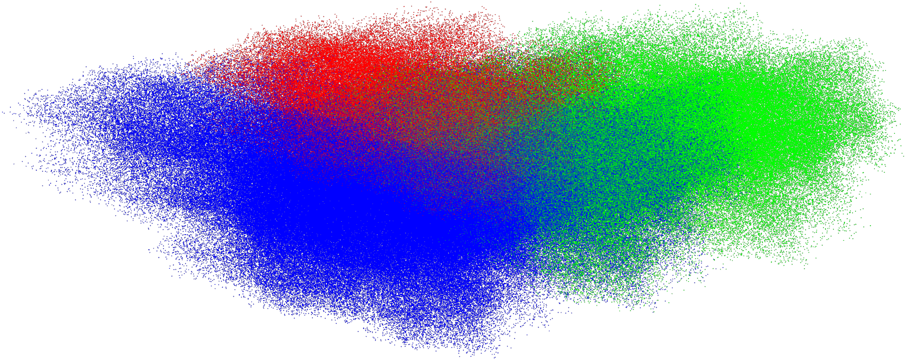


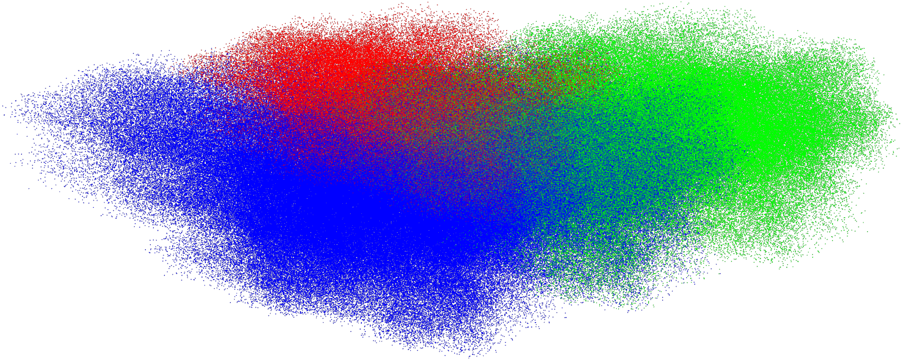


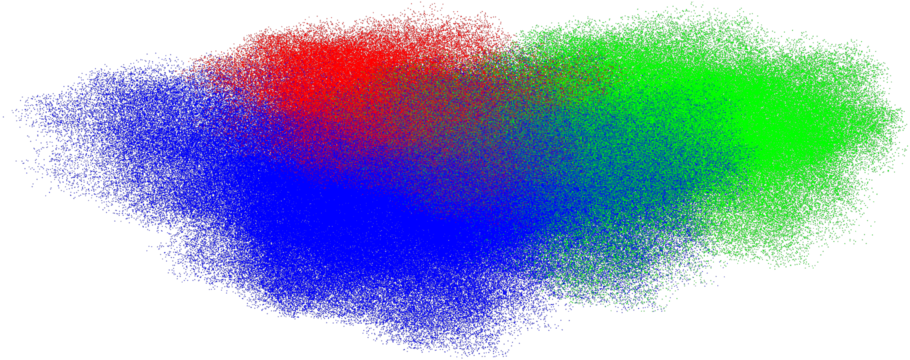


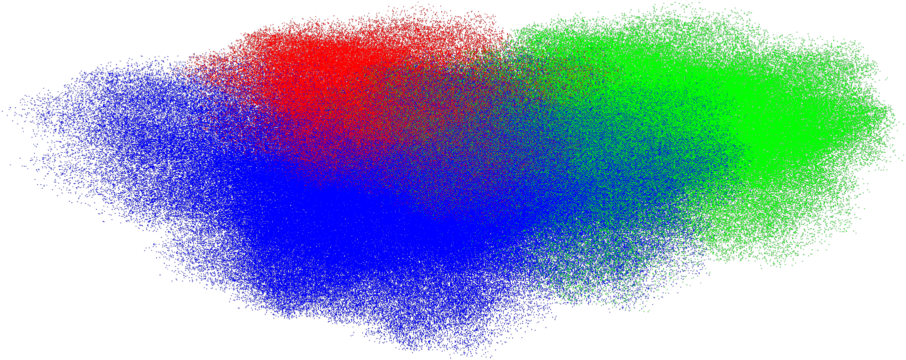


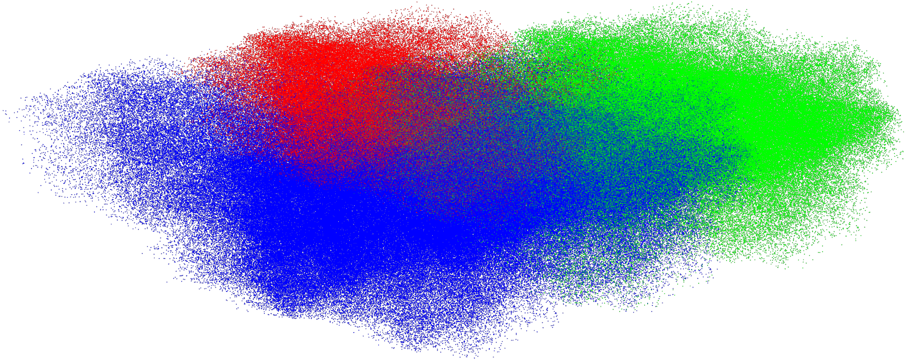


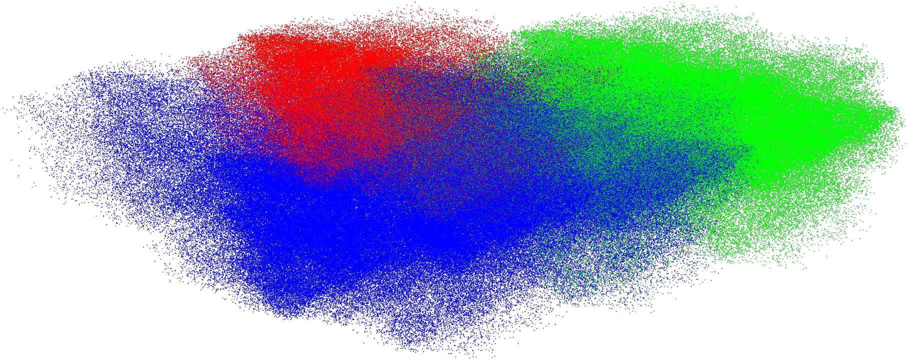


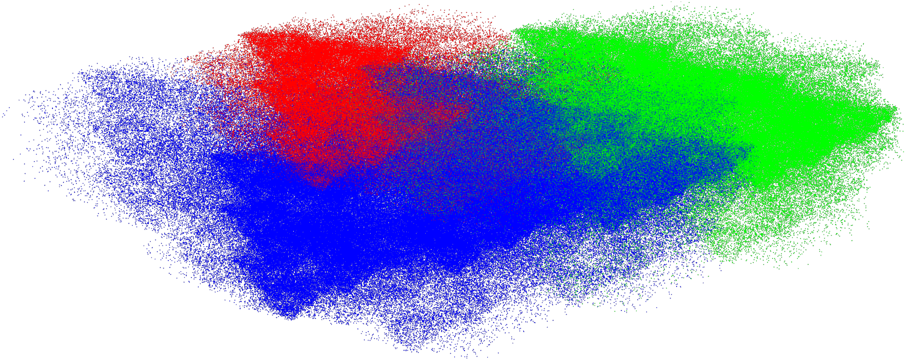


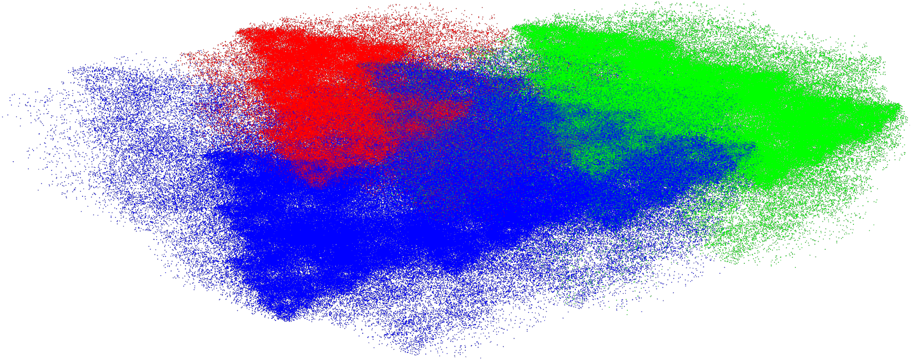


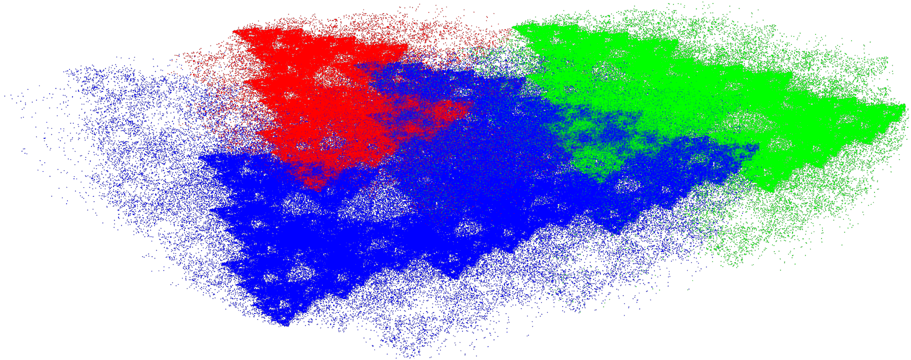


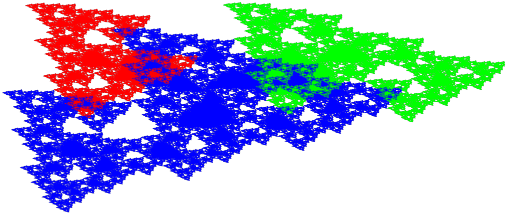


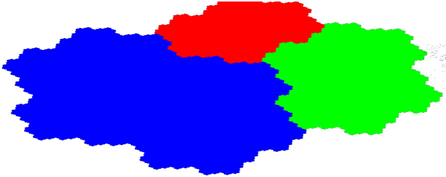
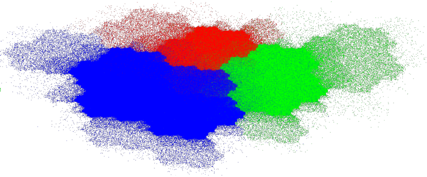
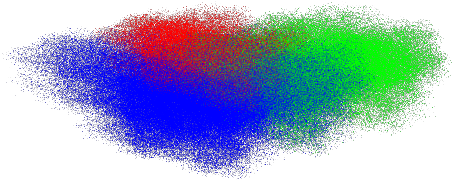
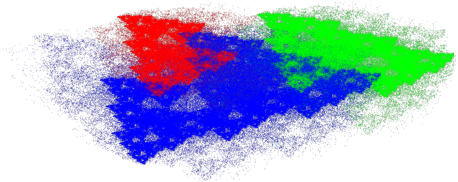
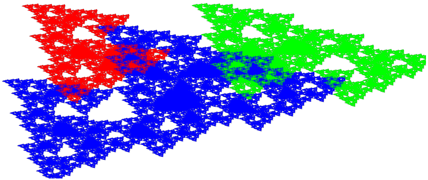










- Topologically, $\mathcal{R}(\vartheta, p)$ does not depend on $p \in (0, 1)$.
- Difference in images is therefore indicating the differences of the induced measures on $\mathcal{R}(\vartheta, p)$ as p changes.
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- Difference in images is therefore indicating the differences of the induced measures on $\mathcal{R}(\vartheta, p)$ as p changes.
- No longer the attractor of a GIFS, but rather a 'Galton–Watson' GIFS.
- Potentially offers a new approach to the Pisot conjecture:
  - Construct $\tilde{\theta}$ which is of 'Barge type' and $M_\theta = M_{\tilde{\theta}}$ (always exists).
  - Construct random substitution ϑ which is a local mixture of θ and $\tilde{\theta}$.
  - We know that $\mathcal{R}(\tilde{\theta})$ tiles the plane [Barge, '16].
  - Show that tilability of $\mathcal{R}(\vartheta)$ (or an analogous metric property) is invariant as p ranges smoothly from 1 to 0.
  - Conclude that $\mathcal{R}(\theta)$ tiles the plane.

We'll see!

Why λ_2?

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The length of the word $\vartheta^n(a_j)$ is given by

$$|\vartheta^n(a_j)| = \sum_{i=1}^{d} \langle e_i, M^n e_j \rangle$$

and the number of times a_i appears in $\vartheta^n(a_j)$ is $|\vartheta^n(a_j)|_{a_i} = \langle e_i, M^n e_j \rangle$. To make things simple, suppose M is diagonalisable. Then,

$$e_j = C_j \mathbf{R} + \sum_{k=2}^{d} c_{kj} v_k \implies M^n e_j = \lambda_{PF}^n C_j \mathbf{R} + O(\lambda_2^n)$$

$$\implies |\vartheta^n(a_j)| = \lambda_{PF}^n C_j \mathbf{R}_i + O(\lambda_2^n)$$

$$\implies \left| |\vartheta^n(a_j)|_{a_i} - |\vartheta^n(a_j)| \mathbf{R}_i \right| = |\lambda_{PF}^n C_j \mathbf{R}_i - \lambda_{PF}^n C_j \mathbf{R}_i| + O(\lambda_2^n).$$

So, λ_2 regulates how far letter-counts can deviate from expected counts.

Topological mixing

A subshift X is **topologically mixing** if for all $u, v \in \mathcal{L}$, there is an $N \geq 0$ such that for all $n \geq N$, $uwv \in \mathcal{L}$ with $|w| = n$. So,

$$
\begin{array}{ll}
\times & u\square\square\square\square v \\
\checkmark & u\square\square\square\square\square v \\
\checkmark & u\square\square\square\square\square\square v \\
\checkmark & u\square\square\square\square\square\square\square v \\
\checkmark & u\square\square\square\square\square\square\square\square v \\
\checkmark & u\square\square\square\square\square\square\square\square\square v \\
\checkmark & \vdots
\end{array}
$$

where $\checkmark$ means legal.

Let ϑ be a primitive, compatible, locally recognisable random substitution .

Theorem (Miro, R., Sadun, Tadeo '20)

(1) *If X_ϑ is C-balanced (inlc. $|\lambda_2| < 1$), then X_ϑ is not top. mixing.*

(2) *X_ϑ top. mixing $\implies$ $\mathrm{gcd}\{|\vartheta^k(a)| : a \in \mathcal{A}\} = 1, \forall k \geq 1$.*

(3) *If $|\lambda_2| > 1$ and $\#\mathcal{A} = 2$, then '$\Longleftarrow$' holds also.*

(4) *If $|\lambda_2| = 1$, then both can happen.* [Dekking–Keane '78]

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• In the deterministic case, (1) with $|\lambda_2| < 1$ is much simpler because

$$|\lambda_2| < 1 \quad \implies \quad \text{not top. weak mixing} \quad \overset{*}{\implies} \quad \text{not top. mixing}$$

but $*$ requires X_ϑ to be minimal.

• The proof of (3) required developing a new classification of periodic 2-letter substitutions (probably folklore).

Open: Classify mixing when (i) $|\lambda_2| > 1$, $\#\mathcal{A} > 2$ (don't have IVT)
(ii) $|\lambda_2| = 1$, non-C-balanced.

What about random Fibonacci?! It's **not** locally recognisable!

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Definition

We say a word $u \in \mathcal{L}_\vartheta$ is **recognisable** if there is a unique way of decomposing u into inflation words and it has a unique preimage (up to boundary).

Ex. aba is not recognisable because $ab\,a$ and $a\,ba$ are decomps.

$abbaaaabbaa$ is recognisable because only decomp. is

$$ab\ ba\ a\ a\ ab\ ba\ a$$

and only preimages are $(a)abbaa(a)$ or $(a)abbaa(b)$ — only disagree on boundary.

Theorem (Miro, R., Sadun, Tadeo '20)

Let ϑ be a primitive compatible random substitution with X_ϑ C-balanced. There exists an integer N_ϑ with the following property: If there exists a ***single*** *recognisable word for ϑ^n with $n \geq N_\vartheta$, then X_ϑ is not topologically mixing.*

For random Fibonacci, there exist ϑ^n_{Fib}-recognisable words for all $n \geq 1$ (constructive argument) and so...

Corollary

The random Fibonacci RS-subshift $X_{\vartheta_{Fib}}$ is not topologically mixing.

Proof sketch

- Let $u \in \mathcal{L}$ with $|u| = m$. C-balanced $\implies$ there are at most $2C$ possible values of $|u|_{a_i}$ and so $(2C)^d$ possibilities for $[u]$.
- Pick $N = N_\vartheta$ so that $(2C)^d < \lambda_{PF}^N$.
- Pick a ϑ^n-recognisable u for some $n \geq N$ and WLOG u is an exact level-N inflation word. So any word uwu must actually be $u\vartheta^N(v)u$.
- Consider the set $A = \{|\vartheta^N(v)| \mid u\vartheta^N(v)u \in \mathcal{L}\}$ - gaps between u.
- X_ϑ mixing $\implies$ A cofinite $\implies$ $\mathrm{dens}(A) = 1$.
- Let's estimate $\# \underbrace{\{|\vartheta^N(v)| \mid v \in \mathcal{L}\}}_{B \subset A} \cap [0, m]$ as $m \to \infty$.
- WTS:

$$\mathrm{dens}(A) \leq \limsup_{m \to \infty} \frac{B \cap [0, \lambda_{PF}^N m]}{\lambda_{PF}^N m} < 1$$

Proof sketch

- By lin. alg., exists k (uniform in m) s.t. for all $m' > m + k$, if $|v| = m'$, then $|\vartheta^N(v)| > \lambda_{PF}^N m$.
- So we only need to look at words v at most of length $m + k$.
- $$\text{dense}(A) \leq \limsup_{m\to\infty} \frac{B \cap [0, \lambda_{PF}^N m]}{\lambda_{PF}^N m} \leq \limsup \frac{(2C)^d (m+k)}{\lambda_{PF}^N m}$$
  $$= \limsup_{m\to\infty} \underbrace{\frac{(2C)^d}{\lambda_{PF}^N}}_{<1 \text{ by choice of } N} + \underbrace{\frac{(2C)^d k}{\lambda_{PF}^N m}}_{\to 0} < 1$$

□

Other open questions

Just a few of the questions still to be fully tackled

- What kinds of subshifts can appear as RS-subshifts? - already know that we can get all mixing SFTs [Gohlke, R., Spindeler, '19])
- Can we classify when $\mathrm{Per}(X_\vartheta) = \varnothing$? - Some criteria [R., 19']
- When $\mathrm{Per}(X_\vartheta) \neq \varnothing$, how many periodic points of period p are there?
- Measure theoretic entropy (topic of Mitchell's thesis)
- Are 'suitably nice' RS-subshifts intrinsically ergodic? (unique mme)
- Essentially nothing has been done in $\mathbb{R}^2$ or higher.