The Cauchy Problem for a One Dimensional Nonlinear Peridynamic Model

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Introduction

The Peridynamic Equation

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Simplifications

- Dimension $= 1$
- $f(\eta, \zeta) = \alpha(\zeta)w(\eta)$ with α even, w odd, $w(0) = 0$.

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Cauchy Problem

$$u_{tt} = \int_{\mathbb{R}} \alpha(y - x)w(u(y,t) - u(x,t))dy, \qquad x \in \mathbb{R}, \quad t > 0$$

$$u(x,0) = \varphi(x), \qquad u_t(x,0) = \psi(x), \qquad x \in \mathbb{R}.$$

Questions

- Local well-posedness of the Cauchy problem,
- Existence of a global solution
- Conditions for finite-time blow-up of the solution.

Local well-posedness:

Theorem

Assume that $\alpha \in L^1(\mathbb{R})$, $w \in C^1(\mathbb{R})$ (or w is locally Lipschitz). Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0, T], X)$ for initial data $\varphi, \psi \in X$ with

$$\begin{aligned} X &= C_b(\mathbb{R}) \\ X &= L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}). \end{aligned}$$

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$$\begin{aligned} X &= C_b^1(\mathbb{R}) \\ X &= W^{1,p}(\mathbb{R}). \end{aligned}$$

Idea of Proof

Solution satisfies

$$\begin{aligned} u(x,t) &= \varphi(x) + t\psi(x) \\ &+ \int_0^t (t-\tau) \int_{\mathbb{R}} \alpha(y-x) w(u(y,\tau) - u(x,\tau)) dy d\tau \end{aligned}$$

Let

$$(Ku)(x,t) = \int_{\mathbb{R}} \alpha(y-x) w(u(y,t) - u(x,t)) dy.$$

Show that $K : X \to X$ is locally Lipscitz.

Local Existence

Theorem: The general peridynamic problem

Assume that $f(0,\eta) = 0$ and $f(\zeta,\eta)$ is continuously differentiable in η for almost all ζ. Moreover, suppose that for each $R > 0$, there are integrable functions $\Lambda_1^{\mathbb{R}}$, Λ_2^R satisfying

$$|f(\zeta,\eta)| \leq \Lambda_1^R(\zeta), \qquad |f_\eta(\zeta,\eta)| \leq \Lambda_2^R(\zeta)$$

for almost all ζ and for all $|\eta| \leq 2R$. Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0,T], C_b(\mathbb{R}))$ for initial data $\varphi, \psi \in C_b(\mathbb{R})$.

Theorem

Assume that $\alpha \in L^1(\mathbb{R})$, $w(\zeta) = \zeta^k$.Then there is some $T > 0$ such that the Cauchy problem is well posed with solution in $C^2([0, T], H^s(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ for initial data $\varphi, \psi \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $s > 0$.

Global Existence

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Theorem: Sub-linear Case

If the nonlinear term w satisfies $|w(\eta)| \leq a|\eta| + b$ for all $\eta \in \mathbb{R}$, then there is a global solution.

Global Existence

Energy Identity

Assume that $\alpha \in L^1(\mathbb{R})$ and $w \in C^1(\mathbb{R})$. If u satisfies the Cauchy problem on $[0, T)$ with initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then the energy

$$E(t) = \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^2} \alpha(y-x) \; W(u(y,t) - u(x,t)) \, dydx,$$

is constant for $t \in [0, T)$, where $W(\eta) = \int_0^\eta w(\rho) d\rho$.

Global Existence

Theorem

Assume that $\alpha \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\alpha \geq 0$ a.e.; $w \in C^1(\mathbb{R})$ and $W \geq 0$. If there is some $q \geq \frac{4}{3}$ and $C > 0$ so that

$$|w(\eta)|^q \leq CW(\eta) \qquad (*)$$

for all $\eta \in \mathbb{R}$, then there is a global solution for initial data $\varphi, \psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

For $w(\eta) = |\eta|^{\nu-1}\eta$, $\quad (*)$ is satisfied if and only if $\nu \leq 3$.

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Idea of proof: Energy density function

$e(x,t) = \frac{1}{2}(u_t(x,t))^2 + \int_{\mathbb{R}} \alpha(y-x)\ W\left(u\left(y,t\right) - u\left(x,t\right)\right) dy.$

Blow-up

Theorem

Let $\alpha \geq 0$ a.e. If there is some $\nu > 0$ such that

$$\eta w(\eta) \leq 2(1+2\nu) W(\eta) \quad \text{for all} \quad \eta \in \mathbb{R},$$

and $E(0) < 0$ then the solution u blows up in finite time.

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Lemma (Levine 1974)

Suppose $H(t)$, $t \geq 0$ is a positive, C^2 function satisfying $H''(t)H(t) - (1+\nu)(H'(t))^2 \geq 0$ for some $\nu > 0$. If $H(0) > 0$ and $H'(0) > 0$, then $H(t) \to \infty$ as $t \to t_1$ for some $t_1 \leq H(0)/\nu H'(0)$.