Global existence and blow-up results for some problems in nonlinear nonlocal elasticity

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Outline

- Nonlinear Nonlocal Elasticity.
- Three Nonlinear Nonlocal Wave Equations.
  - Longitudinal and Transverse Wave Motions.
  - Anti-Plane Shear Motion.
- Cauchy Problems.
  - Local well-posedness, Global existence, Blow-up.
- Ongoing studies / Future work.

Assumptions and Notation

- An isotropic, homogeneous hyperelastic medium.
- A stress-free undistorted state as the reference configuration.
- Position in the reference configuration: $\mathbf{X} = (X_1, X_2, X_3)$.
- Position at time t: $\mathbf{x}(\mathbf{X}, t) = (x_1(\mathbf{X}, t), x_2(\mathbf{X}, t), x_3(\mathbf{X}, t))$.
- Displacement: $\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$.

Local Theory of Elasticity

Constitutive Equation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{A}) \equiv \partial W(\mathbf{A})/\partial \mathbf{A}$$

$\boldsymbol{\sigma}$: nominal stress tensor (transpose of the first Piola-Kirchhoff stress tensor)

$W(\mathbf{A})$: strain energy density function

$\mathbf{A}(\mathbf{X}, t) = \text{Grad } \mathbf{x}(\mathbf{X}, t)$: deformation gradient

Equation of Motion

$$\rho_0 \ddot{\mathbf{x}} = \text{Div } \boldsymbol{\sigma}$$

ρ_0 : mass density, (no body forces)

(The symbol $\dot{}$ indicates the material time derivative)

Local Theory of Elasticity

Major Drawbacks of Local Theory

- Absence of any intrinsic length scale.
- Neglects the long range forces (important especially at small scales).
- It is incapable of predicting, for instance,
  - the dispersive nature of harmonic waves in crystal lattices,
  - the boundedness of the stress field near the tip of a crack.

Generalized Theories of Elasticity

- Micropolar theories.
- Strain elasticity (or higher-order gradient) theories.
- Peridynamic theory.
- Nonlocal elasticity theory. (Kröner, Eringen, Edelen, Kunin, Rogula)

Nonlocal Theory of Elasticity

Constitutive Equation

Stress at a point depends on the strain field at every point in the body.

$$\mathbf{S} = \mathbf{S}(\mathbf{X}, t) \equiv \int \beta(|\mathbf{X} - \mathbf{Y}|)\sigma(\mathbf{A}(\mathbf{Y}, t))d\mathbf{Y}$$

$\mathbf{S}$: stress tensor, $\beta(|\mathbf{X} - \mathbf{Y}|)$: kernel function

The only difference between the two theories is due to the constitutive equations.

Equation of Motion

$$\rho_0 \ddot{\mathbf{x}} = \text{Div } \mathbf{S}$$

Henceforth, all quantities appear in non-dimensional form and $\rho_0 = 1$

Case 1: Longitudinal Motion

Equation of Motion

Consider the longitudinal motion:

$$x_1 = X_1 + U(X_1, t), \qquad x_2 = X_2, \qquad x_3 = X_3$$

The displacement field:

$$u_1 \equiv U(X_1, t), \qquad u_2 = u_3 \equiv 0$$

Equation of motion:

$$U_{tt} = (S(U_x))_x$$

with $x \equiv X_1$ and stress component S.

Case 1: Longitudinal Motion

Constitutive Equation in Local Theory

$$\sigma(U_x)(x,t) = W'(U_x(x,t))$$

W: strain energy function (with $W(0) = W'(0) = 0$)

Constitutive Equation in Nonlocal Theory

$$S(U_x)(x,t) = \int_{-\infty}^{\infty} \beta(x-y) W'(U_x(y,t)) \mathrm{d}y$$

Case 1: Longitudinal Motion

Nonlocal Nonlinear PDE for Longitudinal Waves

1D equation of motion:

$$U_{tt} = \left(\int_{-\infty}^{\infty} \beta(x-y)W'(U_x(y,t))\mathrm{d}y\right)_x$$

Differentiate w.r.t. x

$$U_{xtt} = \left(\int_{-\infty}^{\infty} \beta(x-y)W'(U_x(y,t))\mathrm{d}y\right)_{xx}$$

Change variables $U_x = u$, and write $W'(u) = W'(0)[u + g(u)]$

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Assumption

$$0 \leq \hat{\beta}(\xi) \leq C(1 + \xi^2)^{-r/2}$$

Dirac Measure

$$\beta = \text{Dirac measure}, \; r = 0.$$

$$u_{tt} - u_{xx} = g(u)_{xx}.$$

- Equation of motion is a nonlinear wave equation.

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Triangular Kernel

$$\beta(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| \geq 1. \end{cases}$$

- $\hat{\beta}(\xi) = \frac{4}{\xi^2} \sin^2(\frac{\xi}{2})$, $r = 2$.
- $(\beta * v)_{xx} = v(x+1) - 2v(x) + v(x-1)$.
- Equation of motion is a differential-difference equation.

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Exponential Kernel

$$\beta(x) = \frac{1}{2} e^{-|x|}.$$

- $\hat{\beta}(\xi) = (1 + \xi^2)^{-1}$, $r = 2$.
- $(\beta * v)_{xx} = (1 - D_x^2)^{-1} v_{xx}$.
- Equation of motion: Improved Boussinesq equation

$$u_{tt} - u_{xx} - u_{xxtt} = (g(u))_{xx}$$

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Double Exponential Kernel

$$\beta(x) = \frac{1}{2(c_1^2 - c_2^2)}(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}).$$

- $\hat{\beta}(\xi) = (1 + \gamma_1 \xi^2 + \gamma_2 \xi^4)^{-1}$, $r = 4$.
- $(\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}$.
- Equation of motion: Higher-order Boussinesq equation (Duruk, Erkip, Erbay *IMA J. Appl. Math.* (2009))

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = (g(u))_{xx}$$

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Gaussian Kernels

$$\beta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \hat{\beta}(\xi) = e^{-\xi^2/2}.$$

$$\beta(x) = \frac{1}{\sqrt{2\pi}} (1 - x^2) e^{-x^2/2}, \qquad \hat{\beta}(\xi) = \xi^2 e^{-\xi^2/2}.$$

- Equation of motion: an integro-differential equation.

Examples of 1D Kernel Functions

Nonlinear Nonlocal Wave Equation

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

Gaussian Kernels

$$\beta(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \hat{\beta}(\xi) = e^{-\xi^2/2}.$$

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- Equation of motion: an integro-differential equation.

All these are Boussinesq type nonlocal nonlinear PDEs.

Equation of Motion

Consider the transverse motion:

$$x_1 = X_1, \qquad x_2 = X_2 + U(X_1, t), \qquad x_3 = X_3 + V(X_1, t)$$

The displacement field:

$$u_1 \equiv 0, \qquad u_2 \equiv U(X_1, t), \qquad u_3 \equiv V(X_1, t)$$

Equation of motion:

$$\begin{aligned} U_{tt} &= (P(U_x, V_x))_x \\ V_{tt} &= (Q(U_x, V_x))_x \end{aligned}$$

with $x \equiv X_1$ and stress components P, Q.

Case 2: Transverse Motion

Constitutive Equation in Nonlocal Theory

$$P(U_x, V_x) = \int_{-\infty}^{\infty} \beta(x-y) \frac{\partial W(U_x, V_x)}{\partial U_x} \mathrm{d}y$$
$$Q(U_x, V_x) = \int_{-\infty}^{\infty} \beta(x-y) \frac{\partial W(U_x, V_x)}{\partial V_x} \mathrm{d}y$$

W: strain energy function (with $W(0,0) = 0$, $\nabla W(0,0) = 0$)
For isotropic case $W = F(U_x^2 + V_x^2)$.

Case 2: Transverse Motion

Nonlocal Nonlinear PDE for Transverse Waves

$$u_{tt} = \left(\beta * \frac{\partial F}{\partial u}\right)_{xx}$$
$$v_{tt} = \left(\beta * \frac{\partial F}{\partial v}\right)_{xx}$$

where $u = U_x$, $v = V_x$.

Case 3: Anti-Plane Shear Motion

Equation of Motion

Consider the anti-plane shear motion:

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + w(X_1, X_2, t)$$

The displacement field:

$$u_1 = u_2 \equiv 0, \quad u_3 = w(X_1, X_2, t)$$

Equation of motion:

$$w_{tt} = (P(w_x, w_y))_x + (Q(w_x, w_y))_y$$

with $x \equiv X_1$, $y \equiv X_2$ and stress components P, Q.

Case 3: Anti-Plane Shear Motion

Constitutive Equation in Nonlocal Theory

$$P(w_x, w_y) = \int_{-\infty}^{\infty} \beta(x-y) \frac{\partial W(w_x, w_y)}{\partial w_x} dy$$
$$Q(w_x, w_y) = \int_{-\infty}^{\infty} \beta(x-y) \frac{\partial W(w_x, w_y)}{\partial w_y} dy$$

W: strain energy function (with $W(0,0) = 0$, $\nabla W(0,0) = 0$)
For isotropic case $W = F(w_x^2 + w_y^2)$.

Nonlocal Nonlinear PDE for Shear Waves

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y$$

Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y$$

Assumption

$$0 \leq \hat{\beta}(\xi) \leq C(1 + |\xi|^2)^{-r/2}$$

The Gaussian Kernel

$$\beta(x, y) = (2\pi)^{-1} e^{-(x^2+y^2)/2}$$

- $\widehat{\beta}(\xi_1, \xi_2) = e^{-(\xi_1^2+\xi_2^2)/2}$
- Take any r.

Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y$$

The Modified Bessel Function Kernel

$$\beta(x,y) = (2\pi)^{-1} K_0(\sqrt{x^2+y^2})$$

(K_0 : the modified Bessel function of the second kind of order zero)

- $\widehat{\beta}(\xi_1, \xi_2) = (1 + \xi_1^2 + \xi_2^2)^{-1}$
- $r = 2$
- Letting $F(s) = \frac{1}{2}s + G(s)$

$$w_{tt} - \Delta w - \Delta w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

Examples of 2D Kernel Functions

Nonlinear Nonlocal PDE for Shear Motion

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y$$

The bi-Helmholtz Type Kernel

$$\beta(x,y) = \frac{1}{2\pi(c_1^2 - c_2^2)}[K_0(\sqrt{x^2+y^2}/c_1) - K_0(\sqrt{x^2+y^2}/c_2)]$$

- $\widehat{\beta}(\xi_1, \xi_2) = [1 + \gamma_1(\xi_1^2 + \xi_2^2) + \gamma_2(\xi_1^2 + \xi_2^2)^2]^{-1}$
- $r = 4$.

$$w_{tt} - \Delta w - \gamma_1 \Delta w_{tt} + \gamma_2 \Delta^2 w_{tt} = \left(\frac{\partial G}{\partial w_x}\right)_x + \left(\frac{\partial G}{\partial w_y}\right)_y .$$

Cauchy Problems

Problem 1: Longitudinal Motion

$$u_{tt} = [\beta * (u + g(u))]_{xx}, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

Duruk, Erbay, Erkip *Nonlinearity* (2010)

Cauchy Problems

Problem 2: Transverse Motion

$$
\begin{aligned}
u_{1tt} &= (\beta * (u_1 + g_1(u_1, u_2)))_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0 \\
u_{2tt} &= (\beta * (u_2 + g_2(u_1, u_2)))_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0 \\
u_1(x,0) &= \varphi_1(x), \quad u_{1t}(x,0) = \psi_1(x) \\
u_2(x,0) &= \varphi_2(x), \quad u_{2t}(x,0) = \psi_2(x).
\end{aligned}
$$

Duruk, Erbay, Erkip *J. Differential Equations* (2011)

Exactness Condition

$$\frac{\partial g_1}{\partial u_2} = \frac{\partial g_2}{\partial u_1}$$

Cauchy Problems

Problem 3: Anti-Plane Shear Motion

$$w_{tt} = \left(\beta * \frac{\partial F}{\partial w_x}\right)_x + \left(\beta * \frac{\partial F}{\partial w_y}\right)_y, \quad (x, y) \in \mathbb{R}^2, \quad t > 0,$$
$$w(x, y, 0) = \varphi(x, y), \quad w_t(x, y, 0) = \psi(x, y)$$

Erbay, Erbay, Erkip *Nonlinearity* Submitted

Local Existence for Problem 1

$H^s \cap L^\infty$ valued ODE system

$$
\begin{aligned}
u_t &= v, \qquad u(0) = \varphi \\
v_t &= [\beta * (u + g(u))]_{xx}, \qquad v(0) = \psi \ .
\end{aligned}
$$

Local Bound and Lipschitz Condition

Lemma: Let $g \in C^{[s]+1}(R)$, $s \geq 0$. Then there is some constant $K(M)$ such that for all $u \in H^s \cap L^\infty$ with $\|u\|_\infty \leq M$, we have

$$\|g(u)\|_s \leq K(M)\|u\|_s \ ,$$

and some other constant $J(M)$ such that for all $u, v \in H^s \cap L^\infty$ with $\|u\|_\infty + \|u\|_s \leq M$, $\|v\|_\infty + \|v\|_s \leq M$ we have

$$\|g(u) - g(v)\|_s \leq J(M)\|u - v\|_s \ .$$

Local Existence for Problem 1

Local Well-Posedness Theorem

Consider the Cauchy problem Let $g \in C^{[s]+1}(R) \quad g(0) = 0$.
There is some $T > 0$ such that the Cauchy problem is well-posed with solution $u \in C^2([0, T], H^s \cap L^\infty)$ for initial data $\varphi, \psi \in H^s \cap L^\infty$ in any of the following cases.

- Case 1: $s > 1/2$ and $r \geq 2$,
- Case 2: $s \geq 0$ and $r > 5/2$,
- Case 3: $s \geq 0$ and β_{xx} is a finite measure on R.

Local Well-Posedness Theorem for Problem 2

A similar theorem holds for Problem 2, the coupled system of transverse motion.

Triangular kernel

$$\beta(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & |x| \geq 1. \end{cases}$$

- $r = 2, \quad (\beta * v)_{xx} = v(x+1) - 2v(x) + v(x-1).$
- Case 3 applies for $s \geq 0$ (β_{xx} is a finite measure).

Local Existence for Problems 1 and 2

Triangular kernel

$$\beta(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & |x| \ge 1. \end{cases}$$

- $r = 2, \quad (\beta * v)_{xx} = v(x+1) - 2v(x) + v(x-1).$
- Case 3 applies for $s \ge 0$ (β_{xx} is a finite measure).

Exponential kernel

$$\beta(x) = \frac{1}{2} e^{-|x|}.$$

- $r = 2, \quad (\beta * v)_{xx} = \beta * v - v.$
- Case 3 applies for $s \ge 0$ (β_{xx} is a finite measure).

Local Existence for Problem 1 and 2

Double exponential kernel

$$\beta(x) = \frac{1}{2(c_1^2 - c_2^2)}(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}).$$

- $r = 4, \quad (\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}.$
- Case 2 applies for $s \geq 0 \;\; (r > 5/2)$.

Local Existence for Problem 1 and 2

Double exponential kernel

$$\beta(x) = \frac{1}{2(c_1^2 - c_2^2)}(c_1 e^{-|x|/c_1} - c_2 e^{-|x|/c_2}).$$

- $r = 4, \quad (\beta * v)_{xx} = (1 - \gamma_1 D_x^2 + \gamma_2 D_x^4)^{-1} v_{xx}.$
- Case 2 applies for $s \geq 0 \ \ (r > 5/2)$.

Gaussian kernels

$$\beta(x) = e^{-x^2/2}.$$

$$\beta(x) = (1 - x^2)e^{-x^2/2}.$$

- Case 2 applies for $s \geq 0 \ \ (r > 5/2)$.

Local Existence for Problem 3

Local Well-Posedness Theorem A

Suppose $s > 2$, $r \geq 2$ and $\varphi, \psi \in H^s(\mathbb{R}^2)$. Then there is some $T > 0$ s.t. the Cauchy problem is well posed with $w(x, y, t)$ in $C^2([0, T], H^s(\mathbb{R}^2))$.

Local Well-Posedness Theorem B

Suppose $s \geq 1$, $r > 3$ and $\varphi, \psi \in X^s$. Then there is some $T > 0$ s.t. the Cauchy problem is well posed with $w(x, y, t)$ in $C^2([0, T], X^s)$ where

$$X^s = \{w \in H^s(\mathbb{R}^2);\ w_x, w_y \in L^\infty(\mathbb{R}^2)\},$$

with the norm

$$\|w\|_{s,\infty} = \|w\|_s + \|w_x\|_\infty + \|w_y\|_\infty.$$

Global Existence for Problem 1

- There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} (\|u(t)\|_{H^s \cap L^\infty} + \|u_t(t)\|_{H^s \cap L^\infty}) < \infty \ .$$

- There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} (\|u(t)\|_{H^s \cap L^\infty} + \|u_t(t)\|_{H^s \cap L^\infty}) < \infty .$$

Theorem: Blow up is in L^∞

There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} \|u(t)\|_\infty < \infty .$$

Global Existence for Problem 1

Conservation of energy

Lemma: Let

$$G(u) = \int_0^u g(p)dp.$$

For a solution u of the integro-differential equation, the energy

$$E(t) = \|Pu_t\|^2 + \|u\|^2 + 2\int_R G(u(x,t))dx$$

is constant where

$$Pv = \mathcal{F}^{-1}(|\xi|^{-1}\hat{\beta}(\xi))^{-1/2}\hat{v}(\xi).$$

Global Existence for Problem 1

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$(P^2(\beta * v)_{xx} = -v)$

Global Existence for Problem 1

Theorem A

Let $s \geq 0$ and $r > 3$. Let $\varphi, \psi \in H^s \bigcap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some k such that

$$G(u) \geq -ku^2 \qquad \text{for all} \qquad u \in R \ ,$$

then the Cauchy problem has a global solution in $C^2([0,\infty)\,, H^s)$.

Global Existence for Problem 1

Theorem A

Let $s \geq 0$ and $r > 3$. Let $\varphi, \psi \in H^s \bigcap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some k such that

$$G(u) \geq -ku^2 \qquad \text{for all} \qquad u \in R \ ,$$

then the Cauchy problem has a global solution in $C^2([0,\infty)\,, H^s)$.

Theorem B

Let $s \geq 0$ and $\beta_{xx} * v = h * v - \lambda v$ for some $\lambda > 0$ and for some $h \in L^1 \cap L^\infty$. Let $\varphi, \psi \in H^s \cap L^\infty$, $P\psi \in L^2$ and $G(\varphi) \in L^1$. If there is some $C > 0$ and $q > 1$ so that

$$|g\,(r)|^q \leq CG\,(r)$$

for all $r \in R$; then the Cauchy problem has a global solution in $C^2([0,\infty)\,, H^s)$.

Global Existence for Problem 2

- Let $s > 1/2$, $r \geq 2$. There is a global solution if and only if

  for any $T < \infty$, we have $\limsup_{t \to T^-} (\|u_1(t)\|_\infty < \infty + \|u_2(t)\|_\infty < \infty$.

Energy identity

For solutions (u_1, u_2) of integro-differential equations, the energy

$$E(t) = \|Pu_{1t}\|^2 + \|Pu_{2t}\|^2 + \|u_1\|^2 + \|u_2\|^2 + 2\int_R G(u_1, u_2)dx$$

is constant.

Global Existence for Problem 2

Theorem A

Let $s > 1/2$, $r > 3$. Let $\varphi_i, \psi_i \in H^s$, $\quad P\psi_i \in L^2$ $(i = 1, 2)$ and $G(\varphi_1, \varphi_2) \in L^1$. If there is some $k > 0$ so that

$$G(a, b) \geq -k(a^2 + b^2),$$

for all $a, b \in R$, then the Cauchy problem has a global solution u_1, u_2 in $C^2([0, \infty), H^s)$.

Global Existence for Problem 2

Theorem B

- Let $s > 1/2$, $h \in L^1 \cap L^\infty$.
- Let $(\beta * v)_{xx} = h * v - \lambda v$ with $\lambda > 0$.
- Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \in H^s$, $P\psi_1, P\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$.

If there is some $C > 0$, $k \geq 0$ and $q_i > 1$ so that

$$|g_i(a, b)|^{q_i} \leq C[G(a, b) + k(a^2 + b^2)]$$

for all $i = 1, 2$, $a, b \in R$, then the Cauchy problem has a global solution u_1, u_2 in $C^2([0, \infty), H^s)$.

Global Existence for Problem 3

- There is a global solution if and only if for any $T < \infty$, we have

$$\limsup_{t \to T^-} \|w_x(t)\|_\infty + \|w_y(t)\|_\infty < \infty \ .$$

Conservation of energy

Define the linear operator $R^{\alpha}u = \mathcal{F}^{-1}\left((\widehat{\beta}(\xi))^{-\frac{\alpha}{2}}\widehat{u}(\xi)\right)$. Then $R^{-2}u = \beta * u$, and equation of motion takes the form

$$R^2 w_{tt} = (F_{w_x})_x + (F_{w_y})_y.$$

Lemma: Suppose that the solution of the Cauchy problem problem exists on $[0, T)$. If $R\psi \in L^2$ and $F(|\nabla\phi|^2) \in L^1$, then for any $t \in [0, T)$ the energy

$$E(t) = \frac{1}{2}\|Rw_t(t)\|^2 + \int_{\mathbb{R}^2} F(w_x^2(t) + w_y^2(t))dxdy$$

is constant in $[0, T)$.

Global Existence for Problem 3

Theorem

Let $s \geq 1$ and $r > 4$. Let $\varphi, \psi \in X^s$, $R\psi \in L^2$ and $F(|\nabla\phi|^2) \in L^1$. If there is some $k > 0$ so that $F(u) \geq -ku$ for all $u \geq 0$, then the Cauchy problem has a global solution in $C^2([0, \infty), X^s)$.

Blow-up in Finite Time for Problem 1

Theorem

Suppose $P\varphi, P\psi \in L^2$ and $G(\varphi) \in L^1$. If $E(0) < 0$ and there is some $\nu > 0$ such that

$$pF'(p) \leq 2(1+2\nu)F(p) \text{ for all } p \in R,$$

where $F(u) = G(u) + u^2/2$. Then the solution u of the Cauchy problem blows up in finite time.

Theorem

- Let $s > 1/2$ and $r \geq 2$.
- Suppose that $P\varphi_1, P\varphi_2, P\psi_1, P\psi_2 \in L^2$ and $G(\varphi_1, \varphi_2) \in L^1$.
- Take $F(u_1, u_2) = \frac{1}{2}(u_1^2 + u_2^2) + G(u_1, u_2)$ and $f_i = \frac{\partial F}{\partial u_i}$ $(i = 1, 2)$.

If $E(0) < 0$ and there exists some $\nu > 0$ satisfying

$$u_1 f_1(u_1, u_2) + u_2 f_2(u_1, u_2) \leq 2(1 + 2\nu) F(u_1, u_2),$$

then the solution (u_1, u_2) blows up in finite time.

Blow-up in Finite Time for Problem 3

Theorem

Suppose that the solution, w, of the CP exists, $R\varphi$, $R\psi \in L^2$ and $F(|\nabla\phi|^2) \in L^1$. If there exists $\nu > 0$ s.t.

$$uF'(u) \leq (1+2\nu)F(u) \quad \text{for all } u \geq 0,$$

and

$$E(0) = \frac{1}{2}\|R\psi\|^2 + \int_{\mathbb{R}^2} F(|\nabla\phi|^2)dxdy < 0 \ ,$$

then the solution $w(x, y, t)$ blows up in finite time.

Ongoing studies / future work

Small amplitude solutions

$$u_{tt} = [\beta * (u + g(u))]_{xx}, \quad x \in R, \quad t > 0,$$
$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

Questions:

- For small initial data, is there a global solution ?
- What happens as t goes to infinity ? (scattering problem)
- Energy wells ?

Ongoing studies / future work

Travelling waves

$$u_{tt} = [\beta * (u + g(u))]_{xx}$$

has a travelling wave solution $u = \phi(x - ct)$ if

$$c^2\phi = \beta * (\phi + g(\phi))$$

Questions:

- When do travelling waves exist ?
- Are travelling waves stable ?

Ongoing studies / future work

Double dispersive equations

$$u_{tt} = [\beta * (u + Lu + g(u))]_{xx},$$

where L is a suitable (pseudo) differential operator in x.

- Example: For $Lu = -u_{xx}$ and $\beta(x) = \frac{1}{2}e^{-|x|}$ we get

$$u_{tt} - u_{xx} + u_{xxxx} - u_{xxtt} = (g(u))_{xx}$$

- Weak dispersive limits.

- The case $r < 2$.