Global existence and blow-up results for some problems in nonlinear nonlocal elasticity

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Outline

- Nonlinear Nonlocal Elasticity.

- Three Nonlinear Nonlocal Wave Equations.
  - Longitudinal and Transverse Wave Motions.
  - Anti-Plane Shear Motion.

- Cauchy Problems.
  - Local well-posedness, Global existence, Blow-up.

- Ongoing studies / Future work.
Local Theory of Elasticity

Assumptions and Notation

- An isotropic, homogeneous hyperelastic medium.
- A stress-free undistorted state as the reference configuration.
- Position in the reference configuration: \( \mathbf{X} = (X_1, X_2, X_3) \).
- Position at time \( t \): \( \mathbf{x}(\mathbf{X}, t) = (x_1(\mathbf{X}, t), x_2(\mathbf{X}, t), x_3(\mathbf{X}, t)) \).
- Displacement: \( \mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \).
Local Theory of Elasticity

Constitutive Equation

\[ \sigma = \sigma(A) \equiv \frac{\partial W(A)}{\partial A} \]

\( \sigma \): nominal stress tensor (transpose of the first Piola-Kirchhoff stress tensor)

\( W(A) \): strain energy density function

\( A(X, t) = \text{Grad } x(X, t) \): deformation gradient

Equation of Motion

\[ \rho_0 \ddot{x} = \text{Div } \sigma \]

\( \rho_0 \): mass density, (no body forces)

(The symbol \( \dot{} \) indicates the material time derivative)
Major Drawbacks of Local Theory

- Absence of any intrinsic length scale.
- Neglects the long range forces (important especially at small scales).
- It is incapable of predicting, for instance,
  - the dispersive nature of harmonic waves in crystal lattices,
  - the boundedness of the stress field near the tip of a crack.

Generalized Theories of Elasticity

- Micropolar theories.
- Strain elasticity (or higher-order gradient) theories.
- Peridynamic theory.
- Nonlocal elasticity theory. (Kröner, Eringen, Edelen, Kunin, Rogula)
Nonlocal Theory of Elasticity

Constitutive Equation

Stress at a point depends on the strain field at every point in the body.

\[ S = S(X, t) \equiv \int \beta(|X - Y|) \sigma(A(Y, t)) \, dY \]

\( S \): stress tensor, \( \beta(|X - Y|) \): kernel function

The only difference between the two theories is due to the constitutive equations.

Equation of Motion

\[ \rho_0 \ddot{x} = \text{Div } S \]

Henceforth, all quantities appear in non-dimensional form and \( \rho_0 = 1 \)
Case 1: Longitudinal Motion

Equation of Motion

Consider the longitudinal motion:

\[ x_1 = X_1 + U(X_1, t), \quad x_2 = X_2, \quad x_3 = X_3 \]

The displacement field:

\[ u_1 \equiv U(X_1, t), \quad u_2 = u_3 \equiv 0 \]

Equation of motion:

\[ U_{tt} = (S(U_x))_x \]

with \( x \equiv X_1 \) and stress component \( S \).
Case 1: Longitudinal Motion

Constitutive Equation in Local Theory

\[ \sigma(U_x)(x, t) = W'(U_x(x, t)) \]

\( W \): strain energy function (with \( W(0) = W'(0) = 0 \))

Constitutive Equation in Nonlocal Theory

\[ S(U_x)(x, t) = \int_{-\infty}^{\infty} \beta(x - y)W'(U_x(y, t))dy \]
Case 1: Longitudinal Motion

Nonlocal Nonlinear PDE for Longitudinal Waves

1D equation of motion:

\[ U_{tt} = \left( \int_{-\infty}^{\infty} \beta(x - y) W'(U_x(y, t)) dy \right)_x \]

Differentiate w.r.t. \( x \)

\[ U_{xtt} = \left( \int_{-\infty}^{\infty} \beta(x - y) W'(U_x(y, t)) dy \right)_{xx} \]

Change variables \( U_x = u \), and write \( W'(u) = W'(0)[u + g(u)] \)

Nonlinear Nonlocal Wave Equation

\[ u_{tt} = \left[ \beta * (u + g(u)) \right]_{xx} \]
### Examples of 1D Kernel Functions

#### Nonlinear Nonlocal Wave Equation

\[ u_{tt} = [\beta \ast (u + g(u))]_{xx} \]

#### Assumption

\[ 0 \leq \hat{\beta}(\xi) \leq C(1 + \xi^2)^{-r/2} \]

#### Dirac Measure

\[ \beta = \text{Dirac measure}, \ r = 0. \]

\[ u_{tt} - u_{xx} = g(u)_{xx}. \]

- Equation of motion is a nonlinear wave equation.
Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

\[ u_{tt} = [\beta \ast (u + g(u))]_{xx} \]

Triangular Kernel

\[ \beta(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & |x| \geq 1.
\end{cases} \]

- \[ \hat{\beta}(\xi) = \frac{4}{\xi^2} \sin^2\left(\frac{\xi}{2}\right), \quad r = 2. \]
- \[ (\beta \ast \nu)_{xx} = \nu(x + 1) - 2\nu(x) + \nu(x - 1). \]
- Equation of motion is a differential-difference equation.
Examples of 1D Kernel Functions

**Nonlinear Nonlocal Wave Equation**

\[ u_{tt} = [\beta \ast (u + g(u))]_{xx} \]

**Exponential Kernel**

\[ \beta(x) = \frac{1}{2} e^{-|x|}. \]

- \( \hat{\beta}(\xi) = (1 + \xi^2)^{-1}, \ r = 2. \)
- \( (\beta \ast v)_{xx} = (1 - D_x^2)^{-1} v_{xx}. \)
- Equation of motion: Improved Boussinesq equation

\[ u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx} \]
Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

\[ u_{tt} = [\beta * (u + g(u))]_{xx} \]

Double Exponential Kernel

\[ \beta(x) = \frac{1}{2(c_1^2 - c_2^2)} (c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}). \]

- \( \hat{\beta}(\xi) = (1 + \gamma_1 \xi^2 + \gamma_2 \xi^4)^{-1}, \ r = 4. \)
- \( (\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}. \)
- Equation of motion: Higher-order Boussinesq equation
  (Duruk, Erkip, Erbay *IMA J. Appl. Math.* (2009))

\[ u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxxx} = (g(u))_{xx} \]
Examples of 1D Kernel Functions

### Nonlinear Nonlocal Wave Equation

\[ u_{tt} = [\beta \ast (u + g(u))]_{xx} \]

### Gaussian Kernels

\[ \beta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \hat{\beta}(\xi) = e^{-\xi^2/2}. \]

\[ \beta(x) = \frac{1}{\sqrt{2\pi}} (1 - x^2)e^{-x^2/2}, \quad \hat{\beta}(\xi) = \xi^2 e^{-\xi^2/2}. \]

- Equation of motion: an integro-differential equation.
Nonlinear Nonlocal Wave Equation

\[ u_{tt} = [\beta * (u + g(u))]_{xx} \]

Gaussian Kernels

\[ \beta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \hat{\beta}(\xi) = e^{-\xi^2/2}. \]

\[ \beta(x) = \frac{1}{\sqrt{2\pi}} (1 - x^2) e^{-x^2/2}, \quad \hat{\beta}(\xi) = \xi^2 e^{-\xi^2/2}. \]

• Equation of motion: an integro-differential equation.

All these are Boussinesq type nonlocal nonlinear PDEs.
Consider the transverse motion:

\[ x_1 = X_1, \quad x_2 = X_2 + U(X_1, t), \quad x_3 = X_3 + V(X_1, t) \]

The displacement field:

\[ u_1 \equiv 0, \quad u_2 \equiv U(X_1, t), \quad u_3 \equiv V(X_1, t) \]

Equation of motion:

\[ U_{tt} = (P(U_x, V_x))_x \]
\[ V_{tt} = (Q(U_x, V_x))_x \]

with \( x \equiv X_1 \) and stress components \( P, Q \).
Constitutive Equation in Nonlocal Theory

\[ P(U_x, V_x) = \int_{-\infty}^{\infty} \beta(x - y) \frac{\partial W(U_x, V_x)}{\partial U_x} \, dy \]

\[ Q(U_x, V_x) = \int_{-\infty}^{\infty} \beta(x - y) \frac{\partial W(U_x, V_x)}{\partial V_x} \, dy \]

\( W \): strain energy function (with \( W(0, 0) = 0, \nabla W(0, 0) = 0 \))

For isotropic case \( W = F(U_x^2 + V_x^2) \).
Case 2: Transverse Motion

Nonlocal Nonlinear PDE for Transverse Waves

\[ u_{tt} = \left( \beta \ast \frac{\partial F}{\partial u} \right)_{xx} \]
\[ v_{tt} = \left( \beta \ast \frac{\partial F}{\partial v} \right)_{xx} \]

where \( u = U_x, \ v = V_x. \)
Case 3: Anti-Plane Shear Motion

Equation of Motion

Consider the anti-plane shear motion:

\[ x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1, X_2, t) \]

The displacement field:

\[ u_1 = u_2 \equiv 0, \quad u_3 = w(X_1, X_2, t) \]

Equation of motion:

\[ w_{tt} = (P(w_x, w_y))_x + (Q(w_x, w_y))_y \]

with \( x \equiv X_1, \ y \equiv X_2 \) and stress components \( P, \ Q \).
Case 3: Anti-Plane Shear Motion

Constitutive Equation in Nonlocal Theory

\[ P(w_x, w_y) = \int_{-\infty}^{\infty} \beta(x - y) \frac{\partial W(w_x, w_y)}{\partial w_x} \, dy \]

\[ Q(w_x, w_y) = \int_{-\infty}^{\infty} \beta(x - y) \frac{\partial W(w_x, w_y)}{\partial w_y} \, dy \]

\( W \): strain energy function (with \( W(0, 0) = 0, \nabla W(0, 0) = 0 \))

For isotropic case \( W = F(w_x^2 + w_y^2) \).

Nonlocal Nonlinear PDE for Shear Waves

\[ w_{tt} = \left( \beta \ast \frac{\partial F}{\partial w_x} \right)_x + \left( \beta \ast \frac{\partial F}{\partial w_y} \right)_y \]
Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

\[ w_{tt} = \left( \beta \ast \frac{\partial F}{\partial w_x} \right)_x + \left( \beta \ast \frac{\partial F}{\partial w_y} \right)_y \]

Assumption

\[ 0 \leq \hat{\beta}(\xi) \leq C(1 + |\xi|^2)^{-r/2} \]

The Gaussian Kernel

\[ \beta(x, y) = (2\pi)^{-1} e^{-\frac{x^2 + y^2}{2}} \]

- \( \hat{\beta}(\xi_1, \xi_2) = e^{-\frac{\xi_1^2 + \xi_2^2}{2}} \)
- Take any \( r \).
Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

\[ w_{tt} = \left( \beta \ast \frac{\partial F}{\partial w_x} \right)_x + \left( \beta \ast \frac{\partial F}{\partial w_y} \right)_y \]

The Modified Bessel Function Kernel

\[ \beta(x, y) = (2\pi)^{-1} K_0(\sqrt{x^2 + y^2}) \]

(\( K_0 \): the modified Bessel function of the second kind of order zero)

- \( \hat{\beta}(\xi_1, \xi_2) = (1 + \xi_1^2 + \xi_2^2)^{-1} \)
- \( r = 2 \)
- Letting \( F(s) = \frac{1}{2} s + G(s) \)

\[ w_{tt} - \Delta w - \Delta w_{tt} = \left( \frac{\partial G}{\partial w_x} \right)_x + \left( \frac{\partial G}{\partial w_y} \right)_y. \]
Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

\[
w_{tt} = \left( \beta \ast \frac{\partial F}{\partial w_x} \right)_x + \left( \beta \ast \frac{\partial F}{\partial w_y} \right)_y
\]

The bi-Helmholtz Type Kernel

\[
\beta(x, y) = \frac{1}{2\pi(c_1^2 - c_2^2)}[K_0(\sqrt{x^2 + y^2}/c_1) - K_0(\sqrt{x^2 + y^2}/c_2)]
\]

- \[\hat{\beta}(\xi_1, \xi_2) = [1 + \gamma_1(\xi_1^2 + \xi_2^2) + \gamma_2(\xi_1^2 + \xi_2^2)^2]^{-1}\]
- \[r = 4.\]

\[
w_{tt} - \Delta w - \gamma_1 \Delta w_{tt} + \gamma_2 \Delta^2 w_{tt} = \left( \frac{\partial G}{\partial w_x} \right)_x + \left( \frac{\partial G}{\partial w_y} \right)_y.
\]
Problem 1: Longitudinal Motion

\[ u_{tt} = \left[ \beta \ast (u + g(u)) \right]_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \]

Duruk, Erbay, Erkip *Nonlinearity* (2010)
Problem 2: Transverse Motion

\begin{align*}
    u_{1tt} &= \left( \beta \ast (u_1 + g_1(u_1, u_2)) \right)_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\
    u_{2tt} &= \left( \beta \ast (u_2 + g_2(u_1, u_2)) \right)_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \\
    u_1(x, 0) &= \varphi_1(x), \quad u_{1t}(x, 0) = \psi_1(x) \\
    u_2(x, 0) &= \varphi_2(x), \quad u_{2t}(x, 0) = \psi_2(x).
\end{align*}

Duruk, Erbay, Erkip *J. Differential Equations* (2011)

Exactness Condition

\[
\frac{\partial g_1}{\partial u_2} = \frac{\partial g_2}{\partial u_1}
\]
Problem 3: Anti-Plane Shear Motion

\[ w_{tt} = \left( \beta \ast \frac{\partial F}{\partial w_x} \right)_x + \left( \beta \ast \frac{\partial F}{\partial w_y} \right)_y, \quad (x, y) \in \mathbb{R}^2, \quad t > 0, \]

\[ w(x, y, 0) = \varphi(x, y), \quad w_t(x, y, 0) = \psi(x, y) \]

Erbay, Erbay, Erkip *Nonlinearity* Submitted
Local Existence for Problem 1

$H^s \cap L^\infty$ valued ODE system

\[
\begin{align*}
u_t &= v, \quad u(0) = \varphi \\
v_t &= [\beta \ast (u + g(u))]_{xx}, \quad v(0) = \psi.
\end{align*}
\]
Lemma: Let $g \in C^{[s]+1}(\mathbb{R})$, $s \geq 0$. Then there is some constant $K(M)$ such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leq M$, we have

$$\|g(u)\|_s \leq K(M)\|u\|_s ,$$

and some other constant $J(M)$ such that for all $u, v \in H^s \cap L^\infty$ with $\|u\|_\infty + \|u\|_s \leq M$, $\|v\|_\infty + \|v\|_s \leq M$ we have

$$\|g(u) - g(v)\|_s \leq J(M)\|u - v\|_s .$$
Local Existence for Problem 1

Local Well-Posedness Theorem

Consider the Cauchy problem Let $g \in C^{[s]+1}(\mathbb{R})$, $g(0) = 0$. There is some $T > 0$ such that the Cauchy problem is well-posed with solution $u \in C^2([0, T], H^s \cap L^\infty)$ for initial data $\varphi, \psi \in H^s \cap L^\infty$ in any of the following cases.

- Case 1: $s > 1/2$ and $r \geq 2$,
- Case 2: $s \geq 0$ and $r > 5/2$,
- Case 3: $s \geq 0$ and $\beta_{xx}$ is a finite measure on $\mathbb{R}$.

Local Well-Posedness Theorem for Problem 2

A similar theorem holds for Problem 2, the coupled system of transverse motion.
Local Existence for Problems 1 and 2

Triangular kernel

\[ \beta(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & |x| \geq 1.
\end{cases} \]

- \( r = 2, \quad (\beta \ast v)_{xx} = v(x + 1) - 2v(x) + v(x - 1). \)
- Case 3 applies for \( s \geq 0 \) (\( \beta_{xx} \) is a finite measure).
Local Existence for Problems 1 and 2

Triangular kernel

\[ \beta(x) = \begin{cases} 
1 - |x| & |x| \leq 1 \\
0 & |x| \geq 1.
\end{cases} \]

- \( r = 2, \quad (\beta \ast \nu)_{xx} = \nu(x + 1) - 2\nu(x) + \nu(x - 1). \)
- Case 3 applies for \( s \geq 0 \) (\( \beta_{xx} \) is a finite measure).

Exponential kernel

\[ \beta(x) = \frac{1}{2} e^{-|x|}. \]

- \( r = 2, \quad (\beta \ast \nu)_{xx} = \beta \ast \nu - \nu. \)
- Case 3 applies for \( s \geq 0 \) (\( \beta_{xx} \) is a finite measure).
Double exponential kernel

\[ \beta(x) = \frac{1}{2(c_1^2 - c_2^2)} \left( c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2} \right) . \]

- \( r = 4, \) \((\beta * \nu)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} \nu_{xx} . \)
- Case 2 applies for \( s \geq 0 \) \((r > 5/2) . \)
Local Existence for Problem 1 and 2

**Double exponential kernel**

\[
\beta(x) = \frac{1}{2(c_1^2 - c_2^2)} (c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}).
\]

- \( r = 4, \quad (\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}. \)
- Case 2 applies for \( s \geq 0 \quad (r > 5/2). \)

**Gaussian kernels**

\[
\beta(x) = e^{-x^2/2}.
\]
\[
\beta(x) = (1 - x^2)e^{-x^2/2}.
\]

- Case 2 applies for \( s \geq 0 \quad (r > 5/2). \)
Local Existence for Problem 3

**Local Well-Posedness Theorem A**

Suppose \( s > 2, \ r \geq 2 \) and \( \varphi, \psi \in H^s(\mathbb{R}^2) \). Then there is some \( T > 0 \) s.t. the Cauchy problem is well posed with \( w(x, y, t) \) in \( C^2([0, T], H^s(\mathbb{R}^2)) \).

**Local Well-Posedness Theorem B**

Suppose \( s \geq 1, \ r > 3 \) and \( \varphi, \psi \in X^s \). Then there is some \( T > 0 \) s.t. the Cauchy problem is well posed with \( w(x, y, t) \) in \( C^2([0, T], X^s) \) where

\[
X^s = \{ w \in H^s(\mathbb{R}^2); \ w_x, w_y \in L^\infty(\mathbb{R}^2) \},
\]

with the norm

\[
\| w \|_{s, \infty} = \| w \|_s + \| w_x \|_\infty + \| w_y \|_\infty.
\]
There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} (\|u(t)\|_{H^s \cap L^\infty} + \|u_t(t)\|_{H^s \cap L^\infty}) < \infty.$$
Global Existence for Problem 1

There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} (\|u(t)\|_{H^s \cap L^\infty} + \|u_t(t)\|_{H^s \cap L^\infty}) < \infty.$$ 

Theorem: Blow up is in $L^\infty$

There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} \|u(t)\|_{\infty} < \infty.$$
Global Existence for Problem 1

Conservation of energy

Lemma: Let

\[ G(u) = \int_0^u g(p) dp. \]

For a solution \( u \) of the integro-differential equation, the energy

\[ E(t) = \| Pu_t \|^2 + \| u \|^2 + 2 \int_R G(u(x, t)) dx \]

is constant where

\[ Pv = \mathcal{F}^{-1}(|\xi|^{-1} \hat{\beta}(\xi))^{-1/2} \hat{\nu}(\xi). \]
Conservation of energy

**Lemma:** Let

\[ G(u) = \int_0^u g(p)dp. \]

For a solution \( u \) of the integro-differential equation, the energy

\[ E(t) = \| Pu_t \|^2 + \| u \|^2 + 2 \int_R G(u(x, t))dx \]

is constant where

\[ P\nu = \mathcal{F}^{-1}(|\xi|^{-1} \hat{\beta}(\xi))^{-1/2} \hat{\nu}(\xi). \]

\((P^2(\beta * \nu)_{xx} = -\nu)\)
Theorem A

Let $s \geq 0$ and $r > 3$. Let $\varphi, \psi \in H^s \cap L^\infty$, $P \psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $k$ such that

$$G(u) \geq -ku^2 \quad \text{for all} \quad u \in \mathbb{R},$$

then the Cauchy problem has a global solution in $C^2([0, \infty), H^s)$. 
Global Existence for Problem 1

**Theorem A**

Let \( s \geq 0 \) and \( r > 3 \). Let \( \varphi, \psi \in H^s \cap L^\infty \), \( P\psi \in L^2 \) and \( G(\varphi) \in L^1 \). If there is some \( k \) such that

\[
G(u) \geq -ku^2 \quad \text{for all} \quad u \in R,
\]

then the Cauchy problem has a global solution in \( C^2([0, \infty), H^s) \).

**Theorem B**

Let \( s \geq 0 \) and \( \beta_{xx} * v = h * v - \lambda v \) for some \( \lambda > 0 \) and for some \( h \in L^1 \cap L^\infty \). Let \( \varphi, \psi \in H^s \cap L^\infty \), \( P\psi \in L^2 \) and \( G(\varphi) \in L^1 \). If there is some \( C > 0 \) and \( q > 1 \) so that

\[
|g(r)|^q \leq CG(r)
\]

for all \( r \in R \); then the Cauchy problem has a global solution in \( C^2([0, \infty), H^s) \).
Global Existence for Problem 2

Let \( s > 1/2, \ r \geq 2 \). There is a global solution if and only if for any \( T < \infty \), we have

\[
\limsup_{t \to T^-} \| u_1(t) \|_\infty < \infty + \| u_2(t) \|_\infty < \infty.
\]

**Energy identity**

For solutions \((u_1, u_2)\) of integro-differential equations, the energy

\[
E(t) = \| Pu_1 t \|^2 + \| Pu_2 t \|^2 + \| u_1 \|^2 + \| u_2 \|^2 + 2 \int_R G(u_1, u_2) dx
\]

is constant.
Theorem A

Let $s > 1/2, \ r > 3$. Let $\varphi_i, \psi_i \in H^s, \ P\psi_i \in L^2 \ (i = 1, 2)$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $k > 0$ so that

$$G(a, b) \geq -k(a^2 + b^2),$$

for all $a, b \in \mathbb{R}$, then the Cauchy problem has a global solution $u_1, u_2$ in $C^2([0, \infty), H^s)$. 
Global Existence for Problem 2

Theorem B

Let $s > 1/2$, $h \in L^1 \cap L^\infty$.

Let $(\beta * \nu)_{xx} = h * \nu - \lambda \nu \text{ with } \lambda > 0$.

Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in H^s$, $P\psi_1, P\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$.

If there is some $C > 0$, $k \geq 0$ and $q_i > 1$ so that

$$|g_i(a, b)|^{q_i} \leq C [G(a, b) + k(a^2 + b^2)]$$

for all $i = 1, 2$, $a, b \in R$, then the Cauchy problem has a global solution $u_1, u_2$ in $C^2([0, \infty), H^s)$.
Global Existence for Problem 3

- There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^{-}} \| w_x(t) \|_{\infty} + \| w_y(t) \|_{\infty} < \infty.$$
Conservation of energy

Define the linear operator \( R^\alpha u = \mathcal{F}^{-1} \left( (\hat{\beta}(\xi))^{-\frac{\alpha}{2}} \hat{u}(\xi) \right) \). Then \( R^{-2} u = \beta \ast u \), and equation of motion takes the form

\[ R^2 \ddot{w} = (F_{w_x})_x + (F_{w_y})_y. \]

**Lemma:** Suppose that the solution of the Cauchy problem problem exists on \([0, T)\). If \( R\psi \in L^2 \) and \( F(|\nabla \phi|^2) \in L^1 \), then for any \( t \in [0, T) \) the energy

\[ E(t) = \frac{1}{2} \| Rw_t(t) \|^2 + \int_{\mathbb{R}^2} F(w^2_x(t) + w^2_y(t)) \, dx \, dy \]

is constant in \([0, T)\).
Global Existence for Problem 3

**Theorem**

Let $s \geq 1$ and $r > 4$. Let $\varphi, \psi \in X^s$, $R\psi \in L^2$ and $F(|\nabla \phi|^2) \in L^1$. If there is some $k > 0$ so that $F(u) \geq -ku$ for all $u \geq 0$, then the Cauchy problem has a global solution in $C^2([0, \infty), X^s)$. 
Theorem

Suppose $P\varphi, P\psi \in L^2$ and $G(\varphi) \in L^1$. If $E(0) < 0$ and there is some $\nu > 0$ such that

$$pF'(p) \leq 2(1 + 2\nu)F(p) \text{ for all } p \in R,$$

where $F(u) = G(u) + u^2/2$. Then the solution $u$ of the Cauchy problem blows up in finite time.
**Theorem**

- Let $s > 1/2$ and $r \geq 2$.
- Suppose that $P\varphi_1, P\varphi_2, P\psi_1, P\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$.
- Take $F(u_1, u_2) = \frac{1}{2}(u_1^2 + u_2^2) + G(u_1, u_2)$ and $f_i = \frac{\partial F}{\partial u_i}$ ($i = 1, 2$).

If $E(0) < 0$ and there exists some $\nu > 0$ satisfying

$$u_1 f_1(u_1, u_2) + u_2 f_2(u_1, u_2) \leq 2(1 + 2\nu)F(u_1, u_2),$$

then the solution $(u_1, u_2)$ blows up in finite time.
Theorem

Suppose that the solution, \( w \), of the CP exists, \( R\varphi, \ R\psi \in L^2 \) and \( F(|\nabla \phi|^2) \in L^1 \). If there exists \( \nu > 0 \) s.t.

\[
uf'(u) \leq (1 + 2\nu)F(u) \quad \text{for all } u \geq 0,
\]

and

\[
E(0) = \frac{1}{2} \|R\psi\|^2 + \int_{\mathbb{R}^2} F(|\nabla \phi|^2) dxdy < 0,
\]

then the solution \( w(x, y, t) \) blows up in finite time.
Ongoing studies / future work

Small amplitude solutions

\[ u_{tt} = \left[ \beta \ast (u + g(u)) \right]_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \]
\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \]

Questions:

- For small initial data, is there a global solution?
- What happens as \( t \) goes to infinity? (scattering problem)
- Energy wells?
Ongoing studies / future work

Travelling waves

\[ u_{tt} = \left[ \beta \ast (u + g(u)) \right]_{xx} \]

has a travelling wave solution \( u = \phi(x - ct) \) if

\[ c^2 \phi = \beta \ast (\phi + g(\phi)) \]

Questions:
- When do travelling waves exist?
- Are travelling waves stable?
Ongoing studies / future work

Double dispersive equations

\[ u_{tt} = \left[ \beta \ast (u + Lu + g(u)) \right]_{xx}, \]

where \( L \) is a suitable (pseudo) differential operator in \( x \).

- Example: For \( Lu = -u_{xx} \) and \( \beta(x) = \frac{1}{2}e^{-|x|} \) we get

\[ u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} = (g(u))_{xx} \]

- Weak dispersive limits.

- The case \( r < 2 \).