Dirichlet’s principle and well posedness of steady state solutions in peridynamics

Petronela Radu

Work supported by NSF - DMS award 0908435

January 19, 2011
The steady state peridynamic model

Consider the “elliptic” nonlocal model:

\[
\begin{cases}
\mathcal{L}(u)(x) = b(x), & x \in \Omega \\
u(x) = g(x), & x \in \Gamma,
\end{cases}
\]

where

\[
\mathcal{L}(u)(x) := 2 \int_{\Omega \cup \Gamma} (u(x') - u(x)) \mu(x, x') \, dx',
\]

- \(\Omega\) is an open bounded subset of \(\mathbb{R}^n\)
- \(\Gamma \subset \mathbb{R}^n \setminus \Omega\) denotes a “collar” domain surrounding \(\Omega\) which has nonzero volume
- \(\mu(x, x')\) is nonegative, \(\mu(x, x') = \mu(x', x)\).

Remark: The integral operator is defined on the boundary \(\Gamma\) as well.
Notation

\[ \alpha : \Omega \cup \Gamma \times \Omega \cup \Gamma \to \mathbb{R}^n, \quad u : \Omega \cup \Gamma \to \mathbb{R}^n, \quad f : \Omega \cup \Gamma \times \Omega \cup \Gamma \to \mathbb{R}^n \]

(i) Generalized nonlocal gradient

\[ \mathcal{G}(u)(x, x') := (u(x') - u(x))\alpha(x, x'), \quad x, x' \in \Omega, \]

(ii) Generalized nonlocal divergence

\[ \mathcal{D}(f)(x) := \int_{\Omega \cup \Gamma} (f(x, x')\alpha(x, x') - f(x', x)\alpha(x', x))dx', \quad x \in \Omega, \]

(iii) Generalized nonlocal normal component

\[ \mathcal{N}(f)(x) := -\int_{\Omega \cup \Gamma} (f(x, x')\alpha(x, x') - f(x', x)\alpha(x', x))dx', \quad x \in \Gamma. \]

Note: For the given peridynamic model we will use \( \mu = \alpha^2 \)
Useful nonlocal calculus identities

Gunzburger & Lehoucq:

• \( \mathcal{L} u = \mathcal{D}(\mathcal{G}(u)) \)

• For \( u, v \in L^2(\Omega \cup \Gamma) \)

\[
\int_{\Omega} v \mathcal{D}(\mathcal{G}(u)) \, dx + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(u) \mathcal{G}(v) \, dx' \, dx = \int_{\Gamma} v \mathcal{N}(\mathcal{G}(u)) \, dx.
\]

• For \( u, v \in L^2(\Omega \cup \Gamma) \), \( v = 0 \) over \( \Gamma \)

\[
\int_{\Omega} (\mathcal{L} u) v \, dx = - \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v) \cdot \mathcal{G}(u) \, dx' \, dx.
\]
Nonlocal boundary conditions

Dirichlet Problem:

\[
\begin{align*}
\mathcal{L}u &= b, \quad x \in \Omega \\
u &= g, \quad x \in \Gamma
\end{align*}
\]

Neumann Problem:

\[
\begin{align*}
\mathcal{L}u &= b, \quad x \in \Omega \\
\int_{\Gamma} (u' - u) \mu(x, x') dx' &= g, \quad x \in \Gamma,
\end{align*}
\]

where \( u' = u(x') \), \( u = u(x) \).
Set up for the Dirichlet’s principle. Spaces

Introduce the inner product

\[
\langle u, v \rangle_\mu = \frac{1}{2} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (u(x') - u(x))(v(x') - v(x)) \mu(x, x') \, dx' \, dx
\]

\[+ \langle u, v \rangle_{L^2(\Omega \cup \Gamma)},\]

The space

\[\mathcal{W} := \{ w \in L^2(\Omega) | \langle w, w \rangle_\mu < \infty, w = 0 \text{ on } \Gamma \},\]

endowed with the norm

\[\|w\|_{\mathcal{W}} = \langle w, w \rangle_\mu^{1/2} .\]

is a Banach space whenever \( \mu \) is nonnegative and symmetric.
The energy functional

The energy functional associated with the peridynamic problem (PD) is given by:

$$\mathcal{F}[u] = \frac{1}{2} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (u(x') - u(x))^2 \mu(x, x') dx' dx + \int_{\Omega} b(x) u(x) dx,$$

for $u$ in the class of admissible functions

$$\mathcal{A} = \{ w \in L^2(\Omega) \mid u = g + u_0, \text{ for some } u_0 \in \mathcal{W} \},$$

**Convention:** $\mathcal{A}$ will also be denoted by $g + \mathcal{W}$. 
The Nonlocal Dirichlet’s Principle

Theorem

Let $\mu(x,x')$ be nonnegative and symmetric.

(i) Assume $u$ solves the nonlocal peridynamics problem

$$
\begin{aligned}
\mathcal{L}(u)(x) &= b(x), \quad x \in \Omega \\
u(x) &= g(x), \quad x \in \Gamma.
\end{aligned}
$$

Then

$$
\mathcal{F}[u] \leq \mathcal{F}[v]
$$

for every $v \in \mathcal{A}$.

(ii) Conversely, if $u \in \mathcal{A}$ satisfies $\mathcal{F}[u] \leq \mathcal{F}[v]$ for every $v \in \mathcal{A}$, then $u$ solves the above nonlocal peridynamics problem.
Proof:
(i) If \( w \in A \), then \( u - w = 0 \) over \( \Gamma \). Integration by parts yields:

\[
0 = \int_{\Omega} (Lu - b)(u - w)\,dx
= -\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} G(u) \cdot G(u - w)\,dx' \,dx - \int_{\Omega} b(u - w)\,dx
\]

By Cauchy-Schwarz we obtain:

\[
\frac{1}{2} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} G(u) \cdot G(u)\,dx' \,dx + \int_{\Omega} bud\,dx
\leq \frac{1}{2} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} G(w) \cdot G(w)\,dx' \,dx + \int_{\Omega} bwd\,dx
\]
(ii) Fix $v \in \mathcal{W}$ and write

$$i(\tau) := \mathcal{F}[u + \tau v],$$

where $\tau \in \mathbb{R}$. Since $u + \tau v \in \mathcal{A}$ for each $\tau$, the scalar function $i(\cdot)$ has a minimum at zero. Thus $i'(0) = 0$, ($' = \frac{d}{d\tau}$), provided the derivative exists. Now we have

$$i(\tau) = \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \frac{1}{2} G(u) \cdot G(u) + \tau G(u) \cdot G(v) + \frac{\tau^2}{2} G(v) \cdot G(v) \, dx' \, dx$$

$$+ \int_{\Omega} b(u + \tau v) \, dx$$

Hence, after integration by parts we obtain:

$$0 = i'(0) = \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} G(u) \cdot G(v) \, dx' \, dx + \int_{\Omega} v b \, dx = \int_{\Omega} (\mathcal{L} u - b) v \, dx$$
A particular kernel

**Assumption A.** For every $x' \in \mathcal{H}_x$, there exists a constant $C_0 > 0$ such that $\mu(x, x') \geq C_0$. In other words, for all $x \in \Omega \cup \Gamma$ we have:

$$C_0 \chi_\delta(x, x') \leq \mu(x, x'),$$

where $\chi_\delta(x, x') = \begin{cases} 1, & |x - x'| \leq \delta \\ 0, & |x - x'| > \delta. \end{cases}$

**Remark:** The prototype kernel

$$\mu(x, x') = \begin{cases} \frac{1}{|x - x'|^\beta} & \text{for } |x - x'| \leq \delta \\ 0 & \text{for } |x - x'| > \delta, \end{cases}$$

satisfies this assumption for all $\beta > 0$. 
Prototype kernel:

\[ \mu(x, x') = \begin{cases} 
\frac{1}{|x-x'|^\beta} & \text{for } |x - x'| \geq \delta \\
0 & \text{for } |x - x'| < \delta, 
\end{cases} \]

where \( \beta \geq 0 \).

- \( \beta \geq n \implies \text{strong singularity; work in the framework of Nikolskii spaces } H^{(\beta-n)/2}(\Omega) \)
- \( \beta < n \implies \text{weak singularity; work with } L^2 \text{ spaces (with weights?)} \)
More “elliptic” properties of $\mathcal{L}$

Proposition

The operator $\mathcal{L}$ admits the following properties:

(a) If $u \equiv$ constant then $\mathcal{L}u = 0$.

(b) Let $x \in \Omega \cup \Gamma$. For any maximal point $x_0$ that satisfies $u(x_0) \geq u(x)$, we have $-\mathcal{L}u(x_0) \geq 0$. Similarly, if $x_1$ is a minimal point such that $u(x_1) \leq u(x)$, then $-\mathcal{L}u(x_1) \leq 0$.

(c) $\mathcal{L}u$ is a positive semidefinite operator, i.e. $\langle -\mathcal{L}u, u \rangle \geq 0$.

(d) $\int_{\Omega \cup \Gamma} -\mathcal{L}u(x) \, dx = 0$. 

(e) Weak mean-value inequality.
If $\mu$ satisfies (A) and $u$ is a nonnegative solution of $\mathcal{L}u(x) = 0$ then:

$$u(x) \geq \frac{1}{|\mathcal{H}_x|} \int_{\mathcal{H}_x} u(y) dy$$

(f) Maximum and minimum principle (Rossi)
Assume that $u \in C(\Omega)$ solves (PD) with $f = 0$. If

$$u(x_0) = \max_{x \in \Omega \cup \Gamma} u(x)$$

then $x_0 \in \Gamma$. Similarly, if

$$u(x_0) = \max_{x \in \Omega \cup \Gamma} u(x)$$

then $x_0 \in \Gamma$. 
Lemma
(Nonlocal Poincaré’s Inequality - Rossi, Aksoylu & Parks) If $u \in L^p(\Omega)$, $p > 1$, $m \geq 1$, and $G$ is as defined in (3), then there exist $\lambda_{Pncr} = \lambda_{Pncr}(\Omega, \Gamma, \delta, m) > 0$ and $C_g > 0$ such that the following inequality holds:

$$\lambda_{Pncr} \|u\|_{L^p(\Omega)} \leq \|G(u)\|_{L^p(\Omega \cup \Gamma \times \Omega \cup \Gamma)} + C_g.$$
Wellposedness of the system (PD)

**Theorem**

With $\mathcal{F}[u]$ defined as before and $\mu$ satisfying assumption (A) we have that

$$\inf \{ \mathcal{F}[u] : u \in \mathcal{A} \}$$

attains its minimum, and furthermore this minimizer is unique. Hence, there exists a unique solution to (PD) in $\mathcal{A}$ for every $f, g$ in $\mathcal{A}$. 
Existence of solutions

Follows from convexity of $F$ and coercivity of $F$. First note that since $F[u] \geq 0$ for $u \in L^2(\Omega)$, we have that

$$\inf \{ F[u] : u \in A \} = m \geq 0.$$ 

Let $\{u_\nu\}$ be a minimizing sequence so that $F[u_\nu] \to m$. Thus there exists $M > 0$ such that $F[u_\nu] < M$. We will show that the sequence $u_\nu$ is bounded in $L^2$ (coercivity).

$$M > \frac{1}{2} \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} [G(u_\nu)]^2 dx’dx + \int_\Omega b(x) u_\nu(x) dx$$

$$\geq \frac{1}{2} \|G(u_\nu)\|_{L^2(\Omega \cup \Gamma \times \Omega \cup \Gamma)}^2 - \frac{\varepsilon}{2} \|b\|_{L^2(\Omega)}^2 - \frac{1}{2\varepsilon} \|u_\nu\|_{L^2(\Omega)}^2,$$
Existence contd.

By the nonlocal Poincaré’s inequality we have:

\[ M > C_1 \| u_\nu \|^2_{L^2(\Omega)} + C_2 \| g \|^2_{L^2(\Gamma)} - C_3 \| b \|^2_{L^2(\Omega)} \]

Thus, we can find \( \gamma > 0 \) such that

\[ \| u_\nu \|_{L^2(\Omega)} \leq \gamma. \]

Therefore, we may extract a subsequence \( \{ u_\nu \} \) and find \( \bar{u} \in L^2(\Omega) \) such that \( u_\nu \to \bar{u} \) in \( L^2(\Omega) \). By Mazur’s Lemma we can find a sequence of convex combinations of \( u_\nu \), denoted by \( \bar{u}_\nu \to \bar{u} \) in \( L^2(\Omega) \), hence

\[ \lim_{\nu \to \infty} \mathcal{F}[\bar{u}_\nu] \leq m. \]

By Fatou’s lemma

\[ \mathcal{F}[\bar{u}] \leq \lim_{\nu \to \infty} \mathcal{F}[\bar{u}_\nu], \]

hence, from the above inequalities we have that \( \bar{u} \) is a minimizer since

\[ \mathcal{F}[\bar{u}] \leq m. \]
**Uniqueness of solutions**

Let $\bar{u}, \bar{v} \in L^2(\Omega)$ be minimizers of $\mathcal{F}$ with $\bar{u} \neq \bar{v}$. Set

$$\bar{w} = \frac{\bar{u} + \bar{v}}{2} \in L^2(\Omega).$$

By the strict convexity of the integrand, we have

$$m \leq \mathcal{F}[\bar{w}] \leq \frac{1}{2} \mathcal{F}[\bar{u}] + \frac{1}{2} \mathcal{F}[\bar{v}] = m.$$

Hence $\bar{w}$ is a minimizer of $\mathcal{F}$. This implies that

$$\frac{1}{2} \mathcal{F}[\bar{u}] + \frac{1}{2} \mathcal{F}[\bar{v}] - \mathcal{F}[\bar{w}] = 0.$$

Thus

$$\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \frac{g[\bar{u}]^2}{2} + \frac{g[\bar{v}]^2}{2} - g[\bar{w}] dx' dx = 0.$$

Again, by strict convexity, the integrand is strictly positive – contradiction!!
Consider the nonlocal flux that satisfies a nonlocal Fourier’s law:

\[ Q[u](x, x') = -aG[u](x, x'), \]

The nonlocal conservation law:

\[ u_t = -D(Q). \]

We obtain the nonlocal peridynamic diffusion law

\[ u_t = D(aG(u)) = aLu. \]
Assum that the flux $\mathcal{Q}$ is given by the Cattaneo-Vernotte equation

$$\mathcal{Q}_t + a\mathcal{Q} = -\mathcal{G}(u).$$

Differentiate the conservation law $u_t = -\mathcal{D}(\mathcal{Q})$ with respect to time:

$$u_{tt} = -\mathcal{D}(\mathcal{Q}_t).$$

Substituted into the above equation yields:

$$u_{tt} = -\mathcal{D}(-\mathcal{G}(u) - a\mathcal{Q}) = \mathcal{D}(\mathcal{G}(u)) + \mathcal{D}(a\mathcal{Q}) = \mathcal{L}u - au_t.$$

or

$$u_{tt} - \mathcal{L}u + au_t = 0.$$
Importance of hyperbolic diffusion (classical case)

Consider the classical heat and damped wave equations:

\[
\begin{align*}
&v_t - \Delta v = 0 \text{ in } \mathbb{R}^n \\
&v|_{t=0} = v_0 \text{ in } \mathbb{R}^n.
\end{align*}
\]

\[
\begin{align*}
&u_{tt} - \Delta u + u_t = 0 \text{ in } \mathbb{R}^n \\
&(u, u_t)|_{t=0} = u_0 + u_1 \text{ in } \mathbb{R}^n.
\end{align*}
\]

- hyperbolic diffusion is important in unsteady heat conduction (the second sound of helium)
- the long time behavior of \( u \) with initial conditions \((u_0, u_1)\) can be very well approximated by the long time behavior of \( v \) with initial condition \( v_0 = u_0 + u_1 \) (Abstract Diffusion Phenomenon - JDE)
Questions and future directions

1. physical interpretation of the exponent $\beta$ in $\mu$;
2. what if $\mu$ depends on time? OR the horizon changes with space (different phases of the material)
3. weakly singular kernels ($0 < s < n$): Harnack’s inequality
4. obtain regularity results via Calculus of Variations techniques
5. wellposedness and regularity for time dependent models (diffusion, elasticity,..)
6. nonlinear local problems: very difficult since there is no gain in “smoothness” (higher integrability or more derivatives for the solution)
7. wellposedness in weighted spaces