Variational Theory for Nonlocal Boundary Value Problems

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The scalar nonlocal problem

- We study scalar stationary nonlocal problems formulated as: Given b, find u satisfying certain volume constraints such that

$$\mathcal{L}(u)(\mathbf{x}) = b(\mathbf{x}), \qquad \mathbf{x} \in \Omega,$$

  where the linear nonlocal operator $\mathcal{L}$ is convolution type and is given by

$$\mathcal{L}(u)(\mathbf{x}) := -\int_{\overline{\overline{\Omega}}} C(\mathbf{x} - \mathbf{x}') \left(u(\mathbf{x}') - u(\mathbf{x})\right) d\mathbf{x}'.$$

- Here $\overline{\overline{\Omega}} = \Omega \cup \mathcal{B}\Omega$ where $\Omega \subset \mathbb{R}^d$, a bounded domain with a nonlocal boundary $\mathcal{B}\Omega$.
- We focus our attention when C is radial, locally integrable, compactly supported and $C(r) > 0$ on $[0, \delta)$.

Hilbert Space setting

- The linear operator

$$\mathcal{L} : L^2(\Omega) \to L^2(\Omega)$$

is bounded and self-adjoint, i.e.

$$\|\mathcal{L}u\|_{L^2} \leq C\|u\|_{L^2}$$

This is so because the operator is convolution type and C is locally integrable.

- So given a closed subspace V of $L^2(\overline{\overline{\Omega}})$, if we show that

$$(\mathcal{L}u, u)_{L^2(\overline{\overline{\Omega}})} \geq \lambda \|u\|^2_{L^2(\overline{\overline{\Omega}})}, \quad \text{for all } u \in V$$

for some $\lambda > 0$, then for any $b \in L^2(\Omega)$ the equation $\mathcal{L}u = b$ will have a unique variational solution.

- Indeed, the solution is the minimizer of

$$\min_{u \in V \subset L^2(\overline{\overline{\Omega}})} E(u), \qquad E(u) = (\mathcal{L}u, u)_{L^2} - (b, u)_{L^2}$$

Weak form

Write the weak form:
Given $b \in L^2$ find $u \in V$ such that

$$a(u,v) = (b,v) \qquad \forall v \in V,$$

where the bilinear form

$$a(u,v) := -\int_{\overline{\overline{\Omega}}} \left\{ \int_{\overline{\overline{\Omega}}} C(\mathbf{x},\mathbf{x}') \left[u(\mathbf{x}') - u(\mathbf{x})\right] \, d\mathbf{x}' \right\} v(\mathbf{x}) \, d\mathbf{x}' d\mathbf{x}.$$

Rewrite $a(u,v)$ as

$$a(u,v) = \frac{1}{2} \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} C(|\mathbf{x} - \mathbf{x}'|) \left[u(\mathbf{x}') - u(\mathbf{x})\right] \left[v(\mathbf{x}') - v(\mathbf{x})\right] \, d\mathbf{x}' \; d\mathbf{x},$$

- the bilinear form $a(u,v)$ is symmetric.
- $a(u,u) = (\mathcal{L}u, u)_{L^2}$.

Our interest

- Will give 3 closed subspaces V of L^2 for which Poincaré's inequality (coercivity) holds.
  - The choice of V depends on the boundary condition imposed and
  - will hol for selected L^1 kernels.
- On the way, we (asymptotically) quantify some quantities in terms of the "horizon": the smallest/largest eigenvalue, and so, an upper bound for the condition number as reported in (Aksoylu & Parks).

Spaces of Solutions: V

- *Pure Dirichlet boundary condition for surrounding nonlocal boundary:*

$$V_D^s := \{v \in L^2(\overline{\overline{\Omega}}) : v = 0 \text{ on } \mathcal{B}\Omega\}.$$

for $\mathcal{B}\Omega$ that surround Ω; say
$\mathcal{B}\Omega = \{x \in \mathbb{R}^d \setminus \Omega : \mathrm{dist}(x, \partial\Omega) \leq 1\}$,

- *Pure Dirichlet boundary condition for attached nonlocal boundary*

$$V_D^a := \{v \in L^2(\overline{\overline{\Omega}}) : v = 0 \text{ on } \mathcal{B}\Omega\}.$$

$\mathcal{B}\Omega$ is attached to Ω. Say $\Omega = [0, \pi]$ and $\mathcal{B}\Omega = (-1, 0)$.

- *A zero average condition:*

$$V_N := \{v \in L^2(\overline{\overline{\Omega}}) : \int_{\overline{\overline{\Omega}}} v \, d\mathbf{x} = 0\}.$$

The spaces V_D, V_N are closed subspaces of $L^2(\overline{\overline{\Omega}})$

Indeed,

- if $u_n \in V_D$ such that $u_n \to u \in L^2(\overline{\overline{\Omega}})$, then u vanishes on $\mathcal{B}\Omega$, so $u \in V_D$.
- If $u_n \in V_N$ such that $u_n \to u \in L^2(\overline{\overline{\Omega}})$, then the averages of u_n also converge to the average of u. Hence $u \in V_N$.

The bilinear form is bounded

Lemma

The bilinear form $a(\cdot,\cdot)$ is bounded on $L^2(\overline{\overline{\Omega}})$ with the estimate

$$a(u,v) \le 2\,\overline{\beta}\,\|u\|_{L^2(\overline{\overline{\Omega}})}\|v\|_{L^2(\overline{\overline{\Omega}})},$$

where $\overline{\beta} := \sup_{x\in\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)d\mathbf{x}'$.

$0 < \overline{\beta} < \infty$ because we assumed that C is locally integrable and Ω is bounded.

proof of boundedness

Apply Cauchy-Schwartz.

$$\begin{aligned} a(u,v) &= \frac{1}{2}\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(u(\mathbf{x})-u(\mathbf{x}'))\,(v(\mathbf{x})-v(\mathbf{x}'))\;d\mathbf{x}'\,d\mathbf{x} \\ &\le \frac{1}{2}\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(u(\mathbf{x})-u(\mathbf{x}'))^2\;d\mathbf{x}'\,d\mathbf{x}\right\}^{1/2} \\ &\times\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)(v(\mathbf{x})-v(\mathbf{x}'))^2\;d\mathbf{x}'\,d\mathbf{x}\right\}^{1/2} \end{aligned}$$

Now use the fact that C is locally integrable:

$$\begin{aligned} a(u,v) &\le 2\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)u(\mathbf{x})^2\;d\mathbf{x}'\,d\mathbf{x}\right\}^{1/2} \\ &\quad\times\left\{\int_{\overline{\overline{\Omega}}}\int_{\overline{\overline{\Omega}}} C(|\mathbf{x}-\mathbf{x}'|)v(\mathbf{x})^2\;d\mathbf{x}'\,d\mathbf{x}\right\}^{1/2} \\ &\le 2\,\overline{\beta}\,\|u\|_{L^2(\overline{\overline{\Omega}})}\|v\|_{L^2(\overline{\overline{\Omega}})}, \end{aligned}$$

Coercivity

The following is taken from the work of J.D. Rossi and collaborators.

Lemma

If $\mathcal{B}\Omega$ is surrounding, then there exists $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta, C) > 0$ such that

$$\underline{\lambda}\, \|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u) + \int_{\mathcal{B}\Omega} |u(\mathbf{x})|^2 \, d\mathbf{x} \quad \forall u \in L^2(\overline{\overline{\Omega}}).$$

Corollary

There exists $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta, C) > 0$ such that

$$\underline{\lambda}\, \|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u) \quad \forall u \in V_D^s.$$

Remark: The continuity assumption is not used.

Proof of coercivity

- Divide Ω into strips $\{S_j\}_{j\geq 0}$ of thickness $\delta/2$. Denote $\mathcal{B}\Omega$ by S_{-1}.
- show that there are constants $\alpha_j > 0$ such that

$$\frac{\alpha_j}{2}\int_{S_j}|u|^2 dx \leq 2a(u,u) + \int_{S_{j-1}}|u|^2 dx$$

- Obtain the result from a cascade of inequalities.
-

$$\alpha_j = \min_{x\in\overline{S_j}}\int_{S_{j-1}} C(|\mathbf{x}'-\mathbf{x}|)d\mathbf{x}' > 0$$

Well posedness of the problem

Theorem

The variational problem: given $b \in L^2(\overline{\overline{\Omega}})$ *find* $u \in V_D$ *such that* $a(u, v) = (b, v)$ *for all* $v \in V_D$ *has a unique solution which satisfies the inequality*

$$\|u\|_{L^2} \leq \Lambda \|b\|_{L^2},$$

for some constant $\Lambda = \Lambda(\delta) > 0$.

As a corollary of the above we have the following. Let $g \in L^2(\mathcal{B}\Omega)$ and define the closed and convex subset

$$V_g = \{u \in L^2(\overline{\overline{\Omega}}) : u = g \text{ in } \mathcal{B}\Omega\}$$

Theorem

The variational problem: given $b \in L^2(\overline{\overline{\Omega}})$ find $u \in V_g$ such that $a(u, v) = (b, v)$ for all $v \in V_D^s$ has a unique solution. Moreover, the solution satisfies the inequality: the inequality

$$\|u\|_{L^2} \leq \Lambda(\|b\|_{L^2} + \|g\|_{L^2}),$$

Other volume constraints

- It is not clear if the previous argument is useful in proving coercivity on the subspaces V_D^a and V_N.
- Luckly, we can obtain the coercivity of $a(\cdot,\cdot)$ on V_D^a and V_N for kernels in L^1 satisfying some moment conditions.
- This is possible using the nonlocal Poincaré inequality as proved by A.C. Ponce.
- The inequality is a consequence of the nonlocal characterization of $W^{1,p}$ functions by J. Bourgain, H. Brezis and P. Mironescu.

Lets consider the sequence of radial functions ρ_n satisfying the following conditions:

$$\rho_n \geq 0 \text{ a.e. in } \mathbb{R}^d, \quad \int_{\mathbb{R}^d} \rho_n = 1, \quad \forall n \geq 1,$$

$$\text{and } \lim_{n\to\infty} \int_{|h|>r} \rho_n(h)dh = 0, \quad \forall r > 0.$$

Theorem (Bourgain, Brezis and Mironescu)

If $u \in W^{1,2}(\overline{\overline{\Omega}})$

$$\lim_{n\to\infty} \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}') - u(\mathbf{x})|^2}{|\mathbf{x}' - \mathbf{x}|^2} \rho_n(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' d\mathbf{x} = K_d \int_{\overline{\overline{\Omega}}} |\nabla u|^2 dx.$$

Nonlocal Poincaré for zero mean functions

Theorem (A.C. Ponce, for $p = 2$)

Given $\eta > 0$, there exists n_0 such that

$$\|u\|_{L^2(\overline{\overline{\Omega}})} \leq \left(\frac{c_{pcr}}{K_d} + \eta\right) \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}') - u(\mathbf{x})|^2}{|\mathbf{x}' - \mathbf{x}|^2} \rho_n(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' d\mathbf{x}$$

for all $u \in V_N$ and all $n \geq n_0$.

Here c_{pcr} is the best 'local' Poincaré constant, that depend only on $\overline{\overline{\Omega}}$.

Nonlocal Poincaré for functions vanishing on a subset

The following is our extension of the nonlocal Poincaré's inequality for functions vanishing on a subset positive measure.

Theorem

Given $\eta > 0$, *there exists* n_0 *such that*

$$\|u\|_{L^2(\overline{\overline{\Omega}})} \leq (\frac{c_{pcr}}{K_p} + \eta) \int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}') - u(\mathbf{x})|^2}{|\mathbf{x}' - \mathbf{x}|^2} \rho_n(|\mathbf{x}' - \mathbf{x}|) d\mathbf{x}' d\mathbf{x}$$

for all $u \in V_D^a$ *and all* $n \geq n_0$.

Here c_{pcr} is the best 'local' Poincaré constant, that depend only on $\overline{\overline{\Omega}}$, and the size of $\mathcal{B}\Omega$.

Special choice of radial functions

Let $\gamma : (0, \infty) \to [0, \infty)$ be such that $\gamma(r)r^{d-1} \in L^1_{loc}([0, \infty)$ and

$$\gamma \geq 0, \quad \text{supp}(\gamma) \subset [0, 2),$$
$$\text{and} \int_0^\infty \gamma(r) r^{d+1} \, dr = 1.$$

Then, a simple calculation yields that the sequence $\rho_\delta(r)$ defined by

$$\rho_\delta(r) := \frac{1}{\omega_d \delta^{d+2}} \gamma(\frac{r}{\delta}) r^2$$

satisfies all the required conditions.
[ω_d is a dimensional constant and is the surface area of the unit sphere in $\mathbb{R}^d$.]

For the radial function

$$C(r) = \gamma(\frac{r}{\delta}),$$

then $C(r) \in L^1_{loc}[0,\infty)$ and satisfies a moment condition. Moreover,

$$\int_{\overline{\overline{\Omega}}} \int_{\overline{\overline{\Omega}}} \frac{|u(\mathbf{x}) - u(\mathbf{x}')|^2}{|\mathbf{x} - \mathbf{x}'|^2} \rho_\delta(|\mathbf{x} - \mathbf{x}'|) \, d\mathbf{x}' d\mathbf{x} = \frac{1}{\omega_d \delta^{d+2}} a(u,u).$$

Corollary (Coercivity and well posedness)

For C *as above the bilinear form* $a(\cdot,\cdot)$ *is coercive on* V_D^a *and* V_N*. In fact, there exists* $\delta_0 = \delta_0(\overline{\overline{\Omega}}, \gamma) > 0$ *and* $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta_0)$ *such that for all* $0 < \delta < \delta_0$ *and* $u \in V_D^a$ *or* V_N*:*

$$\underline{\lambda}\, \delta^{d+2} \|u\|^2_{L^2(\overline{\overline{\Omega}})} \leq a(u,u).$$

Theorem

For C *as above, the variational problem* $a(u,v) = (b,v)$ *for all* $v \in V$ *with* $V = V_N$ *or* $V = V_D^a$ *has a unique solution which satisfies the inequality*

$$\|u\|_{L^2} \leq \Lambda \|b\|_{L^2},$$

for some constant $\Lambda > 0$.

Spectral Equivalence:

Theorem

For C as above, there exist $\delta_0 > 0$, $\underline{\lambda} = \underline{\lambda}(\overline{\overline{\Omega}}, \delta_0)$ and $\overline{\lambda} = \overline{\lambda}(\gamma, d)$ such that for all $0 < \delta < \delta_0$ and $u \in V_D^s, V_D^a$, or V_N, we have

$$\underline{\lambda}\,\delta^{d+2}\,\|u\|^2_{L^2(\overline{\overline{\Omega}})} \le a(u,u) \le \overline{\lambda}\delta^d\|u\|^2_{L^2(\overline{\overline{\Omega}})}.$$

Proof.

Recall the boundedness $a(u,v) \le 2\overline{\beta}\|u\|\|v\|$ where

$$\beta \le \sup_{x\in\overline{\overline{\Omega}}} \int_{B(\mathbf{x},R)} C(|\mathbf{x}-\mathbf{x}'|)d\mathbf{x}' = \int_{B(\mathbf{0},R)} C(|\mathbf{x}'|)d\mathbf{x}' \quad R = diam(\overline{\overline{\Omega}}).$$

The last integral can be estimated as

$$\int_{B(\mathbf{0},R)} C(|\mathbf{x}'|)d\mathbf{x}' = \int_{B(\mathbf{0},R)} \gamma(\frac{|\mathbf{x}'|}{\delta})d\mathbf{x}' = \omega_d\delta^d \int_0^{R/\delta} \gamma(s)s^{d-1}ds \le \overline{\lambda}\delta^d.$$

examples of C

- $\gamma(r) = \chi_{(0,1)}(r) \quad C(r) = \chi_{(0,\delta)}(r)$ [used in (Aksoylu & Parks)]
- $\gamma(r) = r^\alpha \chi_{(0,1)}$ for $\alpha > -d$, where d is space dimension.

Condition number upper bound quantified

If $V_h \subset V$ is any finite dimensional subspace and K_h is the stiffness matrix of the operator corresponding to V_h, then

- $cond(K_h) \leq C\delta^{-2}$
- This doesn't say $cond(K_h)$ is independent of h, rather has an upper bounded that is independent of h.

Thank you!