Convergence of the variable two-step BDF time discretisation of nonlinear evolution problems governed by a monotone potential operator

July 31, 2008

ETIENNE EMMRICH[1]

[1] *Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany. email: emmrich@math.tu-berlin.de*

Abstract.

The initial-value problem for a first-order evolution equation is discretised in time by means of the two-step backward differentiation formula (BDF) on a variable time grid. The evolution equation is governed by a monotone and coercive potential operator. On a suitable sequence of time grids, the piecewise constant interpolation and a piecewise linear prolongation of the time discrete solution are shown to converge towards the weak solution if the ratios of adjacent step sizes are close to 1 and do not vary too much.

AMS subject classification (2000): 65M12, 47J35, 35K55, 47H05, 47H50.

Key words: Nonlinear evolution problem, monotone potential operator, time discretisation, linear multistep method, backward differentiation formula, non-uniform time grid, convergence.

1 Introduction

In this paper, we are concerned with the time discretisation of the initial-value problem for a nonlinear evolution equation,

$$u' + Au = f \quad \text{in } (0,T)\,, \quad u(0) = u_0\,. \tag{1.1}$$

The operator $A : V \to V^*$, acting on a Gelfand triple $V \subseteq H \subseteq V^*$, is supposed to be a monotone and coercive potential operator that fulfills a growth condition. A typical example is the p-Laplacian.

The time discretisation under consideration is the two-step backward differentiation formula (BDF) on the variable time grid

$$\begin{aligned}
&\mathbb{I} : 0 = t_0 < t_1 < \cdots < t_N = T \ (N \in \mathbb{N}) \text{ with}\\
&\tau_n := t_n - t_{n-1} \ (n = 1, 2, \ldots, N)\,,\ r_n := \frac{\tau_n}{\tau_{n-1}} \ (n = 2, 3, \ldots, N)\,,\\
&\tau_{\max} := \max_{n=1,2,\ldots,N} \tau_n\,, \quad r_{\max} := \max\Big(\max_{n=2,3,\ldots,N} r_n, 1\Big)\,.
\end{aligned} \tag{1.2}$$

Sometimes, we emphasise the dependence on $\mathbb{I}$ by writing e.g. $\tau_{\max}(\mathbb{I})$. The temporal semidiscretisation of (1.1) for the computation of a time discrete solution

$u^n \approx u(t_n)$ $(n = 1, 2, \ldots, N)$ with an initial implicit Euler step then reads as

(1.3)

$$\mathrm{D}u^n + Au^n = f^n \,,\; n = 1, 2, \ldots, N, \text{ with } \mathrm{D}u^1 := \frac{1}{\tau_1}\left(u^1 - u^0\right),$$

$$\mathrm{D}u^n := \frac{1}{\tau_n}\left(\frac{1 + 2r_n}{1 + r_n}u^n - (1 + r_n)u^{n-1} + \frac{r_n^2}{1 + r_n}u^{n-2}\right),\; n = 2, 3, \ldots, N,$$

where $\{f^n\}_{n=1}^N$ is a given approximation of the right-hand side f and $u^0 \approx u_0$ is a given starting value. With $r_n := 0$, the two-step BDF formally degenerates to an implicit Euler step. The pure implicit Euler method is also covered by our studies although the related results are already known from the analysis of the Rothe method (see e.g. [58]).

Whereas nonlinear evolution problems, which appear in many applications, are well-studied from the analytical point of view (see e.g. the monographs [6, 10, 24, 46, 50, 58, 60, 64, 66] and the references cited therein), their indispensable numerical solution still often lacks a mathematical substantiation and necessitates further studies. This, in particular, applies to the time discretisation on a variable time grid although standard in many implementations.

The time discretisation of linear evolution problems is rather well-understood, and we may refer to the monograph [65] and the references cited therein. The approximation of semilinear evolution equations by means of single- and multistep methods has been considered e.g. in [1, 2, 3, 16, 17, 40, 49, 61, 62], the time discretisation of quasilinear evolution problems has been studied e.g. in [5, 29, 39, 44, 47, 67]. Stability and error estimates for linearly implicit one-step methods applied to nonlinear evolution equations posed in a Gelfand triple are proven in [48] relying on a linearisation. Different time discretisation schemes for fully nonlinear problems, which are governed by a densely defined nonlinear mapping in a Banach space whose first Fréchet derivative is sectorial, have been dealt with, again by linearisation, in [28, 55, 56]. Evolution equations governed by maximal monotone operators and their time discretisation have been studied in [36, 37, 38, 59]. A posteriori error estimates for nonlinear evolution problems governed by an angle-bounded or dissipative operator have been studied in [51, 53, 54], see also [4] for the discretisation of a semilinear equation.

From the many time discetisation schemes available, BDF are somewhat favorable for the integration of stiff problems due to their stability properties. For an analysis of the BDF applied to ordinary differential equations, we refer to [34, 35, 63]. BDF with variable time steps are widely used in computational codes (see e.g. [11, 15]). A variable step size/variable order BDF implementation in combination with a (nonlinear) Galerkin method has been tested in [25, 26] for the time integration of the Kuramoto-Sivashinsky and a reaction-diffusion equation. An early reference for the analysis of BDF with variable step sizes is [8] in which the stability of the corresponding companion matrices and the stability when applying the methods to the scalar test equation $x' + \lambda x = 0$ with λ lying in a sector of the right complex half-plane have been analysed. In particular, it turns out that the two-step BDF becomes stable in a sector with a half-angle of about $\arctan 8.65 \approx 83.4^\circ$ if the ratios of adjacent step sizes are less than $\overline{r} = 1.2$ or with a half-angle $\arctan 2\sqrt{2} \approx 70.5^\circ$ if the ratios are less than $\overline{r} = (1+\sqrt{3})/2 \approx 1.366$. A_0-stability of the variable step size BDF has also been addressed in [14] assuming sufficiently small ratios of adjacent step sizes. Stability and convergence of the variable k-step BDF has been studied extensively in [12, 13, 30, 31, 32, 33], containing e.g. the nowadays classical results on the zero-stability (with $\overline{r} = 1 + \sqrt{2} \approx 2.414$ for $k = 2$).

Two classes of variable step size multistep methods (one of them includes the BDF) applied to a linear evolution problem that is governed by a maximal sectorial operator A in a Hilbert space have been considered in [45]. Stability as well as optimal error estimates for sufficiently smooth solutions are shown for both the methods if they are A- or strongly $A(\vartheta)$-stable with $0 \le \vartheta \le \pi/2$ provided the variable time grid is such that $c\tau \le \tau_n \le \tau$ for some $\tau > 0$ and some $c \in (0,1)$, and such that the ratios of adjacent step sizes are not too large. So, for the variable two-step BDF and a maximal positive operator A, the step size ratios have to be such that for some $C > 0$

$$\sum_{n=2}^{N} |r_n - 1| \le C\,;$$

the stability and error constant then is essentially given by $\exp(cCT)$ with some constant $c > 0$. For a sectorial operator and a strongly $A(\vartheta)$-stable method with $\vartheta > \vartheta_0$ for some $\vartheta_0 \in [0, \pi/2]$, similar results are obtained if

$$|r_n - 1| \le C \frac{\tau_n}{1 + |\log \tau_n|}\,, \quad n = 2, 3, \ldots, N\,.$$

Note that also the stability and error constants appearing in [56] when studying nonlinear parabolic problems depend exponentially on $\sum_n |r_n - 1|$.

For the variable two-step BDF, $A(\vartheta)$-stability-type results with $\vartheta < \pi/3$ have been provided in [31] where the upper bound $\overline{r}$ varies monotonically with ϑ from $\overline{r} = (1 + \sqrt{3})/2$ for $\vartheta = 0$ to $\overline{r} = 1$ for $\vartheta = \pi/3$. From these results, smooth as well as non-smooth data error estimates of optimal order could be derived for a linear homogeneous evolution equation with an operator that is the infinitesimal generator of a holomorphic semigroup. For the same bound $\overline{r} = (1+\sqrt{3})/2$, A_0-stability as well as optimal error estimates for a linear evolution problem with a selfadjoint lower semibounded operator in a Hilbert space have been proven in [32]. Note that the preceding results differ from the results in [45, 57] insofar as the error constant is independent of the time grid and no further restrictions apply to the sequence of time grids in order to have a reasonable stability and error constant.

In [7], the bound $\overline{r}$ could be improved up to $(2 + \sqrt{13})/3 \approx 1.868$ and in [20] (already for mildly semilinear problems) up to 1.91. However, the stability and error constant then depends mildly on the sequence of step size ratios through $\exp(c\Gamma_0)$ with some $c > 0$ and

$$\Gamma_0 := \sum_{n=2}^{N-2} [r_{n+2} - r_n]_-\,, \quad [a]_- := (|a| - a)/2\,. \tag{1.4}$$

A smooth data error estimate in a similar situation, but with more restrictive assumptions on the partition of the time interval, has been obtained in [45].

The two-step BDF with constant time steps (the scheme then is G-stable) has been dealt with for nonlinear problems as e.g. the incompressible Navier-Stokes problem in [18, 19, 22, 27, 41, 42, 43, 44, 52]. For an analysis of the problem under consideration here, in the case of an equidistant time grid, we refer to [21].

In this paper, we prove the convergence of the piecewise constant interpolation and of a piecewise linear prolongation of the discrete numerical solution to (1.3) towards the weak solution to (1.1) (see Theorem 4.1).

In the course of the proof, we make use of the stability of the scheme that is exhibited in terms of algebraic relations and an upper bound on the ratios of adjacent step sizes. Indeed, for a sequence $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ of time grids (1.2), we assume

$$\sup_{k\in\mathbb{N}} r_{\max}(\mathbb{I}_k) < \overline{r} = \frac{1}{\sqrt[3]{4}-1} \approx 1.702 \tag{1.5}$$

in order to derive a priori estimates for the discrete solution. The bound $\overline{r}$ here is smaller than in [7, 20] due to the nonlinear character of the main part A. The method of proof, however, is rather similar to the linear case and relies here on the potential structure of A.

Unfortunately, there appear much more restrictive assumptions on the sequence of time grids in order to prove the coincidence of the weak limits of the piecewise constant interpolants and the piecewise linear prolongations, and in order to prove another auxiliary result that is indispensable in the course of the proof of the main result. These assumptions are fulfilled if

$$\begin{aligned} r_n(\mathbb{I}_k) &= 1 + o(\tau_n(\mathbb{I}_k))\,, \quad n = 2, 3, \ldots, N_k\,, \\ r_{n+1}(\mathbb{I}_k) &= r_n(\mathbb{I}_k) + o(\tau_n(\mathbb{I}_k)^2)\,, \quad n = 2, 3, \ldots, N_k - 1\,, \end{aligned} \tag{1.6}$$

i.e. if we avoid occasional Euler steps and if the time grid is a perturbation of an equidistant grid. The assumptions are also fulfilled if the sequence of time grids is constructed from the relation $\tau_{n+1} = \tau_n + c\tau_n^{2+\varepsilon}$ or even $\tau_{n+1} = \tau_n \exp(c\tau_n^{1+\varepsilon})$ ($n = 1, 2, \ldots, N$, $\tau_1 > 0$, $c \in \mathbb{R}$, $\varepsilon > 0$ given). Such time grids already deviate remarkable from an equidistant time grid.

The proof of the main convergence result relies upon the theory of monotone operators and compactness arguments. In particular, no linearisation and thus no differentiability of the operator A is employed. Note, however, that our assumptions imply global existence of solutions to the original problem which is different from the approach in e.g. [28, 55, 56]. Although the assumptions on the sequence of time grids seem to be rather restrictive (and it may remain for future work to try to weaken these assumptions) this is, to the best knowledge of the author, the first proof of convergence for a rather general class of nonlinear problems avoiding any additional (and in general not known) regularity of the exact solution. The error estimates in [7, 20, 45] for the linear case or in [56] for the nonlinear case, which are valid for more general variable time grids, also yield (pointwise) convergence but only if the exact solution is smooth enough and only if the appearing constants remain bounded. This requires at least the boundedness of Γ_0 and $\sum_n |r_n - 1|$, respectively. To be precise, [56, Thm. 6] provides pointwise convergence in the nonlinear case in some (better) space $D \subset V$ if, in particular, A is twice continuously Fréchet differentiable on some subset of D, if the first Fréchet derivative is sectorial in H, if the exact solution is in $\mathcal{C}^{p+1}([0,T];H)$ for some $p > 1$, and if $\sum_n |r_n - 1|$ remains bounded. Already in the linear case, the regularity assumptions would imply compatibility conditions on the problem data; in the generic case of a general potential operator, [24, VI §2] (see also [58, Thm. 8.16]) only provides $u \in \mathcal{C}^1([0,T];H_w)$ (where H_w means H equipped with the weak topology) under some additional assumptions on the initial data. On the other hand, it is clear that the approach in this paper cannot deliver any order of convergence; in this respect, one has to consult [56].

We may also point out the significant difference between the analysis of a linear and a nonlinear problem within the approach employed in this paper: In the linear case, a priori estimates for the discrete solution also provide stability and thus error estimates. In the nonlinear case, in contrast, a priori estimates rely upon coercivity (and could here be derived already assuming that the ratios of adjacent step sizes are bounded from above appropriately), whereas stability and error estimates would rely upon (uniform) monotonicity of the underlying operator. In the case of the two-step BDF with variable time steps, we need to modify the usual energy technique already in the linear case (see [7]); this could be adopted here for the nonlinear case by employing the potential structure. Stability and error estimates for the nonlinear case, however, cannot be derived in this way. The somewhat severe restrictions on the sequence of time grids result from the proof of convergence and not from deriving a priori estimates. Finally, we should remark that it would be favorable to have convergence for a combination of Euler steps and the two-step BDF but this could not be shown here. An explanation might be that, even in the equidistant case, the two-step BDF is not just a perturbation of the one-step BDF, i.e. the Euler step, and switching between the methods is somewhat a shock.

Results similar to those obtained here have recently been obtained, for a more general class of operators A, in [21] for the two-step backward differentiation formula on an equidistant time grid and in [23] for the ϑ-scheme on a variable grid, see also [58, Ch. 8.2]) for the backward Euler method on an equidistant grid. The results of this paper apply to e.g. the fluid flow of a porous medium as described in [46, pp. 191 ff.] and [24, pp. 72 f.]).

The paper is organised as follows: The notation and the analytical framework for studying (1.1) is described in Section 2. Existence, uniqueness, and a priori estimates of the solution to the temporal semidiscretisation (1.3) are shown in Section 3. The main convergence result is then proven in Section 4.

2 Notation and time continuous problem

Let $V \subseteq H \subseteq V^*$ be a Gelfand triple with a reflexive, separable, real Banach space $(V, \|\cdot\|)$ that is dense and continuously embedded in the Hilbert space $(H, (\cdot,\cdot), |\cdot|)$. The dual V^* of V is equipped with the norm $\|f\|_* :=$

$\sup_{v\in V\setminus\{0\}}\langle f,v\rangle/\|v\|$, where $\langle\cdot,\cdot\rangle$ denotes the dual pairing (sometimes, we explicitly write $\langle\cdot,\cdot\rangle_{V^*\times V}$).

For a Banach space X and the time interval $[0,T]$, let $L^q(0,T;X)$ ($q\in[1,\infty]$) be the Banach space of Bochner integrable (for $q=\infty$ Bochner measurable and essentially bounded) abstract functions with the standard norm $\|\cdot\|_{L^q(0,T;X)}$. In what follows, we always assume $p\in(1,\infty)$ and set $p^*:=p/(p-1)$. The dual pairing between $L^p(0,T;V)$ and $L^{p^*}(0,T;V^*)=(L^p(0,T;V))^*$ is given by

$$\langle f,v\rangle_{L^{p^*}(0,T;V^*)\times L^p(0,T;V)}=\int_0^T\langle f(t),v(t)\rangle_{V^*\times V}\,dt\,,$$

similarly, we have $\big(L^1(0,T;H)\big)^*=L^\infty(0,T;H)$ with

$$\langle f,v\rangle_{L^\infty(0,T;H)\times L^1(0,T;H)}=\int_0^T(f(t),v(t))dt\,.$$

We shall seek solutions to (1.1) in the Banach space

$$\mathcal{W}:=\{v\in\mathcal{X}:v'\in\mathcal{X}^*\}\,,\quad \|v\|_{\mathcal{W}}:=\|v\|_{\mathcal{X}}+\|v'\|_{\mathcal{X}^*}\,,$$

with v' being the distributional time derivative, where

$$\mathcal{X}:=L^p(0,T;V)\cap L^2(0,T;H)\,,\quad \|v\|_{\mathcal{X}}:=\|v\|_{L^p(0,T;V)}+\|v\|_{L^2(0,T;H)}\,,$$

is a reflexive, separable Banach space. Its dual $\mathcal{X}^*$ can be identified with the sum $L^{p^*}(0,T;V^*)+L^2(0,T;H)$, equipped with the norm

$$\|f\|_{\mathcal{X}^*}:=\inf_{\substack{f_1\in L^{p^*}(0,T;V^*),\,f_2\in L^2(0,T;H)\\ f=f_1+f_2}}\max\big(\|f_1\|_{L^{p^*}(0,T;V^*)},\|f_2\|_{L^2(0,T;H)}\big)\,.$$

If f allows the representation $f=f_1+f_2$ with $f_1\in L^{p^*}(0,T;V^*)$, $f_2\in L^2(0,T;H)$ then the dual pairing between $f\in\mathcal{X}^*$ and $v\in\mathcal{X}$ is given by

$$\langle f,v\rangle_{\mathcal{X}^*\times\mathcal{X}}=\int_0^T\big(\langle f_1,v\rangle_{V^*\times V}+(f_2,v)\big)\,dt=\int_0^T\langle f,v\rangle_{V^*\times V}\,dt\,,$$

see e.g. [24] for more details. After all, $\mathcal{X}\subseteq L^2(0,T;H)\subseteq\mathcal{X}^*$ forms a Gelfand triple. In the case $p\geq 2$, we can work with $\mathcal{X}=L^p(0,T;V)$, $\mathcal{X}^*=L^{p^*}(0,T;V^*)$, and the corresponding standard norms. Note that $\mathcal{W}$ is continuously embedded in the space $\mathcal{C}([0,T];H)$ of uniformly continuous abstract functions with values in H.

The structural properties we always assume for the operator A read as follows:

Assumption A. *The mapping $A:V\to V^*$ is a monotone and coercive potential operator such that there is a convex potential $\Phi:V\to\mathbb{R}$ (w.l.o.g. let $\Phi(0)=0$) and, for a suitable $p\in(1,\infty)$, there is a constant $\mu>0$ and for all $v\in V$*

$$\Phi(v)\geq\mu\|v\|^p\,. \tag{2.1}$$

For any $R > 0$ there exists $\alpha = \alpha(R) > 0$ such that for all $v \in V$ with $|v| \le R$

$$\|Av\|_* \le \alpha(R)\left(1 + \|v\|^{p-1}\right) . \tag{2.2}$$

Note that a monotone potential operator is demicontinuous. Moreover, we have for all $v, w \in V$

$$\langle Av, v - w\rangle \ge \Phi(v) - \Phi(w) . \tag{2.3}$$

This relation will be essential when deriving a priori estimates for the time discrete solution. It is, indeed, this relation which is needed for the analysis of the two-step BDF on a variable time grid (but not on an equidistant grid, see [21]). Finally, for all $v \in V$, the potential is given by

$$\Phi(v) = \int_0^1 \langle A(sv), v\rangle ds .$$

Hence, we have for all $v \in V$ with $|v| \le R$ that

$$\Phi(v) \le \alpha(R)\left(1 + \frac{1}{p}\|v\|^{p-1}\right)\|v\| .$$

We shall remark that it is sufficient to require monotonicity and coercivity of $A : V \to V^*$ up to some shift κI with $\kappa > 0$ (see [23, Remark 1]). Under Assumption A, the operator $A : V \to V^*$ extends to a mapping $A : L^p(0,T;V) \cap L^\infty(0,T;H) \subset L^p(0,T;V) \to (L^p(0,T;V))^* = L^{p^*}(0,T;V^*)$ that is monotone, coercive, hemicontinuous, and bounded. Problem (1.1) thus possesses for any $u_0 \in H$ and $f \in \mathcal{X}^*$ a unique solution $u \in \mathcal{W}$ such that the evolution equation holds in $\mathcal{X}^*$ (see [58, Thm. 8.28], [6, Thm. 4.2 on p. 167], [66, Thm. 30.A]).

Examples for operators possessing the above properties can be found e.g. in [24, pp. 68 ff., 215 ff.], [46], [58, pp. 232 ff.], and [66, pp. 567 ff., 590 ff., 779 ff.]). A standard example is the p-Laplacian in a bounded domain supplemented by homogeneous Dirichlet boundary conditions. Another example is the fluid flow through a porous medium when working with the Sobolev space H^{-1} as the pivot space H in the underlying Gelfand triple (see [46, pp. 191 ff.], [24, pp. 72 f.]). Moreover, the diffusion term for some non-Newtonian fluids can be modeled by an operator as above.

3 Time discrete problem and a priori estimates

In what follows, let $c > 0$ be a generic constant that is independent of the time grid (1.2). Moreover, we set $\sum_{j=n_0}^{n} a_j = 0$ for any a_j whatsoever if $n_0 > n$, and we set $\chi_n := 1$ if $r_n > 0$ and $\chi_n := 0$ if $r_n = 0$. Existence, uniqueness and the crucial a priori estimates are provided by the following results.

THEOREM 3.1. *Let Assumption A be fulfilled. For any $u^0 \in H$ and $\{f^n\}_{n=1}^N \subset V^*$ there is a unique solution $\{u^n\}_{n=1}^N \subset V$ to (1.3).*

PROOF. The existence of $u^n \in V$ $(n = 1, 2, \ldots, N)$ follows step-by-step from the well-known Browder-Minty theorem (see e.g. [66, Thm. 26.A]). Its uniqueness follows since the operator appearing in each step is strictly monotone due to the term $\frac{1}{\tau_n}\frac{1+2r_n}{1+r_n}u^n$ that results from the discretisation of the time derivative. □

THEOREM 3.2. *Let Assumption A be fulfilled and suppose that* $r_{\max} < \overline{r} := 1/(\sqrt[3]{4} - 1)$ *and* $\tau_{\max} < 1$. *For given* $u^0 \in H$ *and* $f^n = f_1^n + f_2^n$ *with* $f_1^n \in V^*$, $f_2^n \in H$ $(n = 1, 2, \ldots, N)$, *the solution* $\{u^n\}_{n=1}^N \subset V$ *to (1.3) satisfies the a priori estimate*

$$\begin{aligned}
&\max_{n=1,\ldots,N} |u^n|^2 + \sum_{n=2}^{N-1} (1 - \chi_n)\, |u^n - u^{n-1}|^2 \\
&+ \sum_{n=2}^{N} \frac{r_n^2}{(1 + r_n)^2}\, |u^n - 2u^{n-1} + u^{n-2}|^2 + \mu\sigma(r_{\max}) \sum_{n=1}^{N} \tau_n \|u^n\|^p \\
&\le C \left(|u^0|^2 + \sum_{n=1}^{N} \tau_n \left(\sigma(r_{\max})^{-\frac{1}{p-1}} \|f_1^n\|_*^{p^*} + |f_2^n|^2 \right) \right) =: M,
\end{aligned}$$

where

$$\sigma(r) := 1 + 3r + 3r^2 - 3r^3\,, \quad C = c \exp\left(\frac{T + c\Gamma_0}{1 - \tau_{\max}} \right)$$

with Γ_0 *given by (1.4). Moreover,*

$$\sum_{n=1}^{N} \tau_n \|\mathrm{D}u^n - f_2^n\|_*^{p^*} + \sum_{n=1}^{N} \tau_n \|Au^n\|_*^{p^*} \le M'$$

with M' *being a function in* M *and* $1/\sigma(r_{\max})$ *that is bounded on bounded subsets.*

PROOF. For the initial Euler step, we find

$$|u^1|^2 + |u^1 - u^0|^2 + \tau_1 \Phi(u^1) \le |u^0|^2 + c\tau_1 \|f_1^1\|_*^{p^*} + c\tau_1 |f_2^1|^2 + \tau_1 |u^1|^2$$

from testing the equation by u^1 and employing (2.3), (2.1), Young's inequality, and the relation

$$(a - b)a = \frac{1}{2}\left(a^2 - b^2 + (a - b)^2\right), \quad a, b \in \mathbb{R}\,,$$

which reflects the stability of the implicit Euler method. With $1/(1 - \tau_1) \le \exp(\tau_1/(1 - \tau_{\max}))$, we thus have

$$|u^1|^2 + |u^1 - u^0|^2 + \tau_1 \Phi(u^1) \le c e^{\tau_1/(1-\tau_{\max})} \left(|u^0|^2 + \tau_1 \|f_1^1\|_*^{p^*} + \tau_1 |f_2^1|^2 \right). \tag{3.1}$$

For the following time steps, the main idea due to [7] is to test the n-th equation of (1.3) $(n = 2, 3, \ldots, N)$ by

$$u_\delta^n := u^n + \delta(u^n - u^{n-1}) = (1 + \delta)u^n - \delta u^{n-1}\,,$$

where $\delta \geq 0$ is a parameter that has to be chosen appropriately (see also [20]). Here, we also take into account possible Euler steps by allowing $r_n = 0$ (remember $\chi_n = 0$ for $r_n = 0$ and $\chi_n = 1$ for $r_n > 0$). For $n = 2, 3, \ldots, N$, we multiply (1.3) by $2\tau_n/(1 + r_n)$ and test by u_δ^n. A simple but tedious calculation shows

$$\begin{aligned}
&\frac{2\tau_n}{1+r_n}\,(\mathrm{D}u^n, u_\delta^n) = \frac{\tau_n}{1+r_n}\,\mathrm{D}|u^n|^2 + (1+2\delta)\,\frac{1+2r_n}{(1+r_n)^2}\,|u^n - u^{n-1}|^2 \\
&- \frac{r_n^2}{(1+r_n)^2}\,|u^{n-1} - u^{n-2}|^2 - 2(1+\delta)\,\frac{r_n^2}{(1+r_n)^2}\,\big(u^n - u^{n-1}, u^{n-1} - u^{n-2}\big) \\
&= \frac{\tau_n}{1+r_n}\,\mathrm{D}|u^n|^2 + A_\delta(r_n)\,|u^n - u^{n-1}|^2 - B_\delta(r_n)\,|u^{n-1} - u^{n-2}|^2 \\
&+ (1+\delta)\frac{r_n^2}{(1+r_n)^2}\,|u^n - 2u^{n-1} + u^{n-2}|^2\,,
\end{aligned}$$

where

$$A_\delta(r) := \frac{1 + 2r - r^2 + \delta\,\big(2 + 4r - r^2\big)}{(1+r)^2}\,, \quad B_\delta(r) := (2+\delta)\,\frac{r^2}{(1+r)^2}\,.$$

Summing up gives for $n = 2, 3, \ldots, N$

$$\begin{aligned}
&2\sum_{j=2}^{n} \frac{\tau_j}{1+r_j}\,\Big(\mathrm{D}u^j, u_\delta^j\Big) = \sum_{j=2}^{n} \frac{\tau_j}{1+r_j}\,\mathrm{D}|u^j|^2 \\
&+ (1+\delta)\sum_{j=2}^{n} \frac{r_j^2}{(1+r_j)^2}\,|u^j - 2u^{j-1} + u^{j-2}|^2 + A_\delta(r_n)\,|u^n - u^{n-1}|^2 \\
&+ \sum_{j=2}^{n-1} \big(A_\delta(r_j) - B_\delta(r_{j+1})\big)\,|u^j - u^{j-1}|^2 - B_\delta(r_2)\,|u^1 - u^0|^2\,.
\end{aligned}$$

Observing that $A_\delta(0) = 1 + 2\delta$, $B_\delta(0) = 0$, $r \mapsto A_\delta(r)$ is decreasing whereas $r \mapsto B_\delta(r)$ is increasing for $r \geq 0$, and since $B_\delta(r_{\max}) \leq B_\delta(\overline{r}) = (2+\delta)/\sqrt[3]{4}^{\,2} < 2(2+\delta)/5$, we find

$$A_\delta(r_j) - B_\delta(r_{j+1}) \geq \begin{cases} 1 + 2\delta & \text{if } r_j = 0,\ r_{j+1} = 0, \\ \frac{1}{5} + \frac{8}{5}\delta & \text{if } r_j = 0,\ r_{j+1} \neq 0, \\ A_\delta(r_{\max}) & \text{if } r_j \neq 0,\ r_{j+1} = 0, \\ A_\delta(r_{\max}) - B_\delta(r_{\max}) & \text{if } r_j \neq 0,\ r_{j+1} \neq 0. \end{cases}$$

In order to have

$$A_\delta(r_{\max}) - B_\delta(r_{\max}) = \frac{1 + 2r_{\max} - 3r_{\max}^2 + 2\delta(1 + 2r_{\max} - r_{\max}^2)}{(1+r_{\max})^2} \geq 0\,,$$

we take

$$\delta = \hat{\delta} := -\frac{1}{2}\,\frac{1 + 2r_{\max} - 3r_{\max}^2}{1 + 2r_{\max} - r_{\max}^2} = -\frac{3(r_{\max} - 1)\,\big(r_{\max} + \frac{1}{3}\big)}{2(r_{\max} - 1 - \sqrt{2})(r_{\max} - 1 + \sqrt{2})} \tag{3.2}$$

and remember $1 \le r_{\max} < \overline{r} < 1 + \sqrt{2}$. So it follows for $n = 2, 3, \dots, N$

$$\sum_{j=2}^{n-1} \left(A_\delta(r_j) - B_\delta(r_{j+1})\right) |u^j - u^{j-1}|^2 \ge \frac{1}{5} \sum_{j=2}^{n-1} (1 - \chi_j) \, |u^j - u^{j-1}|^2 .$$

We also have

$$\frac{1}{2} \le A_{\hat\delta}(r_{\max}) \le A_{\hat\delta}(r_n) \,, \quad B_{\hat\delta}(r_2) \le B_{\hat\delta}(r_{\max}) < 2 \,.$$

Moreover, we find

$$\sum_{j=2}^{n} \frac{\tau_j}{1+r_j} \mathrm{D}|u^j|^2 = \sum_{j=2}^{n} \left(\frac{1+2r_j}{(1+r_j)^2} \, |u^j|^2 - |u^{j-1}|^2 + \frac{r_j^2}{(1+r_j)^2} \, |u^{j-2}|^2 \right)$$
$$= \frac{1+2r_n}{(1+r_n)^2} \, |u^n|^2 - \frac{r_{n-1}^2}{(1+r_{n-1})^2} \, |u^{n-1}|^2$$
$$+ \sum_{j=2}^{n-2} \left(\frac{1+2r_j}{(1+r_j)^2} - 1 + \frac{r_{j+2}^2}{(1+r_{j+2})^2} \right) |u^j|^2 - \frac{1+2r_3}{(1+r_3)^2} \, |u^1|^2 + \frac{r_2^2}{(1+r_2)^2} \, |u^0|^2 \,,$$

where

$$\frac{1+2r_j}{(1+r_j)^2} - 1 + \frac{r_{j+2}^2}{(1+r_{j+2})^2} = \frac{r_{j+2}^2}{(1+r_{j+2})^2} - \frac{r_j^2}{(1+r_j)^2}$$
$$= \int_{r_j}^{r_{j+2}} \left(\frac{r^2}{(1+r)^2} \right)' dr \ge -\frac{8}{27} \, [r_{j+2} - r_j]_- \,.$$

In the last step, we have employed the fact that the nonnegative function $r \mapsto \frac{d}{dr}\left(r^2/(1+r)^2\right)$ $(r \ge 0)$ takes its maximum value $8/27$ at $r = 1/2$.

Since $r \mapsto (1+2r)/(1+r)^2$ is decreasing whereas $r \mapsto r^2/(1+r)^2$ is increasing for $r \ge 0$, we, finally, come up with

$$2 \sum_{j=2}^{n} \frac{\tau_j}{1+r_j} \left(\mathrm{D}u^j, u^j_{\hat\delta} \right) \ge \frac{1+2r_{\max}}{(1+r_{\max})^2} |u^n|^2 - \frac{r_{\max}^2}{(1+r_{\max})^2} |u^{n-1}|^2$$
$$+ \frac{1}{2} \, |u^n - u^{n-1}|^2 + \frac{1}{5} \sum_{j=2}^{n-1} (1-\chi_j) \, |u^j - u^{j-1}|^2$$
$$+ \sum_{j=2}^{n} \frac{r_j^2}{(1+r_j)^2} \, |u^j - 2u^{j-1} + u^{j-2}|^2 \tag{3.3}$$
$$- \left(|u^1|^2 + 2 \, |u^1 - u^0|^2 + \frac{8}{27} \sum_{j=2}^{n-2} [r_{j+2} - r_j]_- |u^j|^2 \right) .$$

For the spatial part, (2.3) yields

$$\langle Au^n, u^n_{\hat\delta} \rangle = \langle Au^n, u^n \rangle + \hat\delta \langle Au^n, u^n - u^{n-1} \rangle \ge (1+\hat\delta)\Phi(u^n) - \hat\delta \Phi(u^{n-1})$$

and thus

$$2\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\langle Au^j,u^j_{\hat\delta}\rangle \ge 2(1+\hat\delta)\,\frac{\tau_n}{1+r_n}\,\Phi(u^n)$$
$$+\,2\sum_{j=2}^{n-1}\frac{\tau_j}{1+r_j}\left(1+\hat\delta-\hat\delta\,\frac{r_{j+1}(1+r_j)}{1+r_{j+1}}\right)\Phi(u^j)-2\hat\delta\,\frac{\tau_2}{1+r_2}\Phi(u^1)\,.$$

Since $r\mapsto r/(1+r)$ is increasing, we require

$$\hat\sigma(r_{\max}):=1+\hat\delta-\hat\delta\,r_{\max}>0\,,$$

which is fulfilled if

$$r_{\max}<\frac{1+\hat\delta}{\hat\delta}=\frac{(r_{\max}+1)^2}{3(r_{\max}-1)\left(r_{\max}+\frac{1}{3}\right)}\,.$$

This is, indeed, the case if $r_{\max}<\bar r:=1/(\sqrt[3]{4}-1)$, where $\bar r$ is the real root of the cubic equation

$$(r+1)^2-3r(r-1)\left(r+\frac{1}{3}\right)=1+3r+3r^2-3r^3=0\,.$$

With $\hat\delta\le 3/2$ for $r_{\max}<\bar r$ and

$$0=\hat\sigma(\bar r)<\hat\sigma(r_{\max})\le\hat\sigma(r)=\frac{1+3r+3r^2-3r^3}{2(1+2r-r^2)}\le 1$$

for any $1\le r\le r_{\max}<\bar r$, we end up with

$$2\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\langle Au^j,u^j_{\hat\delta}\rangle\ge\frac{2\hat\sigma(r_{\max})}{1+r_{\max}}\sum_{j=2}^{n}\tau_j\Phi(u^j)-3\tau_2\Phi(u^1)\,, \tag{3.4}$$

where the potential can, finally, be estimated by (2.1).

For the right-hand side, we easily obtain with $\hat\delta\le 3/2$ for $r_{\max}<\bar r$ and Young's inequality

$$2\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\langle f^j,u^j_{\hat\delta}\rangle\le c\sum_{j=2}^{n}\tau_j\left(\|f_1^j\|_*\left(\|u^j\|+\|u^{j-1}\|\right)+|f_2^j|\left(|u^j|+|u^{j-1}|\right)\right)$$
$$\le\sum_{j=2}^{n}\tau_j\left(\frac{\mu\hat\sigma(r_{\max})}{(1+r_{\max})^2}\left(\|u^j\|^p+\|u^{j-1}\|^p\right)+\frac{1+2r_{\max}-r_{\max}^2}{(1+r_{\max})^3}\left(|u^j|^2+|u^{j-1}|^2\right)\right)$$
$$+\,c\sum_{j=2}^{n}\tau_j\left(\hat\sigma(r_{\max})^{-\frac{1}{p-1}}\|f_1^j\|_*^{p^*}+|f_2^j|^2\right),$$

where c depends on $\bar r$ but not on the time grid. Since

$$\sum_{j=3}^{n}\tau_j x_{j-1}=\sum_{j=2}^{n-1}\tau_j r_{j+1}x_j\le r_{\max}\sum_{j=2}^{n-1}\tau_j x_j\,,\quad x_j\in\left\{\|u^j\|^p,|u^j|^2\right\},$$

we find

(3.5)
$$2\sum_{j=2}^{n}\frac{\tau_j}{1+r_j}\,\langle f^j,u^j_{\hat\delta}\rangle \le \frac{\mu\hat\sigma(r_{\max})}{1+r_{\max}}\sum_{j=2}^{n}\tau_j\|u^j\|^p + \frac{1+2r_{\max}-r_{\max}^2}{(1+r_{\max})^2}\sum_{j=2}^{n}\tau_j|u^j|^2$$
$$+\,c\tau_2\left(|u^1|^2+\Phi(u^1)\right)+c\sum_{j=2}^{n}\tau_j\left(\hat\sigma(r_{\max})^{-\frac{1}{p-1}}\|f_1^j\|_*^{p^*}+|f_2^j|^2\right).$$

The estimates (3.3), (3.4), and (3.5) give

$$\frac{1+2r_{\max}}{(1+r_{\max})^2}|u^n|^2+\frac12\,|u^n-u^{n-1}|^2+\frac15\sum_{j=2}^{n-1}(1-\chi_j)\,|u^j-u^{j-1}|^2$$
$$+\sum_{j=2}^{n}\frac{r_j^2}{(1+r_j)^2}\,|u^j-2u^{j-1}+u^{j-2}|^2+\frac{\mu\hat\sigma(r_{\max})}{1+r_{\max}}\sum_{j=2}^{n}\tau_j\|u^j\|^p$$
$$\le c\left(|u^1|^2+|u^1-u^0|^2+\tau_2\Phi(u^1)\right)+c\sum_{j=2}^{n}\tau_j\left(\hat\sigma(r_{\max})^{-\frac{1}{p-1}}\|f_1^j\|_*^{p^*}+|f_2^j|^2\right)$$
$$+\frac{8}{27}\sum_{j=2}^{n-2}[r_{j+2}-r_j]_-|u^j|^2+\frac{1+2r_{\max}-r_{\max}^2}{(1+r_{\max})^2}\sum_{j=2}^{n}\tau_j|u^j|^2+\frac{r_{\max}^2}{(1+r_{\max})^2}|u^{n-1}|^2\,.$$

Taking the maximum on both sides of the inequality, taking into account that

$$\frac{1+2r_{\max}}{(1+r_{\max})^2}-\frac{r_{\max}^2}{(1+r_{\max})^2}\ge\frac{1+2\overline{r}-\overline{r}^2}{(1+\overline{r})^2}>0\,,$$

and applying a discrete Gronwall-type argument (which requires sufficiently small time steps) yields, together with (3.1), the first assertion.

Regarding the second estimate asserted, we observe that

$$\sum_{n=1}^{N}\tau_n\,\|\mathrm{D}u^n-f_2^n\|_*^{p^*}=\sum_{n=1}^{N}\tau_n\|f_1^n-Au^n\|_*^{p^*}\le c\sum_{n=1}^{N}\tau_n\|f_1^n\|_*^{p^*}+c\sum_{n=1}^{N}\tau_n\|Au^n\|_*^{p^*}\,.$$

The growth condition for A yields

$$\sum_{n=1}^{N}\tau_n\|Au^n\|_*^{p^*}\le c\alpha(\sqrt{M})^{p^*}\left(T+\sum_{n=1}^{N}\tau_n\|u^n\|^p\right).$$

This, together with the first estimate, proves the second assertion. □

Note that $\sigma=\sigma(r)$ is decreasing for $r\in[1,\overline{r}]$ such that $\sigma(r_{\max})>\sigma(\overline{r})=0$.

4 Convergence

For a time discrete solution $\{u^n\}_{n=0}^N$ to (1.3) corresponding to the partition $\mathbb{I}$ of $[0,T]$, we construct piecewise polynomial prolongations defined on the entire interval $[0,T]$. We then show their convergence towards a weak solution to (1.1).

Let $u_{\mathbb{I}}$ be the piecewise constant interpolation with $u_{\mathbb{I}}(t) := u^n$ for $t \in (t_{n-1}, t_n]$ and $u_{\mathbb{I}}(0) := u^1$. Furthermore, let $v_{\mathbb{I}}$ be the following piecewise linear function which is continuous on $[0, T]$ and whose slope in (t_{n-1}, t_n) equals the finite difference $\mathrm{D}u^n$ $(n = 1, 2, \ldots, N)$ approximating the time derivative,

$$v_{\mathbb{I}}(t) = (t - t_n)\mathrm{D}u^n + a_n\,, \quad t \in [t_{n-1}, t_n]\ (n = 1, 2, \ldots, N)\,,$$

with

$$a_n = v_{\mathbb{I}}(t_n) = \lim_{t \to t_n + 0} v_{\mathbb{I}}(t) = -\tau_{n+1}\mathrm{D}u^{n+1} + a_{n+1}\,, \quad n = 1, 2, \ldots, N - 1\,.$$

Taking $a_1 = u^1$, i.e. $v_{\mathbb{I}}(0) = -(u^1 - u^0) + a_1 = u^0$, we, immediately, find

$$a_n = u^1 + \sum_{j=2}^{n} \tau_j \mathrm{D}u^j\,, \quad n = 1, 2, \ldots, N\,. \tag{4.1}$$

Unfortunately, the definition of $v_{\mathbb{I}}$ is not local and takes into account the whole history of the time discrete solution. This is different from dealing with the one-step Euler method or the two-step BDF on an equidistant time grid (see [21]).

For a time grid $\mathbb{I}$, we need to define some further characteristic quantities,

$$\Gamma_1 := \sum_{n=2}^{N-1} |r_{n+1} - r_n|\,, \quad \Gamma_{2,n} := \sum_{j=2}^{n-2} \left| \frac{1 + 2r_j}{1 + r_j} - (1 + r_{j+1}) + \frac{r_{j+2}^2}{1 + r_{j+2}} \right|\,,$$

$$\Gamma_2 := \Gamma_{2,N}\,, \quad \Gamma_3 := \sum_{n=2}^{N-1} \tau_{n+1} \left| \frac{r_{n+1}^2}{1 + r_{n+1}} - \frac{r_n}{1 + r_n} \right|^2\,,$$

$$\Gamma_4 := \sum_{n=2}^{N-1} \left| \frac{1 + 2r_{n+1}}{1 + r_{n+1}} - \frac{1 + 2r_n}{1 + r_n} \right|\,, \quad \Gamma_5 := \sum_{n=3}^{N} \chi_n |r_n - 1|\,,$$

$$\Gamma_6 := \frac{1}{4} \sum_{n=2}^{N-1} \left[\frac{1 + 2r_n - r_n^2}{(1 + r_n)^2} - \frac{1 + 2r_{n+1} - r_{n+1}^2}{(1 + r_{n+1})^2} \right]_-\,.$$

We now consider a sequence $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ of time grids (1.2) satisfying

$$\tau_{\max}(\mathbb{I}_k) \to 0 \text{ as } k \to \infty\,, \ \sup_{k\in\mathbb{N}} \tau_{\max}(\mathbb{I}_k) < 1\,, \ \sup_{k\in\mathbb{N}} r_{\max}(\mathbb{I}_k) < \overline{r} := \frac{1}{\sqrt[3]{4} - 1} \tag{4.2}$$

as well as

$$\begin{aligned} &\lim_{k\to\infty} \left(\frac{r_3(\mathbb{I}_k)^2}{1 + r_3(\mathbb{I}_k)} - \frac{r_2(\mathbb{I}_k)}{1 + r_2(\mathbb{I}_k)} \right) = 0\,, \ \sup_{k\in\mathbb{N}} \Gamma_0(\mathbb{I}_k) < \infty\,, \\ &\lim_{k\to\infty} \sum_{n=3}^{N_k} \Gamma_{2,n}(\mathbb{I}_k) = 0\,, \ \lim_{k\to\infty} \Gamma_\ell(\mathbb{I}_k) = 0 \text{ for } \ell \in \{3, 4, 5, 6\}\,. \end{aligned} \tag{4.3}$$

These restrictions on the sequence of time grids are sufficient for proving Lemma 4.3 (uniform boundedness of the corresponding piecewise constant and piecewise linear prolongations in appropriate function spaces and, more severely, the coincidence of their weak limits) and Lemma 4.4 (non-negativeness in the limit of the dual pairing between $v'_{\mathbb{I}_k}$ and $u_{\mathbb{I}_k} - v_{\mathbb{I}_k}$) below.

For the corresponding sequence of time discretisations (1.3) to problem (1.1) with initial value $u_0 \in H$ and right-hand side $f = f_1 + f_2 \in L^{p^*}(0,T;V^*) + L^2(0,T;H)$, we always assume

$$u^0(\mathbb{I}_k) \in H,\ u^0(\mathbb{I}_k) \to u_0 \text{ in } H \text{ as } k \to \infty,\ f^n(\mathbb{I}_k) = \mathrm{R}^n_{\mathbb{I}_k} f \ (n = 1, 2, \ldots, N_k), \tag{4.4}$$

for the starting values $u^0(\mathbb{I}_k)$ and right-hand side $\{f^n(\mathbb{I}_k)\}_{n=1}^{N_k}$, where $\mathrm{R}_{\mathbb{I}_k}$ denotes the natural restriction corresponding to the grid $\mathbb{I}_k$ given by (omitting the subscript $\mathbb{I}_k$ for a moment)

$$\begin{aligned} \mathrm{R}^1 f &:= \frac{1}{\tau_1} \int_0^{t_1} f(t)dt\,, \\ \mathrm{R}^n f &:= \frac{1+2r_n}{(1+r_n)\tau_n} \int_{t_{n-1}}^{t_n} f(t)dt - \frac{r_n}{(1+r_n)\tau_{n-1}} \int_{t_{n-2}}^{t_{n-1}} f(t)dt\,, \\ &\qquad n = 2, 3, \ldots, N\,. \end{aligned} \tag{4.5}$$

From Hölder's inequality, we immediately get

$$\sum_{n=1}^N \tau_n \left(\|\mathrm{R}^n f_1\|_*^{p^*} + |\mathrm{R}^n f_2|^2 \right) \le c \int_0^T \left(\|f_1(t)\|_*^{p^*} + |f_2(t)|^2 \right) dt\,.$$

The main result of the paper can now be formulated as follows.

THEOREM 4.1. *Let $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ be a sequence of time grids that fulfills (4.2), (4.3) and let (4.4) as well as Assumption A be fulfilled. The corresponding sequence $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of piecewise constant interpolants of the discrete solution to (1.3) then converges weakly* in $L^\infty(0,T;H)$ and weakly in $L^p(0,T;V)$ towards the exact solution $u \in \mathcal{W}$ to (1.1). Moreover, the corresponding sequence $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of piecewise linear prolongations converges weakly* in $L^\infty(0,T;H)$ towards u and the sequence $\{v'_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of time derivatives converges weakly in $\mathcal{X}^*$ towards u'.*

REMARK 4.1. *The assumptions (4.3) on the sequence of time grids are fulfilled if (1.6) holds true or if the time grids are constructed from e.g. $\tau_{n+1} = \tau_n(1 + c\tau_n^{1+\varepsilon})$ ($n = 1, 2, \ldots, N$, $\tau_1 > 0$, $c \in \mathbb{R}$, $\varepsilon > 0$ given).*

The foregoing remark can be seen as follows (omitting to write out the dependence on $\mathbb{I}_k$ if possible): Obviously, we have $\Gamma_0 \le c\Gamma_1$ (see (1.4)). Since

$$\begin{aligned} &\left| \frac{1+2r_n}{1+r_n} - (1+r_{n+1}) + \frac{r_{n+2}^2}{1+r_{n+2}} \right| \\ &= \left| -\int_{r_n}^{r_{n+1}} \left(\frac{1+2r}{1+r} \right)' dr + \int_{r_{n+1}}^{r_{n+2}} \left(\frac{r^2}{1+r} \right)' dr \right| \\ &\le |r_{n+1} - r_n| + c|r_{n+2} - r_{n+1}|\,, \end{aligned}$$

we also have $\Gamma_2 \leq c\Gamma_1$. With

$$\left|\frac{1+2r_{n+1}}{1+r_{n+1}} - \frac{1+2r_n}{1+r_n}\right| \left|\int_{r_n}^{r_{n+1}} \left(\frac{r}{1+r}\right)' dr\right| \leq |r_{n+1} - r_n| ,$$

we obtain $\Gamma_4 \leq \Gamma_1$. Moreover, we have

$$\frac{1+2r_n-r_n^2}{4(1+r_n)^2} - \frac{1+2r_{n+1}-r_{n+1}^2}{4(1+r_{n+1})^2} = \int_{r_n}^{r_{n+1}} \frac{r}{(1+r)^3} dr \geq -\frac{8}{28}|r_{n+1}-r_n]_-$$

since the function $r \mapsto r/(1+r)^3$ $(r \geq 0)$ takes its maximum value $8/27$ at $r = 1/2$. Hence, we find $\Gamma_6 \leq c\Gamma_1$. So, $\{\Gamma_0(\mathbb{I}_k)\}_{k\in\mathbb{N}}$ is bounded and $\Gamma_\ell(\mathbb{I}_k) \to 0$ as $k \to \infty$ for $\ell \in \{2, 4, 6\}$ if $\Gamma_1(\mathbb{I}_k) \to 0$ as $k \to \infty$. This is indeed the case if $|r_{n+1}(\mathbb{I}_k) - r_n(\mathbb{I}_k)| = o(\tau_n(\mathbb{I}_k))$. In order to have $\sum_{n=3}^{N_k} \Gamma_{2,n}(\mathbb{I}_k) \to 0$ as $k \to \infty$, we assume, however, $|r_{n+1}(\mathbb{I}_k) - r_n(\mathbb{I}_k)| = o(\tau_n(\mathbb{I}_k)^2)$. Finally, the conditions on Γ_3 and Γ_5 as well as on r_2, r_3 are fulfilled if, in addition, either $r_n = 0$ (Euler step) or $|r_n(\mathbb{I}_k) - 1| = o(\tau_n(\mathbb{I}_k))$. For this, observe that for $n = 2, 3, \ldots, N-1$

$$\frac{r_{n+1}^2}{1+r_{n+1}} - \frac{r_n}{1+r_n} = \frac{r_{n+1}^2}{1+r_{n+1}} - \frac{r_n^2}{1+r_n} + \frac{r_n(r_n-1)}{1+r_n}$$

and thus

$$\Gamma_3 \leq c\tau_{\max}\Gamma_1^2 + \sum_{n=3}^{N} \tau_n \frac{r_n^2(r_n-1)^2}{(1+r_n)^2} .$$

The proof of Theorem 4.1 will be prepared by the following lemmata.

Lemma 4.2. *Under the assumptions of Theorem 4.1 there is a subsequence, denoted by k', such that*

$$u^1(\mathbb{I}_{k'}) - u^0(\mathbb{I}_{k'}) \to 0 \text{ in } H \text{ as } k' \to \infty .$$

The proof is analogous to that of [22, Lemma 4.3] and shall be omitted here.

Lemma 4.3. *Under the assumptions of Theorem 4.1 there is a subsequence, denoted by k', and an element $u \in \mathcal{W}$ such that*

$$\begin{aligned} &u_{\mathbb{I}_{k'}} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H) , \quad u_{\mathbb{I}_{k'}} \rightharpoonup u \text{ in } L^p(0,T;V) , \\ &v_{\mathbb{I}_{k'}} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H) , \quad v'_{\mathbb{I}_{k'}} \rightharpoonup u' \text{ in } \mathcal{X}^* \text{ as } k' \to \infty . \end{aligned}$$

Proof. From (4.2), (4.3) and (4.4), we infer that M and M' in Theorem 3.2 are bounded independently of k, in particular

$$\sigma\left(r_{\max}(\mathbb{I}_k)\right) \geq \sigma\left(\sup_{k\in\mathbb{N}} r_{\max}(\mathbb{I}_k)\right) > \sigma(\overline{r}) = 0 .$$

For readability, we do not emphasise in what follows the dependence of N, τ_n, u^n, and f^n on $\mathbb{I}_k$. From an elementary calculation, we obtain

$$\|u_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} = \max_{n=1,2,\ldots,N} |u^n| , \quad \|u_{\mathbb{I}_k}\|_{L^p(0,T;V)} = \left(\sum_{n=1}^{N} \tau_n \|u^n\|^p\right)^{1/p} .$$

Theorem 3.2 then shows the boundedness of $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ in $L^\infty(0,T;H)$ as well as in $L^p(0,T;V)$, and we conclude by standard arguments (theorems of Eberlein-Šmulyan and Banach-Alaoglu, see e.g. [9, Cor. III.26, Thm. III.27], together with density arguments) with the weak* in $L^\infty(0,T;H)$ and weak in $L^p(0,T;V)$ convergence of a subsequence towards $u \in L^\infty(0,T;H)\cap L^p(0,T;V)$.

With respect to the piecewise linear prolongation, we observe that

$$\|v_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} = \max_{n=0,1,\dots,N} |v_{\mathbb{I}_k}(t_n)| \le \max\left(|u^0|, \max_{n=1,\dots,N}|a_n|\right),$$

with a_n given by (4.1), i.e.

$$a_1 = u^1\,,\quad a_2 = \frac{1+2r_2}{1+r_2}\,u^2 - r_2u^1 + \frac{r_2^2}{1+r_2}u^0\,, \tag{4.6}$$

and for $n=3,4,\dots,N$

$$\begin{aligned} a_n &= a^{(1)} + a_n^{(2)} + a_n^{(3)} \text{ with } a^{(1)} = \left(\frac{r_3^2}{1+r_3} - r_2\right)u^1 + \frac{r_2^2}{1+r_2}u^0\,,\\ a_n^{(2)} &= \sum_{j=2}^{n-2}\left(\frac{1+2r_j}{1+r_j} - (1+r_{j+1}) + \frac{r_{j+2}^2}{1+r_{j+2}}\right)u^j\,,\\ a_n^{(3)} &= \frac{1+2r_n}{1+r_n}\,u^n + \left(\frac{r_{n-1}}{1+r_{n-1}} - r_n\right)u^{n-1}\,. \end{aligned} \tag{4.7}$$

It follows

$$\|v_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} \le c(1+\Gamma_2)\max_{n=0,1,\dots,N}|u^n|\,,$$

where c only depends on $\overline{r}$, and, by the assumptions and Theorem 3.2, $\{v_{\mathbb{I}_k}\}$ is bounded in $L^\infty(0,T;H)$. Furthermore, we find

$$\|v'_{\mathbb{I}_k}\|_{\mathcal{X}^*} \le \max\left(\left(\sum_{n=1}^N \tau_n\,\|\mathrm{D}u^n - f_2^n\|_*^{p^*}\right)^{1/p^*}, \left(\sum_{n=1}^N \tau_n|f_2^n|^2\right)^{1/2}\right)$$

and thus, again by Theorem 3.2 and by the properties of the natural restriction, the asserted boundedness of the sequence of time derivatives in $\mathcal{X}^*$. We, therefore, have a subsequence $\{v_{\mathbb{I}_{k'}}\}$ that converges weakly* in $L^\infty(0,T;H)$ towards a function $v\in L^\infty(0,T;H)$, and the time derivatives $v'_{\mathbb{I}_{k'}}$ converge weakly in $\mathcal{X}^*$ towards v'.

We now prove $v=u$ from which we also obtain that the limit u is in $\mathcal{W}$. From the definition of $u_{\mathbb{I}_k}$ and $v_{\mathbb{I}_k}$, we find (see also (4.6), (4.7))

$$v_{\mathbb{I}_k}(t) - u_{\mathbb{I}_k}(t) = (t-t_n)\mathrm{D}u^n + a_n - u^n\,,\quad t\in(t_{n-1},t_n]\ (n=1,2,\dots,N)\,. \tag{4.8}$$

Some straightforward calculations show for $n=3,4,\dots,N$ that

$$\begin{aligned} a_n^{(3)} - u^n &= \frac{r_n\tau_n\mathrm{D}u^n}{1+2r_n-r_n^2} - \frac{r_n^3\left(u^n - 2u^{n-1} + u^{n-2}\right)}{(1+2r_n-r_n^2)(1+r_n)}\\ &\quad - \left(\frac{r_n^2}{1+r_n} - \frac{r_{n-1}}{1+r_{n-1}}\right)u^{n-1} =: \frac{r_n\tau_n\mathrm{D}u^n}{1+2r_n-r_n^2} + a_n^{(4)} + a_n^{(5)}\,. \end{aligned}$$

This yields (with $a_1 - u^1 = 0$) because of $0 \le r_n/(1+2r_n - r_n^2) \le c$ for $0 \le r_n < \overline{r}$

$$\|v_{\mathbb{I}_k} - u_{\mathbb{I}_k}\|_{\mathcal{X}^*} \le c \max\Bigg(\Bigg(\sum_{n=1}^{N} \tau_n^{1+p^*} \|\mathrm{D}u^n - f_2^n\|_*^{p^*}\Bigg)^{1/p^*},$$

$$\Bigg(\sum_{n=1}^{N} \tau_n^3 |f_2^n|^2\Bigg)^{1/2} + \tau_2^{1/2}\,|a_2 - u^2| + \Bigg(\sum_{n=3}^{N} \tau_n |a^{(1)} + a_n^{(2)} + a_n^{(4)} + a_n^{(5)}|^2\Bigg)^{1/2}\Bigg).$$

Due to the second estimate in Theorem 3.2, the properties of the natural restriction, and since $|a_2 - u^2|$ is bounded, the first three terms on the right-hand side tend to zero as $k \to \infty$. For the last term, we observe that

$$\sum_{n=3}^{N} \tau_n |a^{(1)}|^2 \le T \left|\frac{r_3^2}{1+r_3} - r_2\right| |u^1 - u^0| + T \left|\frac{r_3^2}{1+r_3} - \frac{r_2}{1+r_2}\right| |u^0|\,;$$

the first term on the right-hand side converges towards zero (at least for a subsequence) because of Lemma 4.2 and the last term by assumption. Moreover,

$$\sum_{n=3}^{N} \tau_n |a_n^{(2)}|^2 + \sum_{n=3}^{N} \tau_n |a_n^{(4)}|^2 \le \Gamma_2^2\, TM + c\tau_{\max} M\,,$$

with M being the bound from the first estimate in Theorem 3.2, and again the right-hand side tends to zero as $k \to \infty$. For the remaining term, we get

$$\sum_{n=3}^{N} \tau_n |a_n^{(5)}|^2 \le \Gamma_3\, M$$

which, by assumption, tends to zero as $k \to \infty$. After all, we have shown that

$$v_{\mathbb{I}_{k'}} - u_{\mathbb{I}_{k'}} \to 0 \text{ in } \mathcal{X}^* \text{ as } k' \to \infty\,. \tag{4.9}$$

As $\mathcal{X}$ is dense in $L^1(0,T;H)$, the weak* in $L^\infty(0,T;H)$ limits of an appropriate common subsequence of $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ and $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ coincide. □

Note that we do not have boundedness of $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ and weak convergence of a subsequence towards u in $L^p(0,T;V)$ as this requires further assumptions on the choice of the initial values $u^0(\mathbb{I}_k)$.

Moreover, the difference $a_n - u^n$ $(n = 1, 2, \ldots, N)$ that appears in (4.8) and has to be estimated in order to show the coincidence of the weak limits is nothing else than $v_{\mathbb{I}_k}(t_n) - u_{\mathbb{I}_k}(t_n)$. We also have the representation

$$a_n - u^n = \sum_{j=2}^{n} \frac{r_j}{1+r_j}\left(u^j - (1+r_j)u^{j-1} + r_j u^{j-2}\right), \tag{4.10}$$

as a straightforward calculation shows. Assume for a moment that $u^j = u(t_j)$ with a smooth function u. Then we would come up with

$$a_n - u^n = \frac{1}{2}\sum_{j=2}^{n} \tau_j^2 u''(t_j) + \mathcal{O}(\tau_{\max}^3)\,,$$

and the term would tend to zero for any time grid. Unfortunately, we were not able to make use of the representation (4.10) in the course of the proofs of Lemma 4.3 and the following Lemma 4.4.

LEMMA 4.4. *Under the assumptions of Theorem 4.1 there holds for a subsequence, denoted by k',*

$$\liminf_{k'\to\infty}\langle v'_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}}\rangle \geq 0\,.$$

PROOF. We again omit the subscripts denoting the sequence of time grids if possible. We observe with $a_1 = u^1$ and the definition of $\mathrm{D}u^1$ that

$$\begin{aligned}\langle v'_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}}\rangle &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\mathrm{D}u^n, u^n - (t-t_n)\mathrm{D}u^n - a_n)\,dt \\ &= \frac{1}{2}|u^1-u^0|^2 + \sum_{n=2}^{N}\tau_n\left(\mathrm{D}u^n, u^n + \frac{\tau_n}{2}\mathrm{D}u^n - a_n\right),\end{aligned} \tag{4.11}$$

where a_n is given by (4.1). From (4.1), we obtain for $n = 2, 3, \ldots, N$

$$d_n := u^n + \frac{\tau_n}{2}\mathrm{D}u^n - a_n = u^n + \frac{\tau_n}{2}\mathrm{D}u^n - \sum_{j=2}^{n}\tau_j\mathrm{D}u^j - u^1\,. \tag{4.12}$$

In particular, we find

$$d_2 = \frac{1}{2(1+r_2)}(u^2-u^1) + \frac{r_2^2}{2(1+r_2)}(u^1-u^0)$$

and thus with Young's inequality

$$\begin{aligned}&\tau_2(\mathrm{D}u^2, d_2) \\ &= \frac{1}{2(1+r_2)^2}\Big((1+2r_2)(u^2-u^1) - r_2^2(u^1-u^0), u^2-u^1+r_2^2(u^1-u^0)\Big) \\ &\geq \frac{1+2r_2-r_2^2}{4(1+2r_2)^2}|u^2-u^1|^2 - \frac{(1+2r_2+3r_2^2)r_2^4}{2(1+r_2)^4}|u^1-u^0|^2\,.\end{aligned} \tag{4.13}$$

With (4.7) and (4.12), we find for $n = 3, 4, \ldots, N$

$$\begin{aligned}d_n &= -a^{(1)} - a_n^{(2)} + d'_n \text{ with} \\ d'_n &= \frac{1}{2(1+r_n)}u^n - \left(\frac{1+2r_{n-1}}{1+r_{n-1}} - \frac{1+r_n}{2}\right)u^{n-1} + \frac{r_n^2}{2(1+r_n)}u^{n-2}\,.\end{aligned}$$

From (4.1), (4.6), (4.7), and Theorem 3.2, we obtain

$$\begin{aligned}\left|\sum_{n=3}^{N}\tau_n(\mathrm{D}u^n, a^{(1)})\right| &\leq \left|\sum_{n=3}^{N}\tau_n\mathrm{D}u^n\right||a^{(1)}| = |a_N - a_2||a^{(1)}| \\ &\leq (c+\Gamma_2)M^{1/2}\left(c|u^1-u^0| + \left|\frac{r_3^2}{1+r_3} - \frac{r_2}{1+r_2}\right||u^0|\right),\end{aligned} \tag{4.14}$$

where c only depends on $\overline{r}$. Moreover, we have

$$
\begin{aligned}
&\left|\sum_{n=3}^{N}\tau_n(\mathrm{D}u^n, a_n^{(2)})\right| \\
&\le \max_{n=3,4,\dots,N}\left(\frac{1+2r_n}{1+r_n}|u^n| + (1+r_n)|u^{n-1}| + \frac{r_n^2}{1+r_n}|u^{n-2}|\right)\sum_{n=3}^{N}|a_n^{(2)}| \qquad (4.15)\\
&\le 2(1+\overline{r})M\sum_{n=3}^{N}\Gamma_{2,n}\,.
\end{aligned}
$$

It remains to estimate the terms with d_n'. It is straightforward to show that

$$
d_n' = d_n^{(1)} + d_n^{(2)} + d_n^{(3)} \text{ with } d_n^{(1)} = \left(\frac{1+2r_n}{1+r_n} - \frac{1+2r_{n-1}}{1+r_{n-1}}\right)u^{n-1}\,,
$$
$$
d_n^{(2)} = \frac{(r_n-1)^2}{2(1+r_n)}u^{n-1} + \frac{r_n^2-1}{2(1+r_n)}u^{n-2}\,,\ d_n^{(3)} = \frac{u^n - 2u^{n-1} + u^{n-2}}{2(1+r_n)}\,.
$$

Similarly as before, we come up with

$$
\left|\sum_{n=3}^{N}\left(\tau_n\mathrm{D}u^n, d_n^{(1)}\right)\right| \le cM\Gamma_4\,.
$$

If $r_n = 0$ then $d_n^{(2)} + d_n^{(3)} = (u^n - u^{n-1})/2$ and thus

$$
\tau_n(\mathrm{D}u^n, d_n^{(2)} + d_n^{(3)}) = \frac{1}{2}|u^n - u^{n-1}|^2 \ge \frac{1}{4}\left(|u^n - u^{n-1}|^2 - |u^{n-1} - u^{n-2}|^2\right).
$$

If $r_n \ne 0$, we find

$$
\tau_n\left|(\mathrm{D}u^n, d_n^{(2)})\right| \le cM\frac{(r_n-1)^2 + |r_n^2 - 1|}{2(1+r_n)} \le cM|r_n - 1|
$$

and thus

$$
\left|\sum_{n=3}^{N}\chi_n\tau_n(\mathrm{D}u^n, d_n^{(2)})\right| \le cM\Gamma_5\,.
$$

For the remaining term with $d_n^{(3)}$, we employ the following algebraic relation that is easily proved by Young's inequality for any $a, b, c \in \mathbb{R}$ and $r \ge 0$,

$$
\begin{aligned}
&\left(\frac{1+2r}{1+r}a - (1+r)b + \frac{r^2}{1+r}c\right)\left(a - 2b + c\right) \\
&= \left(\frac{1+2r}{1+r}(a-b) - \frac{r^2}{1+r}(b-c)\right)\left((a-b) - (b-c)\right) \\
&\ge \frac{1+2r-r^2}{2(1+r)}\left((a-b)^2 - (b-c)^2\right).
\end{aligned}
$$

It thus follows

$$\tau_n(\mathrm{D}u^n, d_n^{(3)}) \geq \frac{1+2r_n-r_n^2}{4(1+r_n)^2}\Big(|u^n-u^{n-1}|^2 - |u^{n-1}-u^{n-2}|^2\Big).$$

Altogether, we end up with (note that $(1+2r_n-r_n^2)/(1+r_n)^2 = 1$ for $r_n = 0$)

(4.16)

$$\begin{aligned}\sum_{n=3}^N \tau_n(\mathrm{D}u^n, d_n') &\geq -\left|\sum_{n=3}^N \tau_n(\mathrm{D}u^n, d_n^{(1)})\right| - \left|\sum_{n=3}^N \chi_n\tau_n(\mathrm{D}u^n, d_n^{(2)})\right| \\ &\quad + \sum_{n=3}^N (1-\chi_n)\tau_n(\mathrm{D}u^n, d_n^{(2)}+d_n^{(3)}) + \sum_{n=3}^N \chi_n\tau_n(\mathrm{D}u^n, d_n^{(3)}) \\ &\geq -c(\Gamma_4+\Gamma_5)M + \sum_{n=3}^N \frac{1+2r_n-r_n^2}{4(1+r_n)^2}\Big(|u^n-u^{n-1}|^2 - |u^{n-1}-u^{n-2}|^2\Big) \\ &= -c(\Gamma_4+\Gamma_5)M + \frac{1+2r_N-r_N^2}{4(1+r_N)^2}|u^N-u^{N-1}|^2 - \frac{1+2r_2-r_2^2}{4(1+r_2)^2}|u^2-u^1|^2 \\ &\quad + \frac{1}{4}\sum_{n=2}^{N-1}\left(\frac{1+2r_n-r_n^2}{(1+r_n)^2} - \frac{1+2r_{n+1}-r_{n+1}^2}{(1+r_{n+1})^2}\right)|u^n-u^{n-1}|^2. \\ &\geq -c(\Gamma_4+\Gamma_5+\Gamma_6)M - \frac{1+2r_2-r_2^2}{4(1+r_2)^2}|u^2-u^1|^2.\end{aligned}$$

From (4.13), (4.14), (4.15), and (4.16), we deduce that

$$\begin{aligned}\sum_{n=2}^N \tau_n(\mathrm{D}u^n, d_n) &\geq -c\left(\sum_{n=3}^N \Gamma_{2,n} + \Gamma_4+\Gamma_5+\Gamma_6\right) \\ &\quad - c(1+\Gamma_2)\left(|u^1-u^0|^2 + \left|\frac{r_3^2}{1+r_3} - \frac{r_2}{1+r_2}\right||u^0|\right)\end{aligned} \tag{4.17}$$

with c depending on $\overline{r}$ and M. The assumptions on the sequence of time grids and Lemma 4.2 ensure that the right-hand side in (4.17) converges towards zero, at least for a subsequence. This, together with (4.11), proves the assertion. □

We are now ready to prove the main result.

Proof. [Proof of Theorem 4.1] In what follows, if it is clear from the context, we omit emphasising the dependence (of N, τ_n, u^n, etc.) on $\mathbb{I}_k$.

As a first step, we rewrite the numerical scheme (1.3) corresponding to the time grid $\mathbb{I}_k$ as the differential equation

$$v_{\mathbb{I}_k}' + Au_{\mathbb{I}_k} = \mathrm{R}_{\mathbb{I}_k} f\,, \tag{4.18}$$

where $(\mathrm{R}_{\mathbb{I}_k} f)(t) := \mathrm{R}_{\mathbb{I}_k}^n f$ for $t \in (t_{n-1}(\mathbb{I}_k), t_n(\mathbb{I}_k)]$ $(n = 1, 2, \ldots, N_k)$.

By standard arguments, we get the strong convergence

$$\mathrm{R}_{\mathbb{I}_k} f \to f \text{ in } \mathcal{X}^* \text{ as } k \to \infty\,. \tag{4.19}$$

The second estimate in Theorem 3.2 yields the boundedness of $\{Au_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ in $L^{p^*}(0,T;V^*)$. So there is a subsequence, denoted by k', such that

$$Au_{\mathbb{I}_{k'}} \rightharpoonup a \text{ in } L^{p^*}(0,T;V^*) \text{ as } k' \to \infty \tag{4.20}$$

for an element $a \in L^{p^*}(0,T;V^*)$. The subsequence can be chosen such that also the weak and weak* convergence from Lemma 4.3 take place.

Since $v'_{\mathbb{I}_{k'}} \rightharpoonup u'$ in $\mathcal{X}^*$ as $k' \to \infty$ (see Lemma 4.3), we find

$$0 = v'_{\mathbb{I}_{k'}} + Au_{\mathbb{I}_{k'}} - \mathrm{R}_{\mathbb{I}_{k'}} f \rightharpoonup u' + a - f \text{ in } \mathcal{X}^* \text{ as } k' \to \infty$$

and thus

$$u' + a = f \text{ in } \mathcal{X}^*\,. \tag{4.21}$$

It remains to show that $u \in \mathcal{W}$ fulfills the initial condition and that $a = Au$.

In virtue of (4.4), we have

$$v_{\mathbb{I}_k}(0) = u^0(\mathbb{I}_k) \to u_0 \text{ in } H \text{ as } k \to \infty\,. \tag{4.22}$$

As was already shown in the proof of Lemma 4.3, $\{v_{\mathbb{I}_k}(T)\}_{k\in\mathbb{N}}$ is bounded in H. We thus can choose the subsequence such that

$$v_{\mathbb{I}_{k'}}(T) \rightharpoonup \xi \text{ in } H \text{ as } k' \to \infty \tag{4.23}$$

for some $\xi \in H$.

Since $v_{\mathbb{I}_k} \in \mathcal{W}$, we can employ integration by parts which yields for all $v \in V$ and $\phi \in \mathcal{C}^1([0,T])$

$$\begin{aligned}
&(u(T),v)\phi(T) - (u(0),v)\phi(0) = \int_0^T \Big(\langle u'(t),v\rangle\phi(t) + \langle u(t),v\rangle\phi'(t)\Big)dt \\
&= \int_0^T \Big(\langle f(t) - a(t),v\rangle\phi(t) + \langle u(t),v\rangle\phi'(t)\Big)dt \\
&= \int_0^T \Big(\langle f(t) - \mathrm{R}_{\mathbb{I}_{k'}} f(t) + v'_{\mathbb{I}_{k'}}(t) + Au_{\mathbb{I}_{k'}}(t) - a(t),v\rangle\phi(t) + \langle u(t),v\rangle\phi'(t)\Big)dt \\
&= \int_0^T \Big(\langle f(t) - \mathrm{R}_{\mathbb{I}_{k'}} f(t) + Au_{\mathbb{I}_{k'}}(t) - a(t),v\rangle\phi(t) + \langle u(t) - v_{\mathbb{I}_{k'}}(t),v\rangle\phi'(t)\Big)dt \\
&\qquad\qquad + (v_{\mathbb{I}_{k'}}(T),v)\phi(T) - (v_{\mathbb{I}_{k'}}(0),v)\phi(0)\,.
\end{aligned}$$

Taking the limit on the right-hand side, we come up with

$$(u(T),v)\phi(T) - (u(0),v)\phi(0) = (\xi,v)\phi(T) - (u_0,v)\phi(0)\,.$$

Choosing $\phi(T) = 0$ and $\phi(0) = 0$, respectively, we find that

$$u(0) = u_0\,, \quad u(T) = \xi \tag{4.24}$$

in H since V is dense in H. The limit u thus satisfies the initial condition in (1.1).

Employing the monotonicity of A, we find (with $\langle \cdot, \cdot \rangle$ denoting the dual pairing between $\mathcal{X}^*$ and $\mathcal{X}$) for any $w \in L^p(0,T;V) \cap L^\infty(0,T;H)$

$$\langle Au_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle \geq \langle Au_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle - \langle Au_{\mathbb{I}_{k'}} - Aw, u_{\mathbb{I}_{k'}} - w \rangle = \langle Au_{\mathbb{I}_{k'}}, w \rangle + \langle Aw, u_{\mathbb{I}_{k'}} - w \rangle\,.$$

Testing (4.18) by $u_{\mathbb{I}_{k'}}$ thus yields

(4.25)
$$\begin{aligned} 0 &= \langle v'_{\mathbb{I}_{k'}} + Au_{\mathbb{I}_{k'}} - \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \\ &= \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle + \langle v'_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}} \rangle + \langle Au_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle - \langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \\ &\geq \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle + \langle v'_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}} \rangle + \langle Au_{\mathbb{I}_{k'}}, w \rangle + \langle Aw, u_{\mathbb{I}_{k'}} - w \rangle - \langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle\,. \end{aligned}$$

The problematic term in (4.25) is $\langle v'_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}} \rangle$; it has already been dealt with in Lemma 4.4. Since $u, v_{\mathbb{I}_{k'}} \in \mathcal{W}$, we find from integration by parts

$$\langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle = \frac{1}{2}\left(|v_{\mathbb{I}_{k'}}(T)|^2 - |v_{\mathbb{I}_{k'}}(0)|^2\right), \quad \langle u', u \rangle = \frac{1}{2}\left(|u(T)|^2 - |u(0)|^2\right).$$

Because of (4.22), (4.23), (4.24), we thus obtain

$$\langle u', u \rangle \leq \liminf_{k' \to \infty} \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle\,.$$

Because of (4.20) and Lemma 4.3, we find

$$\langle Au_{\mathbb{I}_{k'}}, w \rangle + \langle Aw, u_{\mathbb{I}_{k'}} - w \rangle \to \langle a, w \rangle + \langle Aw, u - w \rangle\,.$$

With (4.19) and Lemma 4.3, we also have

$$\langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \to \langle f, u \rangle\,.$$

Altogether, we obtain from (4.25) together with (4.21)

$$0 \geq \langle u', u \rangle + \langle a, w \rangle + \langle Aw, u - w \rangle - \langle f, u \rangle = -\langle a, u \rangle + \langle a, w \rangle + \langle Aw, u - w \rangle$$

which yields

$$\langle a, u - w \rangle \geq \langle Aw, u - w \rangle\,.$$

With $w = u \pm sv$ ($v \in L^p(0,T;V) \cap L^\infty(0,T;H)$) and $s \to 0+$, the demicontinuity and thus hemicontinuity of the monotone potential operator A proves, by density, $a = Au$ in $L^{p^*}(0,T;V^*)$. This completes the proof of u being a solution to (1.1).

By contradiction, we can show that the whole sequences $\{u_{\mathbb{I}_k}\}_{k \in \mathbb{N}}$ and $\{v_{\mathbb{I}_k}\}_{k \in \mathbb{N}}$ converge towards u since a solution to (1.1) is unique in $\mathcal{W}$. ◻

REFERENCES

1. G. Akrivis and M. Crouzeix, Linearly implicit methods for nonlinear parabolic equations, *Math. Comp.* **73** (2004) 246, pp. 613–635.
2. G. Akrivis, M. Crouzeix, and C. Makridakis, Implicit-explicit multistep finite element methods for nonlinear parabolic problems, *Math. Comp.* **67** (1998) 222, pp. 457–477.
3. G. Akrivis, M. Crouzeix, and C. Makridakis, Implicit-explicit multistep methods for quasilinear parabolic equations, *Numer. Math.* **82** (1999) 4, pp. 521–541.
4. G. Akrivis, C. Makridakis, and R. H. Nochetto, A posteriori error estimates for the Crank-Nicolson method for parabolic equations, *Math. Comp.* **75** (2006) 254, pp. 511–531.
5. O. Axelsson and S. V. Gololobov, Stability and error estimates for the ϑ-method for strongly monotone and infinitely stiff evolution equations, *Numer. Math.* **89** (2001) 1, pp. 31–48.
6. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff Int. Publ., Leyden, 1976.
7. J. Becker, A second order backward difference method with variable steps for a parabolic problem, *BIT* **38** (1998) 4, pp. 644–662.
8. R.K. Brayton and C.S. Conley, Some results on the stability and instability of the backward differentiation methods with non-uniform time steps, in: *Topics in numerical analysis* (Proc. Roy. Irish Acad. Conf., University Coll., Dublin, 1972), pp. 13–33, Academic Press, London, 1973.
9. H. Brézis, *Analyse fonctionnelle: Théorie et applications*, Dunod, Paris, 1999.
10. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publ. Comp., Amsterdam – London, 1973.
11. G.D. Byrne and A.C. Hindmarsh, Stiff ODE solvers: a review of current and coming attractions, *J. Comput. Phys.* **70** (1987) 1, 1–62.
12. M. Calvo, T. Grande, and R.D. Grigorieff, On the zero-stability of the variable order variable stepsize BDF-formulas, *Numer. Math.* **57** (1990) 1, pp. 39–50.
13. M. Calvo and R.D. Grigorieff, Time discretisation of parabolic problems with the variable 3-step BDF, *BIT* **42** (2002) 4, pp. 689–701.
14. M. Calvo, J.I. Montijano, and L. Rández, A_0-stability of variable stepsize BDF methods, *J. Comput. Appl. Math.* **45** (1993) 1-2, pp. 29–39.
15. M. Calvo, J.I. Montijano, and L. Rández, On the change of step size in multistep codes, *Numer. Algorithms* **4** (1993) 3, pp. 283–304.
16. M. P. Calvo and C. Palencia, A class of explicit multistep exponential integrators for semilinear problems, *Numer. Math.* **102** (2006) 3, pp. 367–381.
17. M. Crouzeix and V. Thomée, On the discretization in time for semilinear parabolic equations with nonsmooth initial data, *Math. Comp.* **49** (1987) 180, pp. 359–377.
18. E. Emmrich, Error of the two-step BDF for the incompressible Navier-Stokes problem, *M2AN Math. Model. Numer. Anal.* **38** (2004) 5, pp. 757–764.
19. E. Emmrich, Stability and convergence of the two-step BDF for the incompressible Navier-Stokes problem, *Int. J. Nonlin. Sci. Numer. Simul.* **5** (2004) 3, pp. 199–210.
20. E. Emmrich, Stability and error of the variable two-step BDF for semilinear parabolic problems, *J. Appl. Math. Comput.* **19** (2005) 1-2, pp. 33–55.
21. E. Emmrich, Two-step BDF time discretisation of nonlinear evolution problems governed by monotone operators with strongly continuous perturbations, TU Berlin, September 2007.
22. E. Emmrich, Convergence of a time discretization for a class of non-Newtonian

fluid flow, TU Berlin, April 2008.
23. E. Emmrich, Variable time-step ϑ-scheme for nonlinear evolution equations governed by a monotone operator, TU Berlin, May 2008.
24. H. Gajewski, K. Gröger, and K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie-Verlag, Berlin, 1974.
25. B. García-Archilla, Some practical experience with the time integration of dissipative equations, *J. Comput. Phys.* **122** (1995) 1, pp. 25–29.
26. B. García-Archilla and J. de Frutos, Time integration of the non-linear Galerkin method, *IMA J. Numer. Anal.* **15** (1995) 2, pp. 221–244.
27. V. Girault and P.-A. Raviart, *Finite element approximation of the Navier-Stokes equations*, Lecture Notes in Mathematics, vol. **749**, Springer, Berlin, 1981.
28. C. González, A. Ostermann, C. Palencia, and M. Thalhammer, Backward Euler discretization of fully nonlinear parabolic problems, *Math. Comp.* **71** (2002) 237, pp. 125–145.
29. C. González and C. Palencia, Stability of Runge-Kutta methods for quasilinear parabolic problems, *Math. Comp.* **69** (2000) 230, pp. 609–628.
30. R.D. Grigorieff, Stability of multistep-methods on variable grids, *Numer. Math.* **42** (1983) 3, pp. 359–377.
31. R.D. Grigorieff, Time discretization of semigroups by the variable two-step BDF method, in: *Numerical treatment of differential equations* (Sel. Pap. 5th Int. Semin., NUMDIFF-5, Halle, 1989), Teubner-Texte Math. 121, 1991, pp. 204–216.
32. R.D. Grigorieff, *On the variable grid two-step BDF method for parabolic equations,* Preprint 426, FB Mathem., TU Berlin, 1995.
33. R.D. Grigorieff and P.J. Paes-Leme, On the zero-stability of the 3-step BDF-formula on nonuniform grids, *BIT* **24** (1984) 1, pp. 85–91.
34. E. Hairer, S.P. Nørsett, and G. Wanner, *Solving ordinary differential equations, I: Nonstiff problems* Springer, Berlin, 1993.
35. E. Hairer and G. Wanner, *Solving ordinary differential equations, II: Stiff problems* Springer, Berlin, 1996.
36. E. Hansen, Runge-Kutta time discretizations of nonlinear dissipative evolution equations, *Math. Comp.* **75** (2006) 254, pp. 631–640.
37. E. Hansen, Convergence of multistep time discretizations of nonlinear dissipative evolution equations, *SIAM J. Numer. Anal.* **44** (2006) 1, pp. 55–65.
38. E. Hansen, Galerkin/Runge-Kutta discretizations of nonlinear parabolic equations, *J. Comput. Appl. Math.* **205** (2007) 2, pp. 882–890.
39. R. Hass and H. Kreth, Stabilität und Konvergenz von Mehrschrittverfahren zur numerischen Lösung quasilinearer Anfangswertprobleme, *ZAMM* **54** (1974), pp. 353–358.
40. A. T. Hill and E. Süli, Upper semicontinuity of attractors for linear multistep methods approximating sectorial evolution equations, *Math. Comp.* **64** (1995) 211, pp. 1097–1122.
41. A. T. Hill and E. Süli, Approximation of the global attractor for the incompressible Navier-Stokes equations, *IMA J. Numer. Anal.* **20** (2000) 4, pp. 633–667.
42. H. Kreth, Time-discretisations for nonlinear evolution equations, in: *Numerical treatment of differential equations in applications* (Proc. Meeting, Math. Res. Center, Oberwolfach, 1977), pp. 57–63, Lecture Notes in Mathematics, vol. **679**, Springer, Berlin, 1978.
43. H. Kreth, Ein Zwei-Schritt-Differenzenverfahren zur Berechnung strömungsabhängiger Ausbreitungsvorgänge, *ZAMP* **29** (1978), pp. 12–22.
44. M.-N. Le Roux, Méthodes multipas pour des équations paraboliques non linéaires,

Numer. Math. **35** (1980), pp. 143–162.
45. M.-N. Le Roux, Variable step size multistep methods for parabolic problems, *SIAM J. Numer. Anal.* **19** (1982) 4, pp. 725–741.
46. J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
47. C. Lubich and A. Ostermann, Runge-Kutta approximation of quasi-linear parabolic equations, *Math. Comp.* **64** (1995) 210, pp. 601–627.
48. C. Lubich and A. Ostermann, Linearly implicit time discretization of non-linear parabolic equations, *IMA J. Numer. Anal.* **15** (1995) 4, pp. 555–583.
49. C. Lubich and A. Ostermann, Runge-Kutta time discretization of reaction-diffusion and Navier-Stokes equations: Nonsmooth-data error estimates and applications to long-time behaviour, *Appl. Numer. Math.* **22** (1996) 1-3, pp. 279–292.
50. A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser, Basel, 1995.
51. C. Makridakis and R. H. Nochetto, A posteriori error analysis for higher order dissipative methods for evolution problems, *Numer. Math.* **104** (2006) 4, pp. 489–514.
52. P. K. Moore, A posteriori error estimation with finite element semi- and fully discrete methods for nonlinear parabolic equations in one space dimension, *SIAM J. Numer. Anal.* **31** (1994) 1, pp. 149–169.
53. R. H. Nochetto and G. Savaré, Nonlinear evolution governed by accretive operators in Banach spaces: error control and applications, *Math. Models Methods Appl. Sci.* **16** (2006) 3, pp. 439–477.
54. R. H. Nochetto, G. Savaré, and C. Verdi, A posteriori error estimates for variable time-step discretizations of nonlinear evolution equations, *Commun. Pure Appl. Math.* **53** (2000) 5, pp. 525–589.
55. A. Ostermann and M. Thalhammer, Convergence of Runge-Kutta methods for nonlinear parabolic equations, *Appl. Numer. Math.* **42** (2002) 1-3, pp. 367–380.
56. A. Ostermann, M. Thalhammer, and G. Kirlinger, Stability of linear multistep methods and applications to nonlinear parabolic problems, *Appl. Numer. Math.* **48** (2004) 3-4, pp. 389–407.
57. C. Palencia and B. García-Archilla, Stability of linear multistep methods for sectorial operators in Banach spaces, *Appl. Numer. Math.* **12** (1993) 6, pp. 503–520.
58. T. Roubíček, *Nonlinear partial differential equations with applications*, Birkhäuser, Basel, 2005.
59. J. Rulla, Error analysis for implicit approximations to solutions to Cauchy problems, *SIAM J. Numer. Anal.* **33** (1996) 1, pp. 68–87.
60. G. R. Sell and Y. You, *Dynamics of evolutionary equations*, Springer, New York, 2002.
61. M. Slodička, Smoothing effect and discretization in time to semilinear parabolic equations with nonsmooth data, *Comment. Math. Univ. Carolinae* **32** (1991) 4, pp. 703–713.
62. M. Slodička, Semigroup formulation of Rothe's method: application to parabolic problems, *Comment. Math. Univ. Carolinae* **33** (1992) 2, pp. 245–260.
63. A.M. Stuart and A.R. Humphries, *Dynamical systems and numerical analysis*, Cambridge University Press, Cambridge, 1996.
64. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer, New York, 1997.
65. V. Thomée, *Galerkin finite element methods for parabolic problems*, Springer, Berlin, 2006.

66. E. Zeidler, *Nonlinear functional analysis and its applications, II/A: Linear monotone operators, II/B: Nonlinear monotone operators*, Springer, New York, 1990.
67. M. Zlámal, Finite element methods for nonlinear parabolic equations, *R.A.I.R.O. Anal. Numér.* **11** (1977) 1, pp. 93–107.