Mathematical and numerical analysis of peridynamic models

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Peridynamics

Collaborators:

- Kun Zhou (who couldn't come due to visa issues)
  1. Linear PD bond model, Cauchy problem (ESIAM-M2AN 2011)
  2. Initial-boundary value problems, finite element approximation, error estimates and condition numbers (SINUM 2010)
- Max Gunzburger, Rich Lehoucq and Kun Zhou:
  3. Nonlocal laws and nonlocal vector calculus (Sandia preprint)
  4. Linear nonlocal BVPs with volumic constraints (almost done)
  5. Variational formulation of PD state model (in progress)
- Lili Ju, Li Tian and Kun Zhou:
  6. Finite element approximations to linear PD models, a posterior error analysis (in progress), adaptive methods

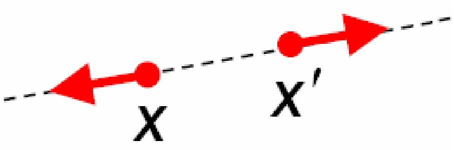
Linear bond-based PD:

$\mathbf{u}$: displacement $\quad \mathbf{y} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$

$$\mathbf{u}_{tt}(\mathbf{x}, t) = L_\delta \mathbf{u}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t)$$

$$L_\delta \mathbf{u}(\mathbf{x}) = c_\delta \int_{B_\delta(\mathbf{x})} \frac{(\mathbf{x}' - \mathbf{x}) \otimes (\mathbf{x}' - \mathbf{x})}{\sigma(|\mathbf{x}' - \mathbf{x}|)} (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}'$$

model for a spring network



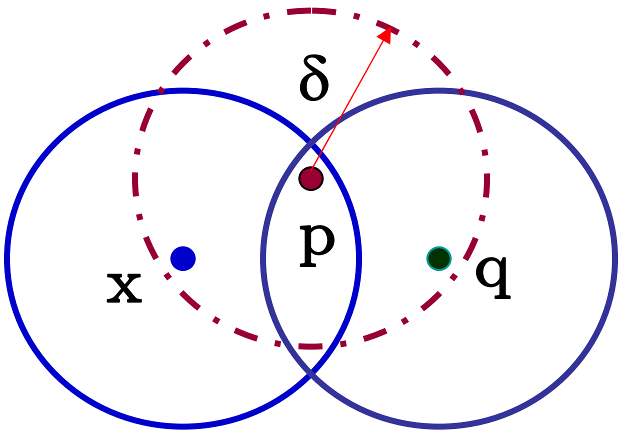
σ: kernel function, related to spring constants

δ : PD horizon

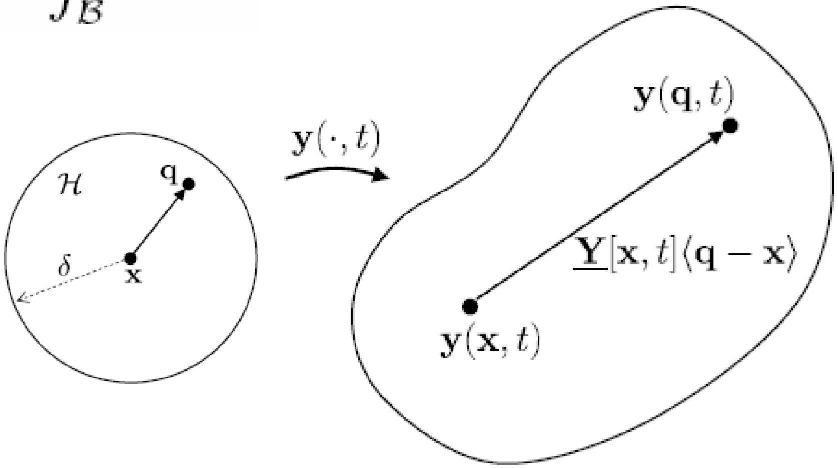
Linear Peridynamic State : Silling 2009

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x},\mathbf{q})\mathbf{u}(\mathbf{q},t)\, dV_{\mathbf{q}} - \mathbf{P}_0(\mathbf{x})\mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)$$

where $$\mathbf{C}_0(\mathbf{x},\mathbf{q}) = \int_{\mathcal{B}} \Big(\underline{\mathbb{K}}[\mathbf{x}]\langle \mathbf{p}-\mathbf{x}, \mathbf{q}-\mathbf{x}\rangle\big) - \underline{\mathbb{K}}[\mathbf{p}]\langle \mathbf{x}-\mathbf{p}, \mathbf{q}-\mathbf{p}\rangle + \underline{\mathbb{K}}[\mathbf{q}]\langle \mathbf{x}-\mathbf{q}, \mathbf{p}-\mathbf{q}\rangle \Big)\, dV_{\mathbf{p}}$$



$$\mathbf{P}_0(\mathbf{x}) = \int_{\mathcal{B}} \mathbf{C}_0(\mathbf{x},\mathbf{q})\, dV_{\mathbf{q}}$$



$\underline{\mathbb{K}}^{\dagger} = \underline{\mathbb{K}}$ if elastic

- Examples: bond-based, PD fluid, PD solid, …

Mathematical issues:

- Increasing popularity of PD based simulations demands better mathematical and numerical analysis
- Rigorous mathematical framework
  - Are all formulations used in practice well-posed?
  - Are singularities appearing from models or numerics?
- Numerical analysis
  - Stability (CFL, log dependence on mesh?)
  - Conditioning (independent of mesh?)
  - Error (uniform in δ?)

  on discretization and model parameters

Existing studies motivated our works:

- Silling 2000, 2009
- Silling/Epton/Weckner/Xu/Askari 2007
- Silling/Lehoucq 2005, 2007
- Weckner/Brunk/Epton/Silling/Askari 2009

- **Modeling**

- Emmrich/Weckner, 2006 2007
- Alali/Lipton 2009
- Gunzburger/Lehoucq 2009

- **Analysis**

- Bobaru/Yang/Alves/Silling/Askari/Xu 2009
- Aksoylu/Parks 2009
- Chen/Gunzburger 2009
- Seleson/Parks/Gunzburger/Lehoucq 2009

- **Numerics**

+ others discussed in the workshop

Nonlocal PD & PDEs: how different/similar are they

Consider an Initial Boundary Value Problem of PD:

$$\mathbf{u}_{tt}(\mathbf{x},t) = \mathcal{L}_\delta \mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)$$

together with initial and boundary conditions.

The linear bond-based PD operator

$$\mathcal{L}_\delta \mathbf{u}(\mathbf{x}) = c_\delta \int_{B_\delta(\mathbf{x})} \frac{(\mathbf{x}'-\mathbf{x}) \otimes (\mathbf{x}'-\mathbf{x})}{\sigma(|\mathbf{x}'-\mathbf{x}|)} (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) d\mathbf{x}'$$

- PD horizon parameter δ, spherical neighborhood $B_\delta(\mathrm{x})$
- Kernel function $\sigma = \sigma(|\mathbf{x}'-\mathbf{x}|)$
- c_δ a normalization constant

Nonlocal PD: boundary conditions

A simple 1-d example: equation defined on $[0, \pi]$

Zhou-D. SINUM 2010

$$u \text{ is odd in } (-\delta, \delta) \text{ and } (\pi - \delta, \pi + \delta)$$

Practical application: none, but it allows Fourier analysis (O. Weckner), offers much insight and is one of the natural extensions of its local limit: homogeneous Dirichlet BC

$$-\mathcal{L}_\delta^o u(x) = -c_\delta \int_{x-\delta}^{x+\delta} \frac{|x' - x|^2}{\sigma(|x' - x|)} (u(x') - u(x)) dx'$$

Linear bond-based PD: assumptions/symbols

- Assumptions

$$\sigma(\mathbf{x}) > 0,\ \forall \mathbf{x} \in B_\delta(0)$$

$$\frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)} \geq \rho(\mathrm{y}) \in L^1(B_\delta(0))$$

ρ non-negative

(can be relaxed to include negative part, see for example D-G-L-Z 2011 for an illustration of sign-changing kernels)

$$\tau_\delta := c_\delta \int_{B_\delta(0)} \frac{|\mathbf{x}|^4}{\sigma(|\mathbf{x}|)} d\mathbf{x} < \infty$$

(necessary for well-defined elastic modulus)

Linear bond-based PD: symbols

- Fourier symbol: consider $u(x) = \sum_k u_k^o \sin(kx)$

$$u_k^o = \frac{2}{\pi}\int_0^\pi u(x)\sin(kx)dx$$

$$-\mathcal{L}_\delta^o u(x) = \sum_k \eta_\delta(k) u_k^o \sin(kx)$$

$$\eta_\delta(k) = c_\delta \int_{-\delta}^{\delta} (1-\cos(ky))\frac{|y|^2}{\sigma(|y|)}dy$$

- In comparison

$$-u''(x) = \sum_k^o k^2 u_k^o \sin(kx)$$

Nonlocal BVP for Linear bond-based PD

- Natural spaces for PD operator L_δ: for any *s*

$$M_\sigma^{so} = \left\{ u : \|u\|^2_{M_\sigma^{so}} = \sum_k \eta_\delta^s(k) u_k^{o2} < \infty \right\}$$

are generalized energy spaces, in particular, *s*=1

$$\|u\|_{M_\sigma^o} = [-\frac{2}{\pi}(\mathcal{L}_\delta^o u, u)]^{\frac{1}{2}} = \left\{ \sum_k \eta_\delta(k) u_k^{o2} \right\}^{1/2}$$

M_σ^{-so} : the dual space of M_σ^{so}

Nonlocal BVP for Linear bond-based PD

- L_δ is an self-adjoint operator: M_σ^o to M_σ^{-o}

  And an isometry: by the Riemann Lemma

$$\lim_{k\to\infty}\int_{-\delta}^{\delta}\rho(|y|)\cos(ky)dy = 0$$

$$\begin{cases} \eta_\delta(k) \ge \inf\limits_{k\ge 1} c_\delta \displaystyle\int_{-\delta}^{\delta}(1-\cos(ky))\rho(y)dy > 0 \\ 0 \le \tau_\delta k^2 - \eta_\delta(k) \le \dfrac{k^4\delta^2\tau_\delta}{12} . \end{cases}$$

$$\longrightarrow H_o^1 \hookrightarrow M_\sigma^o \hookrightarrow L_o^2 .$$

Nonlocal BVP for Linear bond-based PD

- If in addition $\dfrac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)} \in L^1(B_\delta(0))$

$$0 < \inf_{k \geq 1} \eta_\delta(k) \leq \eta_\delta(k) \leq 4c_\delta \int_0^\delta \frac{y^2}{\sigma(y)} dy$$

$$\longrightarrow \qquad M_\sigma^{2o} = M_\sigma^o = L_o^2$$

L^2 theory: Emmrich/Weckner, Alali/Lipton

the corresponding linear bond-based PD models no longer have smoothing effect

Linear bond-based PD:

- Other kernels:

$$\sigma(|y|) \geq \gamma_1 |y|^{3+\alpha} \longrightarrow \quad 0 \leq \eta_\delta(k) \leq C_1^\delta(\alpha)^2 k^\alpha$$

$$\alpha \in (0,2) \qquad H_o^{\alpha/2} \hookrightarrow M_\sigma^o$$

$$\sigma(|y|) \leq \gamma_2 |y|^{3+\beta} \longrightarrow \quad \eta_\delta(k) \geq C_2^\delta(\beta)^2 k^\beta$$

$$\beta < 2 \qquad M_\sigma^o \hookrightarrow H_o^{\beta/2}$$

Linear bond-based PD:

- Elliptic regularity:

$$-\mathcal{L}_\delta^o u = f$$

Theorem: (Zhou-D. SINUM 2010) For $\alpha=\beta$,

$$C_2^\delta(\beta)^2 k^\beta \leq \eta_\delta(k) \leq C_1^\delta(\beta)^2 k^\beta \qquad \beta \in (0,2)$$

For $f \in H_o^m$, $m \geq -\alpha$, we have $u \in H_o^{m+\beta}$

which implies "smoothing effect" of the PD model

The dependence on horizon is *explicitly* estimated.

Connection to DE and standard Sobolev spaces

ODE $\quad -u'' = f$

$$u(0) = u(\pi) = 0$$

PD $\quad -\mathcal{L}_\delta^o u = f$

$$u \text{ is odd in } (-\delta, \delta) \text{ and } (\pi - \delta, \pi + \delta)$$

Notice that $\quad 0 \le \tau_\delta k^2 - 2\eta_\delta(k) \le \dfrac{k^4 \delta^2 \tau_\delta}{12}$

When $\quad \tau_\delta \to 2 \quad$ we have $\quad u_\delta^o - u \to 0 \;\; in \; M_\sigma^{2o}$

Convergence in conventional norms follows from space equivalence

Are these effects due to the special 1-d BC?

$$u \;\text{ is odd in } (-\delta, \delta) \text{ and } (\pi - \delta, \pi + \delta)$$

No, the results are similar when using other (more conventional) nonlocal BCs

For example,

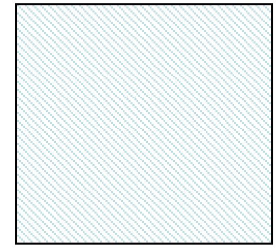
- for scalar equation, same results were shown in any dim with general nonlocal volumic constraints (D-G-L-Z [4]);

  Null kernel + compactness, similar to works of Rossi

- similar results can be shown in higher dim for linear PD bond system (Zhou-D. SINUM 2010) and state (D-G-L-Z [5])

Linear bond-based PD: I(nonlocal)BVP

$$\begin{cases} \mathbf{u}_{tt}(\mathbf{x},t) = L_\delta \mathbf{u}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)\,, & t \in (0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{g}(\mathbf{x})\,, & \{t=0\} \\ \mathbf{u}_t(\mathbf{x},0) = \mathbf{h}(\mathbf{x})\,, & \{t=0\} \end{cases}$$



$$\mathcal{L}_\delta \mathbf{u}(\mathbf{x}) = c_\delta \int_{B_\delta(\mathbf{x})} \frac{(\mathbf{x}'-\mathbf{x}) \otimes (\mathbf{x}'-\mathbf{x})}{\sigma(|\mathbf{x}'-\mathbf{x}|)} (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\,d\mathbf{x}'$$

The operator has a non-diagonal tensor kernel

A nonlocal BC for a square: $\mathbf{u} = \sum_{k,l=1}^{\infty} \begin{pmatrix} u_{kl} \sin(kx_1)\cos(lx_2) \\ v_{kl}\cos(kx_1)\sin(lx_2) \end{pmatrix}$

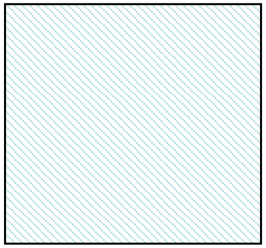
Alternatingly odd in one direction, even in the other

Linear bond-based PD: I(nonlocal)BVP

Alternatingly odd and even in variables: a strange BC?

Not totally. Again, it is a natural nonlocal analog of the Navier equation with the well-known greased wall BC and allows a block diagonalization via Fourier

$$\begin{cases} \mathbf{u}_{tt} = \mu\triangle\mathbf{u} + 2\mu\nabla\nabla\cdot\mathbf{u} + \mathbf{b}\,, \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x})\,, \quad \mathbf{u}_t(0,\mathbf{x}) = \mathbf{h}(\mathbf{x})\,, \\ u_1(\mathbf{x}) = \frac{\partial}{\partial n}u_2(\mathbf{x}) = 0\,, \quad x_2 \in \{0,\pi\} \\ u_2(\mathbf{x}) = \frac{\partial}{\partial n}u_1(\mathbf{x}) = 0\,, \quad x_1 \in \{0,\pi\} \end{cases}$$



Linear bond-based PD:

For
$$\mathbf{u} = \sum_{k,l=1}^{\infty} \begin{pmatrix} u_{kl} \sin(kx_1) \cos(lx_2) \\ v_{kl} \cos(kx_1) \sin(lx_2) \end{pmatrix}$$

Let
$$M_{\delta,kl} = c_\delta \int_{B_\delta(0)} \frac{1-\cos(k\xi_1 + l\xi_2)}{\sigma(|\boldsymbol{\xi}|)} \begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix} d\boldsymbol{\xi} \,.$$

$$-\mathcal{L}_\delta \mathbf{u} = \sum_{k,l=1}^{\infty} \begin{pmatrix} \sin(kx_1)\cos(lx_2) & 0 \\ 0 & \cos(kx_1)\sin(lx_2) \end{pmatrix} M_{\delta,kl} \begin{pmatrix} u_{kl} \\ v_{kl} \end{pmatrix}$$

Local limit
$$M_{0,kl} = \begin{pmatrix} (2\mu+\lambda)k^2 + \mu l^2 & (\mu+\lambda)kl \\ (\mu+\lambda)kl & \mu k^2 + (2\mu+\lambda)l^2 \end{pmatrix}$$

$$\mu = \lambda = \lim_{\delta\to 0} \frac{c_\delta}{2} \int_{B_\delta(0)} \frac{\xi_1^2\xi_2^2}{\sigma(|\boldsymbol{\xi}|)} d\boldsymbol{\xi} = \lim_{\delta\to 0} \frac{\tau_\delta}{16}$$

Linear bond-based PD:

The matrix symbols commute (!) and

$$M_{\delta,kl} \leq c_\delta \left(\int_{B_\delta(0)} \frac{1-\cos(k\xi_1 + l\xi_2)}{\sigma(|\boldsymbol{\xi}|)} |\boldsymbol{\xi}|^2 d\boldsymbol{\xi} \right) \mathbf{I}$$

$$= c_\delta \left(\int_{B_\delta(0)} \frac{1-\cos(k\xi_1)\cos(l\xi_2)}{\sigma(|\boldsymbol{\xi}|)} |\boldsymbol{\xi}|^2 d\boldsymbol{\xi} \right) \mathbf{I}$$

$$\leq c_\delta \left(\int_{B_\delta(0)} \frac{(k^2 + l^2)}{2\sigma(|\boldsymbol{\xi}|)} |\boldsymbol{\xi}|^4 d\boldsymbol{\xi} \right) \mathbf{I} \leq \frac{\tau_\delta}{2\mu} M_{0,kl}$$

The difference between the two symbols is

$$Z_{kl} = \frac{c_\delta}{24} \int_{B_\delta(0)} \frac{(k\xi_1 + l\xi_2)^4}{\sigma(|\boldsymbol{\xi}|)} \boldsymbol{\xi} \otimes \boldsymbol{\xi} \cos(\theta)\, d\boldsymbol{\xi}$$

$$|Z_{kl}| \leq \frac{\tau_\delta}{24} \delta^2 (k^2 + l^2)^2 \mathbf{I}$$

Leads to order of convergence for small δ

Linear bond-based PD:

Define energy spaces accordingly M_σ^{soe}

Theorem: (Zhou-D. SINUM 2010)

Well-posedness of IBVP in

$$C([0,T], M_\sigma^{oe}) \cap H^1(0,T; L^2_{oe})$$

and convergence to local limit

For system, the PD bond operator is not of a diagonal form, one can utilize the symmetry in the tensor and variables to achieve diagonalization, which helps establishing the analytical framework

Finite dimensional approximation

Internal approximations: dense subspace

- Truncated Fourier spaces
- Conforming finite element of piecewise polynomials of degree m

Galerkin-Ritz $$u_n = \mathrm{argmin}_{v_n \in V_n} \{ \frac{1}{2} \|v_n\|^2_{M_\sigma^o} - (v_n, f)_{L^2} \}$$

Theorem: (Zhou-D. SINUM 2010) For $f \in M_\sigma^{-o}$

best approximation $$\|u - u_n\|_{M_\sigma^o} \leq \min_{v_n \in V_n} \|u - v_n\|_{M_\sigma^o} \to 0$$

\+ A priori error estimates for nonlocal BVP/IBVP

One-d nonlocal BVP: error estimates

For smooth solutions (Zhou-D 2010 SINUM)

Theorem: $C_2^\delta(\beta)^2 k^\beta \le \eta_\delta(k) \le C_1^\delta(\beta)^2 k^\beta$

Fourier spectral with n modes $f \in H_o^m$, $m \ge -\beta$

$$\|u-u_n\|_\alpha \le C_2^\delta(\beta)^{-2} n^{-m-\beta+\alpha} \|f\|_m \qquad \alpha \le m+\beta$$

Theorem: $C_2^\delta(\beta)^2 k^\beta \le \eta_\delta(k) \le C_1^\delta(\alpha)^2 k^\alpha$, $0 \le \beta \le \alpha \in (0,2)$

FEM with conforming elements of degree m (continuity not required for $\alpha<1$), if $f \in H_o^{m'-\alpha}$, $0 \le m' \le m+1$, $s \in [0, \beta/2]$

$$\|u-u_n\|_s \le c_s C_1^\delta(\alpha)^{1+s'} C_2^\delta(\beta)^{-3-s'} h^{m'-\alpha/2+(\beta-\alpha/2)s'} \|f\|_{m'-\beta}$$

$$\sigma(y) = |y|^3 \qquad \|u-u_n\|_0 \le O(h^{m'-\alpha/2}\delta^{-1+\alpha/2})$$

Numerics: Chen-Gunzburger 2010

Finite Element Stiffness Matrix $A^o = ((\phi_i, \phi_j)_{M_\sigma^o})$

Theorem : (Zhou-D. 2010 SINUM) Finite element with conforming elements of degree m, quasi-uniform mesh

$$0 < cC_2^\delta(\beta)^2 h \le \lambda_1 \le \lambda_n \le cC_1^\delta(\alpha)^2 h^{1-\alpha}$$

$$\mathrm{cond}(A^o) \le c(C_1^\delta(\alpha)/C_2^\delta(\beta))^2 h^{-\alpha}$$

Eg $\sigma(y) = |y|^2$ $\mathrm{cond}(A^o) \le c\min\{\delta^{-2}, h^{-2}\}$

Estimates/numerical observation $\mathrm{cond}(A^o) \le c\delta^{-2}$ Aksoylu-Parks 2009

Eg $\sigma(|y|) = |y|^3$ $\mathrm{cond}(A^o) \le c\min\{h^{-\alpha}\delta^{\alpha-2}, h^{-2}\}$

Estimates/numerical observation Seleson-Parks-Gunzburger-Lehoucq 2009

Same type estimates hold for more general scalar diffusion equation with volumic constraints, but dependence not so precise (D-G-L-Z [4])

Linear peridynamic model

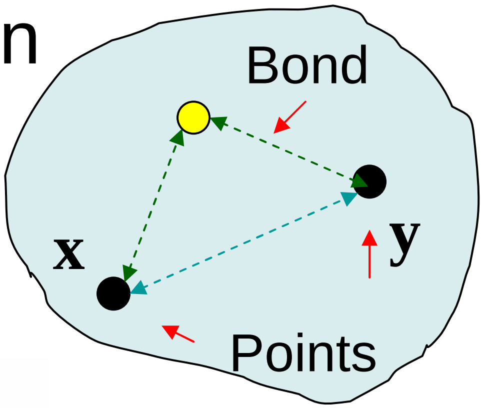
- The discussion so far is for linear problems with a special, but nonlocal, type of BC.
- Still, similar results holds for Cauchy problems (D.-Zhou ESIAM-M2AN)
- Similar results for other nonlocal BVPs (see D-G-L-Z [4])
- For systems: fewer results, but they set PD apart from existing studies of other nonlocal/integral equations; simultaneous diagonalization may not be valid in general, but can still be a helpful tool (well-posedness for state models of PD solid, D-G-L-Z, [5])

PD state of solids via nonlocal calculus

- Notation: $\otimes$ Kronecker (tensor) product

  $\bullet$ Frobenius (scalar) product

  $\Omega \subset \mathbb{R}^d$ a given material domain



D-G-L-Z [3]: nonlocal vector calculs

- Point functions $\mathbf{u} : \Omega \rightarrow \mathbb{R}^r$

  Inner product: $(\mathbf{u}_1 , \mathbf{u}_2) = \int_\Omega \mathbf{u}_1(\mathbf{x}) \cdot \mathbf{u}_2(\mathbf{x}) d\mathbf{x}$

- Two-point (bond) functions $\mathbf{v} : \Omega \times \Omega \rightarrow \mathbb{R}^m$

  Inner product: $< \mathbf{v}_1 , \mathbf{v}_2 >= \int_\Omega \int_\Omega \mathbf{v}_1(\mathbf{x}, \mathbf{y}) \cdot \mathbf{v}_2(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}$

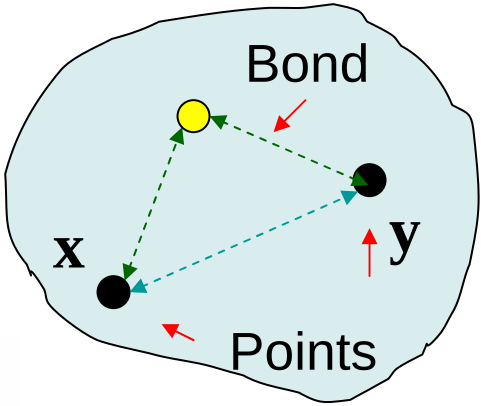
PD state solid via nonlocal calculus

- Nonlocal point gradient and its adjoint

$$\mathcal{G}(\eta)(\mathbf{x}) = -\int_{\Omega} \big(\eta(\mathbf{x},\mathbf{y})+\eta(\mathbf{y},\mathbf{x})\big)\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})\,d\mathbf{y}$$

$$\mathcal{G}^*(\mathbf{v})(\mathbf{x},\mathbf{y}) = \big(\mathbf{v}(\mathbf{y})-\mathbf{v}(\mathbf{x})\big)\cdot\boldsymbol{\alpha}(\mathbf{x},\mathbf{y})$$

- Weighted gradient and adjoint



$$\mathcal{G}_{\omega}(u)(\mathbf{x}) = \mathcal{G}(\omega(\mathbf{x},\mathbf{y})u(\mathbf{x}))(\mathbf{x})$$

$$\mathcal{G}^*_{\omega}(\mathbf{v})(\mathbf{x}) = \int_{\Omega} \mathcal{G}^*(\mathbf{v})(\mathbf{x},\mathbf{y})\,\omega(\mathbf{x},\mathbf{y})\,d\mathbf{y}$$

Linear state-based PD solid

dilatation $\hat{\theta} = \mathcal{G}^*_w(\mathbf{u})$ Volumic strain

isotropic part of the extension scalar state $\underline{e}^i = \mathcal{G}^*_w(\mathbf{u})\,|\mathbf{y}-\mathbf{x}|/d$

deviatoric part $\underline{e}^d = \mathcal{G}^*(\mathbf{u}) - \mathcal{G}^*_w(\mathbf{u})\,|\mathbf{y}-\mathbf{x}|/d$

- Energy: $E(\mathbf{u})$

$$\int_\Omega \frac{k(\mathbf{x})\,\hat{\theta}^2}{2}\,d\mathbf{x} + \int_\Omega\int_\Omega \frac{\eta(\mathbf{x})}{2}\underline{\omega}(\mathbf{x},\mathbf{y})(\underline{e}^d)^2\,d\mathbf{y}d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\mathbf{b}\,d\mathbf{x}$$

$|u|^2$

$U(\Omega)$: Energy space with bounded |u|

$$\mathcal{Z} = \{\mathbf{u} : |\mathbf{u}| = 0\}$$

$U_0(\Omega)$: $\mathbf{v} \in U(\Omega)$ with homogenous volumic constraint

Linear state-based PD solid

Well-posedness: (D-G-L-Z [5])

For a Hilbert space $U(\Omega)$ with a Poincare on $U_0(\Omega)$, the nonlocal BVP for the linear state-based peridynamic solids is well-posed provided that $\eta(\mathbf{x}) \geq \eta_0 > 0$ and $k(\mathbf{x}) \geq k_0 > 0$ for $\mathbf{x} \in \Omega$.

An example: when ω is square integrable and we have a uniformly positive definite acoustic tensor P_0, the energy space is L^2, thus the linear PD state model is well-posed with a L^2 well-defined homogeneous volumic constraint .

Summary: linear peridynamic model

- Linear problems provide foundation to nonlinear problems, and for linear PD setting, there are many open questions.
- Results are dependent on micromodulus functions, we may see “smoothing” vs “no-smoothing”. Even for the latter case, interests might be on the horizon dependence of the regularity
- Ongoing works: for nonlinear PD, studying steady states; for linear PD, speed of propagation, material stability; for fem, characterizing mesh dependence, a posterior estimates/adaptivity; for nonlocal calculus, application to shape analysis/geometry; …