The Cauchy Problem for a One Dimensional Nonlinear Peridynamic Model

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The Peridynamic Equation

\[ u_{tt} = \int f(u(y, t) - u(x, t), y - x)dy \]
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Simplifications

- Dimension = 1
- \( f(\eta, \zeta) = \alpha(\zeta)w(\eta) \) with \( \alpha \) even, \( w \) odd, \( w(0) = 0 \).
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Cauchy Problem

\[ u_{tt} = \int_{\mathbb{R}} \alpha(y - x)w(u(y, t) - u(x, t))dy, \quad x \in \mathbb{R}, \quad t > 0 \]

\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}. \]
Questions

- Local well-posedness of the Cauchy problem,
- Existence of a global solution
- Conditions for finite-time blow-up of the solution.
Local well-posedness:

**Theorem**

Assume that $\alpha \in L^1(\mathbb{R})$, $w \in C^1(\mathbb{R})$ (or $w$ is locally Lipschitz). Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0, T], X)$ for initial data $\varphi, \psi \in X$ with

$$
X = C_b(\mathbb{R})
$$

$$
X = L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}).
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Theorem
Assume that $\alpha \in L^1(\mathbb{R})$, $w \in C^2(\mathbb{R})$ (or $w'$ is locally Lipschitz). Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0, T], X)$ for initial data $\varphi, \psi \in X$ with

\[ X = C^1_b(\mathbb{R}) \]
\[ X = W^{1,p}(\mathbb{R}). \]
Solution satisfies

\[ u(x, t) = \varphi(x) + t\psi(x) \]
\[ + \int_0^t (t - \tau) \int_\mathbb{R} \alpha(y - x)w(u(y, \tau) - u(x, \tau)) dy d\tau \]

Let

\[ (Ku)(x, t) = \int_\mathbb{R} \alpha(y - x)w(u(y, t) - u(x, t)) dy. \]

Show that \( K : X \to X \) is locally Lipschitz.
Theorem: The general peridynamic problem

Assume that \( f(0, \eta) = 0 \) and \( f(\zeta, \eta) \) is continuously differentiable in \( \eta \) for almost all \( \zeta \). Moreover, suppose that for each \( R > 0 \), there are integrable functions \( \Lambda_1^R, \Lambda_2^R \) satisfying

\[
|f(\zeta, \eta)| \leq \Lambda_1^R(\zeta), \quad |f_\eta(\zeta, \eta)| \leq \Lambda_2^R(\zeta)
\]

for almost all \( \zeta \) and for all \( |\eta| \leq 2R \). Then there is some \( T > 0 \) such that the Cauchy problem is well posed with solution in \( C^2([0, T], C_b(\mathbb{R})) \) for initial data \( \varphi, \psi \in C_b(\mathbb{R}) \).
Assume that $\alpha \in L^1(\mathbb{R})$, $w(\zeta) = \zeta^k$. Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0, T], H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ for initial data $\varphi, \psi \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $s > 0$. 
Theorem

Blow up occurs only if \( \limsup_{t \to T_{\text{max}}} \| u(t) \|_{\infty} = \infty. \)
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Theorem: Sub-linear Case
If the nonlinear term $w$ satisfies $|w(\eta)| \leq a |\eta| + b$ for all $\eta \in \mathbb{R}$, then there is a global solution.
Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^1(\mathbb{R})$. If $u$ satisfies the Cauchy problem on $[0, T)$ with initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the energy

$$E(t) = \frac{1}{2} \| u_t(t) \|_2^2 + \frac{1}{2} \int_{\mathbb{R}^2} \alpha(y-x) \ W(u(y,t)-u(x,t)) \, dy \, dx,$$

is constant for $t \in [0, T)$, where $W(\eta) = \int_0^\eta w(\rho) \, d\rho$. 
Global Existence

Theorem

Assume that $\alpha \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\alpha \geq 0$ a.e.; $w \in C^1(\mathbb{R})$ and $W \geq 0$. If there is some $q \geq \frac{4}{3}$ and $C > 0$ so that

$$|w(\eta)|^q \leq CW(\eta) \quad (*)$$

for all $\eta \in \mathbb{R}$, then there is a global solution for initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

For $w(\eta) = |\eta|^{\nu-1}\eta$, $\quad (*)$ is satisfied if and only if $\nu \leq 3$. 

Idea of proof: Energy density function $e(x,t) = \frac{1}{2}u_t(x,t)^2 + \int_{\mathbb{R}} \alpha(y-x)W(u(y,t) - u(x,t)) dy$. 

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Global Existence

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$$e(x, t) = \frac{1}{2}(u_t(x, t))^2 + \int_\mathbb{R} \alpha(y - x) W(u(y, t) - u(x, t)) \, dy.$$
Theorem
Let $\alpha \geq 0$ a.e. If there is some $\nu > 0$ such that
\[ \eta w(\eta) \leq 2(1 + 2\nu) W(\eta) \quad \text{for all} \quad \eta \in \mathbb{R}, \]
and $E(0) < 0$ then the solution $u$ blows up in finite time.
Theorem

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\]

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Lemma (Levine 1974)

Suppose \( H(t), t \geq 0 \) is a positive, \( C^2 \) function satisfying

\[
H''(t)H(t) - (1 + \nu)(H'(t))^2 \geq 0 \quad \text{for some} \quad \nu > 0.
\]

If \( H(0) > 0 \) and \( H'(0) > 0 \), then \( H(t) \to \infty \) as \( t \to t_1 \) for some \( t_1 \leq H(0)/\nu H'(0) \).