VARIABLE TIME-STEP ϑ-SCHEME FOR NONLINEAR EVOLUTION EQUATIONS GOVERNED BY A MONOTONE OPERATOR

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Abstract. The single-step ϑ-scheme on a variable time grid is employed for the approximate solution of the initial-value problem for a nonlinear first-order evolution equation. The evolution equation is supposed to be governed by a possibly time-dependent hemicontinuous operator that is (up to a shift) monotone and coercive, and fulfills a growth condition.

A piecewise constant as well as piecewise linear prolongation of the time-discrete solution is shown to converge towards the exact solution if $\vartheta \geq 1/2$ (including the Crank-Nicolson scheme). In the appearance of a strongly continuous perturbation of the monotone main part, the method is still convergent if $\vartheta > 1/2$ and if the ratio of adjacent step sizes is bounded from above by a power of $\vartheta/(1-\vartheta)$.

Besides convergence also well-posedness of the time-discrete problem as well as a priori error estimates are studied. **Keywords:** Evolution equation, monotone operator, time discretisation, ϑ-scheme, non-uniform grid, convergence.

1. Introduction

Although nonlinear evolution problems appear in many applications and are well-studied from the analytical point of view (see the many monographs as e.g. [6, 7, 14, 27, 31, 38, 40, 43, 45] and the references cited therein), their numerical analysis still offers many open questions.

In this paper, we are concerned with the time discretisation of the initial-value problem for a nonlinear evolution equation,

$$u' + Au = f \quad \text{in } (0,T)\,, \quad u(0) = u_0\,. \tag{1.1}$$

The operator A is supposed to be the Nemytskii operator corresponding to a family of hemicontinuous operators $A(t) : V \to V^*$ ($t \in [0,T]$) acting on a Gelfand triple $V \subseteq H \subseteq V^*$. The main assumption is that $A(t) + \kappa I : V \to V^*$ (with I being the identity) is, uniformly in $t \in [0,T]$, coercive and monotone for some $\kappa \geq 0$. Moreover, $A(t) : V \to V^*$ is assumed to fulfill a growth condition.

Besides, we also consider the problem

$$u' + Au + Bu = f \quad \text{in } (0,T)\,, \quad u(0) = u_0\,, \tag{1.2}$$

where B is the Nemytskii operator corresponding to a family of strongly continuous operators $B(t) : V \to V^*$ ($t \in [0,T]$).

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The time discretisation under consideration is the single-step ϑ-scheme on the variable time grid

$$
\begin{cases}
\mathbb{I} : 0 = t_0 < t_1 < \cdots < t_N = T \ (N \in \mathbb{N}) \text{ with} \\
\tau_n := t_n - t_{n-1} \ (n = 1, 2, \ldots, N)\,, \\
r_n := \dfrac{\tau_n}{\tau_{n-1}} \ (n = 2, 3, \ldots, N)\,, \\
\tau_{\max} := \max\limits_{n=1,2,\ldots,N} \tau_n\,, \quad r_{\max} := \max\limits_{n=2,3,\ldots,N} r_n\,.
\end{cases} \tag{1.3}
$$

Sometimes, we emphasise the dependence on $\mathbb{I}$ by writing e.g. $\tau_{\max}(\mathbb{I})$. The temporal semidiscretisation of (1.1) for the computation of a time-discrete solution $u^n \approx u(t_n)$ $(n = 1, 2, \ldots, N)$ then reads as

$$
\begin{cases}
\dfrac{1}{\tau_n}(u^n - u^{n-1}) + A(t_{n-1+\vartheta})u^{n-1+\vartheta} = f^{n-1+\vartheta} \\
\qquad\qquad\qquad\qquad\qquad\qquad (n = 1, 2, \ldots, N) \text{ with} \\
u^{n-1+\vartheta} := \vartheta u^n + (1-\vartheta)u^{n-1}\,,\ t_{n-1+\vartheta} := \vartheta t_n + (1-\vartheta)t_{n-1}\,,
\end{cases} \tag{1.4}
$$

where $\{f^{n-1+\vartheta}\}_{n=1}^N$ is a given approximation of the right-hand side f and $u^0 \approx u_0$ is a given starting value. Here, $\vartheta \in [0, 1]$ is a fixed parameter. For $\vartheta = 0$ and $\vartheta = 1$, we obtain the explicit and implicit Euler scheme, respectively, whereas $\vartheta = 1/2$ corresponds to the Crank-Nicolson (or midpoint) scheme. Note that, in the case of an equidistant time grid, the scheme is (strongly) A-stable if $\vartheta \geq 1/2$ $(\vartheta > 1/2)$.

The time discretisation of linear evolution problems is rather well-understood, and we may refer to the monograph [44] and the references cited therein.

The approximation of semilinear evolution equations by means of single-step methods has been considered e.g. in [9, 30, 41, 42], and by means of linear multistep methods e.g. in [22]. In [8], explicit multistep exponential integrators have been studied. Implicit-explicit multistep methods have been considered in [1, 2, 3].

The time discretisation for a class of quasilinear evolution problems by multistep schemes has been studied in [21, 26, 46]. An analysis of Runge-Kutta methods can be found in [17, 28]. Stability and error estimates for linearly implicit one-step methods applied to nonlinear evolution equations posed in a Gelfand triple are proven in [29] relying on a linearisation. The two-step backward differentiation formula has been dealt with for nonlinear problems as e.g. the incompressible Navier-Stokes problem in [10, 11, 12, 15, 23, 24, 25, 26, 33].

The backward Euler, strongly $A(\vartheta)$-stable Runge-Kutta discretisations, and linear multistep methods for fully nonlinear problems, which are governed by a densely defined nonlinear mapping in a Banach space whose first Fréchet derivative is sectorial, have been dealt with, again by linearisation, in [16, 36, 37].

Evolution equations governed by maximal monotone operators and their time discretisation have been studied in [18, 19, 20, 39].

A posteriori error estimates for the time discretisation of nonlinear evolution problems (governed by an angle-bounded or dissipative operator) have been studied in [32, 34, 35]. Recently, also a posteriori error estimates for the Crank-Nicolson scheme have been derived in [4] for a semilinear evolution problem with a positive definite self-adjoint main part and a locally Lipschitz-continuous perturbation.

In [5], the ϑ-scheme has been considered for the quite restrictive class of strongly monotone operators. Stability and error estimates of order 4/3 in terms of the discrete $L^\infty(H)$-norm in case of a smooth solution u with $u''' \in L^\infty(H)$ have been derived for all times $t > 0$ if ϑ is sufficiently large. Optimal second order error estimates are obtained under further assumptions on the nearly equidistant time grid.

In this paper, we prove, for a rather large class of nonlinear evolution problems, the convergence of piecewise polynomial prolongations of the discrete numerical solution towards the weak solution without requiring any additional and in general not known regularity of the exact solution. A priori estimates for the numerical solution are the essential prerequisite for the convergence.

The importance of these new convergence results lies in the theoretical substantiation of the numerical approximation even in the case when the exact solution does not exhibit the regularity that is assumed for deriving standard error estimates. In this sense, these results are complementary to error estimates providing optimal or suboptimal order of convergence. Note that, in the nonlinear case considered here, convergence without additional regularity assumptions cannot be derived from error estimates for smooth solutions. This is in contradiction to the linear case, where density arguments and interpolation allow to obtain convergence results even for non-smooth solutions from the standard error estimates. We shall also remark that in the generic case, i.e., when the problem admits only a weak solution without additional smoothness, one cannot expect convergence better than the one provided here.

In the case of a non-monotone perturbation of the monotone main part, it turns out that, in opposite to fully implicit methods as the backward Euler or the two-step backward differentiation formula on an equidistant grid, the convergence can only be proved for $\vartheta > 1/2$ and requiring an additional assumption on the quotient of adjacent step sizes. The reason for this is that otherwise we are not able to prove a strong convergence of the numerical solution that is necessary to handle the perturbation.

Moreover, we show stability of the numerical solution with respect to the data. These estimates then allow to derive local a priori error estimates of optimal order for sufficiently regular solutions.

The method of proof relies upon the theory of monotone operators and compactness arguments together with algebraic relations that describe properties of the temporal discretisation. So, no linearisation and thus no differentiability of the operator is employed. Note, however, that our assumptions imply global well-posedness of the original problem which is different from the approach in e.g. [16, 36, 37].

Results similar to those obtained here have recently been obtained in [13] for the two-step backward differentiation formula on an equidistant grid and can be found in [38, Ch. 8.2]) for the backward Euler method.

All the results of this paper apply to e.g. the fluid flow of a porous medium as described in [27, pp. 191 ff.] and [14, pp. 72 f.]). Unfortunately, a direct application of the results to incompressible fluid flow problems described by the Navier-Stokes equations or the equations for generalised Newtonian fluids is not possible. Nevertheless, the methods developed here can, with some modifications regarding the convection term, also be used to study these problems. This will be conducted in forthcoming research.

The paper is organised as follows: The notation and the analytical framework for studying (1.1) is described in Section 2. Existence, uniqueness, and a priori estimates of the solution to the temporal semidiscretisation (1.4) are shown in Section 3. The main convergence result is then proven in Section 4. Stability and a priori error estimates in case of a sufficiently smooth exact solution are derived in Section 5. The generalisation of the results to the perturbed problem (1.2) is, finally, discussed in Section 6.

2. Notation and time continuous problem

Let $(V, \|\cdot\|)$ be a reflexive, separable, real Banach space that is dense and continuously embedded in the Hilbert space $(H, (\cdot,\cdot), |\cdot|)$. The dual V^* of V is equipped with the norm $\|f\|_* := \sup_{v\in V\setminus\{0\}} \langle f, v\rangle / \|v\|$. Here, $\langle\cdot,\cdot\rangle$ denotes the dual pairing; sometimes, we call the spaces in a subscript as in $\langle\cdot,\cdot\rangle_{V^*\times V}$. So, $V \subseteq H \subseteq V^*$ is a Gelfand triple.

For a Banach space X and the time interval $[0,T]$, let $L^r(0,T;X)$ ($r \in [1,\infty]$) denote the Banach space of Bochner integrable (for $r=\infty$ Bochner measurable and essentially bounded) abstract functions with the standard norm denoted by $\|\cdot\|_{L^r(0,T;X)}$. In what follows, we always assume $p \in (1,\infty)$ and set $p^* := p/(p-1)$. The dual pairing between $L^p(0,T;V)$ and $L^{p^*}(0,T;V^*) = (L^p(0,T;V))^*$ is given by

$$\langle f, v\rangle_{L^{p^*}(0,T;V^*)\times L^p(0,T;V)} = \int_0^T \langle f(t), v(t)\rangle_{V^*\times V} dt\,.$$

Similarly, we have $\big(L^1(0,T;H)\big)^* = L^\infty(0,T;H)$ with

$$\langle f, v\rangle_{L^\infty(0,T;H)\times L^1(0,T;H)} = \int_0^T (f(t), v(t)) dt\,.$$

The inner product in $L^2(0,T;H)$ is denoted by $(\cdot,\cdot)_{L^2(0,T;H)}$.

The space

$$\mathcal{X} := L^p(0,T;V) \cap L^2(0,T;H)\,, \quad \|v\|_{\mathcal{X}} := \|v\|_{L^p(0,T;V)} + \|v\|_{L^2(0,T;H)}\,,$$

is a reflexive, separable Banach space. Its dual $\mathcal{X}^*$ can be identified with the sum $L^{p^*}(0,T;V^*) + L^2(0,T;H)$, equipped with the norm

$$\|f\|_{\mathcal{X}^*} := \inf_{\substack{f_1\in L^{p^*}(0,T;V^*),\, f_2\in L^2(0,T;H)\\ f=f_1+f_2}} \max\big(\|f_1\|_{L^{p^*}(0,T;V^*)}, \|f_2\|_{L^2(0,T;H)}\big)\,.$$

If f allows the representation $f = f_1 + f_2$ with $f_1 \in L^{p^*}(0,T;V^*)$, $f_2 \in L^2(0,T;H)$ then the dual pairing between $f \in \mathcal{X}^*$ and $v \in \mathcal{X}$ is given by

$$\langle f, v\rangle_{\mathcal{X}^*\times\mathcal{X}} = \int_0^T \big(\langle f_1, v\rangle_{V^*\times V} + (f_2, v)\big)\, dt = \int_0^T \langle f, v\rangle_{V^*\times V} dt\,,$$

see e.g. [14] for more details. After all, $\mathcal{X} \subseteq L^2(0,T;H) \subseteq \mathcal{X}^*$ forms a Gelfand triple. In the case $p \ge 2$, we can work with $\mathcal{X} = L^p(0,T;V)$, $\mathcal{X}^* = L^{p^*}(0,T;V^*)$, and the corresponding standard norms.

For $p \in (1,\infty)$, the Banach space

$$\mathcal{W} := \{v \in \mathcal{X} : v' \in \mathcal{X}^*\}\,, \quad \|v\|_{\mathcal{W}} := \|v\|_{\mathcal{X}} + \|v'\|_{\mathcal{X}^*}\,,$$

with v' being the distributional time derivative, is continuously embedded in the space $\mathcal{C}([0,T];H)$ of uniformly continuous functions with values in H.

The structural properties we always assume for A read as follows:

Assumption A. *$\{A(t)\}_{t\in[0,T]}$ is a family of hemicontinuous operators $A(t): V \to V^*$ such that for all $v \in V$ the mapping $t \mapsto A(t)v : [0,T] \to V^*$ is continuous for almost all $t \in [0,T]$. There is a constant $\kappa \geq 0$ such that $A(t) + \kappa I : V \to V^*$ is monotone. For a suitable $p \in (1,\infty)$, there are constants $\mu > 0$, $\lambda \geq 0$ such that for all $t \in [0,T]$ and $v \in V$*

$$\langle (A(t) + \kappa I)v, v\rangle \geq \mu \|v\|^p - \lambda\,.$$

For any $R > 0$ there exists $\alpha = \alpha(R) > 0$ such that for all $t \in [0,T]$ and $v \in V$ with $|v| \leq R$

$$\|A(t)v\|_* \leq \alpha(R)\left(1 + \|v\|^{p-1}\right)\,.$$

With $\{A(t)\}_{t\in[0,T]}$, we associate the Nemytskii operator A that is defined by $(Av)(t) := A(t)v(t)$ ($t \in [0,T]$) for a function $v : [0,T] \to V$.

Under Assumption A, the Nemytskii operator A maps $L^p(0,T;V) \cap L^\infty(0,T;H) \subset L^p(0,T;V)$ into $(L^p(0,T;V))^* = L^{p^*}(0,T;V^*)$ and is hemicontinuous and bounded. Moreover, $A + \kappa I : L^p(0,T;V) \cap L^\infty(0,T;H) \subset L^p(0,T;V) \to L^{p^*}(0,T;V^*)$ is monotone and satisfies for all $v \in L^p(0,T;V) \cap L^\infty(0,T;H)$

$$\langle (A + \kappa I)v, v\rangle \geq \mu \|v\|^p_{L^p(0,T;V)} - \lambda T\,.$$

Problem (1.1) then possesses for any $u_0 \in H$ and $f \in \mathcal{X}^*$ a unique solution $u \in \mathcal{W}$ such that the evolution equation holds in $\mathcal{X}^*$ (see e.g. [38, Thm. 8.28] or, with a growth condition that is more restrictive and independent of R, [6, Thm. 4.2 on p. 167], [45, Thm. 30.A], see also [14, Satz 1.1 on p. 201, Bem. 1.5 on p. 210]).

Examples for operators possessing the above properties can be found e.g. in [14, pp. 68 ff., 215 ff.], [27], [38, pp. 232 ff.], and [45, pp. 567 ff., 590 ff., 779 ff.]). A standard example is the p-Laplacian in a bounded domain supplemented by homogeneous Dirichlet boundary conditions. Another example is the fluid flow through a porous medium when working with the Sobolev space H^{-1} as the pivot space H in the underlying Gelfand triple (see [27, pp. 191 ff.], [14, pp. 72 f.]).

3. Time discrete problem and a priori estimates

Theorem 3.1. *Let Assumption A be fulfilled. For any $u^0 \in V$ and $\{f^{n-1+\vartheta}\}_{n=1}^N \subset V^*$ there is a unique solution $\{u^n\}_{n=1}^N \subset V$ to (1.4) if $\tau_{\max} < 1/(\vartheta\kappa)$.*

Proof. If $\vartheta = 0$, existence and uniqueness are obvious. Otherwise, we can rewrite each step of (1.4) as

$$\frac{1}{\vartheta\tau_n}u^{n-1+\vartheta} + A(t_{n-1+\vartheta})u^{n-1+\vartheta} = f^{n-1+\vartheta} + \frac{1}{\vartheta\tau_n}u^{n-1}\,,\ n = 1,2,\ldots,N\,.$$

The existence of $u^{n-1+\vartheta}$ and thus of u^n now follows step-by-step from the well-known Browder-Minty theorem (see e.g. [45, Thm. 26.A]). The uniqueness follows since the operator $\frac{1}{\vartheta\tau_n}I + A(t_{n+1-\vartheta}) : V \to V^*$ ($n = 1,2,\ldots,N$) is strictly monotone if $\frac{1}{\vartheta\tau_n} > \kappa$. $\square$

In what follows, let $c > 0$ be a generic constant that is independent of the time grid (1.3). For proving a priori estimates, we need the following Gronwall-type argument.

Lemma 3.2. *Let* $\{a_n\}_{n=0}^N, \{b_n\}_{n=1}^N, \{c_n\}_{n=1}^N \subset \mathbb{R}_0^+$ *and* $\gamma > 0$. *If* $\tau_{\max} < 1/(\gamma\vartheta)$ *then*

$$\frac{a_n - a_{n-1}}{\tau_n} + b_n \le c_n + \gamma(\vartheta a_n + (1-\vartheta)a_{n-1}) \,, \quad n = 1, 2, \dots, N \,,$$

implies

$$a_n + \sum_{j=1}^n \frac{\tau_j b_j}{1 + \gamma(1-\vartheta)\tau_{\max}} \le \exp\left(\frac{\gamma T}{1 - \gamma\vartheta\tau_{\max}}\right)\left(a_0 + \sum_{j=1}^n \tau_j c_j\right) , \quad n = 1, 2, \dots, N \,.$$

Proof. With

$$\tilde{a}_n := a_n \prod_{l=1}^n \omega_l^{-1} \ (n = 0, 1, \dots, N) \,, \quad \omega_n := \frac{1 + \gamma(1-\vartheta)\tau_n}{1 - \gamma\vartheta\tau_n} \ (n = 1, 2, \dots, N) \,,$$

we easily obtain for $n = 1, 2, \dots, N$

$$\frac{\tilde{a}_n - \tilde{a}_{n-1}}{\tau_n} \le (1 + \gamma(1-\vartheta)\tau_n)^{-1}\left(\prod_{l=1}^{n-1} \omega_l^{-1}\right)(c_n - b_n) \,.$$

Some elementary manipulations and summation over n (note $\tilde{a}_0 = a_0$) lead, together with

$$\prod_{l=1}^N \omega_l \le \exp\left(\frac{\gamma T}{1 - \gamma\vartheta\tau_{\max}}\right) ,$$

to the assertion. $\square$

Lemma 3.3. *Let Assumption A be fulfilled and assume* $\tau_{\max} < 1/(2\vartheta\kappa)$. *For given* $u^0 \in V$ *and* $f^{n-1+\vartheta} = f_1^{n-1+\vartheta} + f_2^{n-1+\vartheta}$ *with* $f_1^{n-1+\vartheta} \in V^*$, $f_2^{n-1+\vartheta} \in H$ *(*$n = 1, 2, \dots, N$*), the solution* $\{u^n\}_{n=1}^N \subset V$ *to (1.4) satisfies the following a priori estimates:*

$$\max_{n=1,\dots,N} |u^n|^2 + (2\vartheta - 1)\sum_{n=1}^N |u^n - u^{n-1}|^2 + \sum_{n=1}^N \tau_n \|u^{n-1+\vartheta}\|^p$$
$$\le c\left(|u^0|^2 + \sum_{n=1}^N \tau_n\left(\|f_1^{n-1+\vartheta}\|_*^{p^*} + |f_2^{n-1+\vartheta}|^2\right) + \lambda T\right) =: M \,,$$

$$\sum_{n=1}^N \tau_n \left\|\frac{1}{\tau_n}\left(u^n - u^{n-1}\right) - f_2^{n-1+\vartheta}\right\|_*^{p^*} + \sum_{n=1}^N \tau_n \|A(t_{n-1+\vartheta})u^{n-1+\vartheta}\|_*^{p^*} \le M' \,,$$

where M' *is a function in* M *that is bounded on bounded subsets.*

Proof. Testing (1.4) by $u^{n-1+\vartheta}$, employing the algebraic relation

$$(a-b)(\vartheta a + (1-\vartheta)b) = \frac{1}{2}\left(a^2 - b^2 + (2\vartheta-1)(a-b)^2\right) , \quad a, b \in \mathbb{R} \,,$$

and the coercivity as well as Young's inequality leads for $n = 1, 2, \dots, N$ to

$$\frac{1}{2\tau_n}\left(|u^n|^2 - |u^{n-1}|^2 + (2\vartheta-1)|u^n - u^{n-1}|^2\right) + \mu\|u^{n-1+\vartheta}\|^p - \lambda - \kappa|u^{n-1+\vartheta}|^2$$
$$\le c\|f_1^{n-1+\vartheta}\|_*^{p^*} + \frac{\mu}{2}\|u^{n-1+\vartheta}\|^p + c|f_2^{n-1+\vartheta}|^2 + \varepsilon|u^{n-1+\vartheta}|^2 \,,$$

where $\varepsilon > 0$ is supposed to be sufficiently small such that $2(\varepsilon + \kappa)\vartheta\tau_{\max} < 1$. Convexity gives

$$|u^{n-1+\vartheta}|^2 \leq \vartheta|u^n|^2 + (1-\vartheta)|u^{n-1}|^2 .$$

The first estimate now follows from applying Lemma 3.2.

For $n = 1, 2, \ldots, N$, the growth condition yields

$$\sum_{j=1}^{n} \tau_j \|A(t_{j-1+\vartheta})u^{j-1+\vartheta}\|_*^{p^*} \leq c\alpha(\sqrt{M})^{p^*} \left(T + \sum_{j=1}^{n} \tau_j \|u^{j-1+\vartheta}\|^p\right).$$

Moreover, we have

$$\frac{1}{\tau_n}(u^n - u^{n-1}) = f_1^{n-1+\vartheta} + f_2^{n-1+\vartheta} - A(t_{n-1+\vartheta})u^{n-1+\vartheta}$$

and thus

$$\begin{aligned}\sum_{j=1}^{n} \tau_j \left\|\frac{1}{\tau_j}(u^j - u^{j-1}) - f_2^{j-1+\vartheta}\right\|_*^{p^*} &= \sum_{j=1}^{n} \tau_j \|f_1^{j-1+\vartheta} - A(t_{j-1+\vartheta})u^{j-1+\vartheta}\|_*^{p^*} \\ &\leq c\sum_{j=1}^{n} \tau_j \|f_1^{j-1+\vartheta}\|_*^{p^*} + c\sum_{j=1}^{n} \tau_j \|A(t_{j-1+\vartheta})u^{j-1+\vartheta}\|_*^{p^*} .\end{aligned}$$

This, together with the first estimate, proves the second assertion. □

Lemma 3.4. *In addition to the assumptions of Lemma 3.3 suppose that* $\vartheta \in (1/2, 1]$ *and that there is some* $r \geq 1$ *such that*

$$r_{\max} \leq r < \left(\frac{\vartheta}{1-\vartheta}\right)^p . \tag{3.1}$$

Then also

$$\sum_{n=1}^{N} \tau_n \|u^n\|^p \leq c\left(M + \tau_1\|u^0\|^p\right)$$

with c *depending on* r*.*

Proof. For $n = 1, 2, \ldots, N$, we have

$$\|u^n\| \leq \frac{1}{\vartheta}\|u^{n-1+\vartheta}\| + \frac{1-\vartheta}{\vartheta}\|u^{n-1}\| .$$

With Minkowski's inequality we thus obtain

$$\begin{aligned}\left(\sum_{n=1}^{N} \tau_n \|u^n\|^p\right)^{1/p} &\leq \frac{1}{\vartheta}\left(\sum_{n=1}^{N} \tau_n \|u^{n-1+\vartheta}\|^p\right)^{1/p} + \frac{1-\vartheta}{\vartheta}\left(\sum_{n=1}^{N} \tau_n \|u^{n-1}\|^p\right)^{1/p} \\ &\leq \frac{1}{\vartheta}\left(\sum_{n=1}^{N} \tau_n \|u^{n-1+\vartheta}\|^p\right)^{1/p} + \frac{1-\vartheta}{\vartheta}\left(\tau_1\|u^0\|^p + \sum_{n=1}^{N-1} r_{n+1}\tau_n \|u^n\|^p\right)^{1/p} \\ &\leq \frac{1}{\vartheta}\left(\sum_{n=1}^{N} \tau_n \|u^{n-1+\vartheta}\|^p\right)^{1/p} + \frac{1-\vartheta}{\vartheta} r^{1/p} \left(\tau_1^{1/p}\|u^0\| + \left(\sum_{n=1}^{N-1} \tau_n \|u^n\|^p\right)^{1/p}\right).\end{aligned}$$

The assertion follows with Lemma 3.3 if (3.1) is fulfilled. □

If $\vartheta = 1$ then $u^0 \in H$ suffices in Theorem 3.1 as well as in Lemma 3.3 and 3.4 since u^0 does not appear as an argument of $A(\cdot)$.

For simplicity, we only consider, for the rest of the paper, the natural restriction

$$\mathrm{R}^{n-1+\vartheta} f := \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} f(t) dt\,, \quad n = 1, 2, \ldots, N\,, \tag{3.2}$$

as the right-hand side in (1.4). If $f = f_1 + f_2$ with $f_1 \in L^{p^*}(0,T;V^*)$, $f_2 \in L^2(0,T;H)$, we obtain from standard arguments

$$\sum_{j=1}^{n} \tau_j \left(\left\| \mathrm{R}^{j-1+\vartheta} f_1 \right\|_*^{p^*} + \left| \mathrm{R}^{j-1+\vartheta} f_2 \right|^2 \right) \le c \int_0^{t_n} \left(\| f_1(t) \|_*^{p^*} + |f_2(t)|^2 \right) dt\,.$$

4. Convergence

From the discrete solution $\{u^n\}_{n=0}^N$ of (1.4) corresponding to the partition $\mathbb{I}$ of $[0,T]$ (see (1.3)), we now construct the piecewise constant function $u_{\mathbb{I}}$ with $u_{\mathbb{I}}(t) := u^{n-1+\vartheta}$ for $t \in (t_{n-1}, t_n]$ and $u_{\mathbb{I}}(0) := u^{0+\vartheta}$ as well as the piecewise linear function $v_{\mathbb{I}}$ that interpolates the points (t_n, u^n) $(n = 0, 1, \ldots, N)$. Note that the slope of $v_{\mathbb{I}}$ in $(t_{n-1}, t_n]$ is $(u^n - u^{n-1})/\tau_n$ $(n = 1, 2, \ldots, N)$.

We now consider a sequence $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ of time grids (1.3) such that the corresponding sequence $\{\tau_{\max}(\mathbb{I}_k)\}_{k\in\mathbb{N}}$ of maximum time steps is a null sequence with $\sup_{k\in\mathbb{N}} \tau_{\max}(\mathbb{I}_k) < 1/(2\vartheta\kappa)$. For brevity, such a sequence is said to be *admissible.* For the corresponding sequence of time discretisations (1.4) to problem (1.1) with initial value $u_0 \in H$ and right-hand side $f = f_1 + f_2 \in L^{p^*}(0,T;V^*) + L^2(0,T;H)$, we always assume for the starting values $u^0(\mathbb{I}_k)$ and right-hand side $\{f^{n-1+\vartheta}(\mathbb{I}_k)\}_{n=1}^{N_k}$ that

$$\begin{cases} u^0(\mathbb{I}_k) \in V\,, \quad u^0(\mathbb{I}_k) \to u_0 \text{ in } H \text{ as } k \to \infty\,, \\ f^{n-1+\vartheta}(\mathbb{I}_k) = \mathrm{R}_{\mathbb{I}_k}^{n-1+\vartheta} f_1 + \mathrm{R}_{\mathbb{I}_k}^{n-1+\vartheta} f_2 \ (n = 1, 2, \ldots, N_k)\,, \end{cases} \tag{4.1}$$

where $\mathrm{R}_{\mathbb{I}_k}$ denotes the natural restriction corresponding to the grid $\mathbb{I}_k$. From (4.1), we infer in particular that M and M' in Lemma 3.3 can be bounded independently of k.

Lemma 4.1. *Let $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ be an admissible sequence of time grids and suppose that (4.1) as well as Assumption A is fulfilled. If $\vartheta \in [1/2, 1]$ then there is a subsequence, denoted by k', and an element $u \in \mathcal{W}$ such that*

$$\begin{aligned} &u_{\mathbb{I}_{k'}} \overset{*}{\rightharpoonup} u \ \text{in } L^\infty(0,T;H)\,, \quad u_{\mathbb{I}_{k'}} \rightharpoonup u \ \text{in } L^p(0,T;V)\,, \\ &v_{\mathbb{I}_{k'}} \overset{*}{\rightharpoonup} u \ \text{in } L^\infty(0,T;H)\,, \quad v'_{\mathbb{I}_{k'}} \rightharpoonup u' \ \text{in } \mathcal{X}^* \ \text{as } k' \to \infty\,. \end{aligned}$$

Proof. For readability, we do not emphasise the dependence of N, τ_n, u^n, and $f^{n-1+\vartheta}$ on $\mathbb{I}_k$. From an elementary calculation, we obtain

$$\begin{aligned} \|u_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} &= \max_{n=1,2,\ldots,N} |u^{n-1+\vartheta}| \le \max_{n=0,1,\ldots,N} |u^n|\,, \\ \|u_{\mathbb{I}_k}\|_{L^p(0,T;V)}^p &= \sum_{n=1}^{N} \tau_n \|u^{n-1+\vartheta}\|^p\,. \end{aligned}$$

Lemma 3.3 then shows the boundedness of $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ in $L^\infty(0,T;H)$ as well as in $L^p(0,T;V)$, and we conclude by standard arguments with the weak* in $L^\infty(0,T;H)$

and weak in $L^p(0,T;V)$ convergence of a subsequence towards $u \in L^\infty(0,T;H) \cap L^p(0,T;V)$.

Similarly, we find

$$\|v_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} = \max_{n=0,1,\ldots,N} |u^n| ,$$

$$\|v'_{\mathbb{I}_k}\|_{\mathcal{X}^*} \leq$$

$$\max\left(\left(\sum_{n=1}^N \tau_n \left\| \frac{1}{\tau_n}\left(u^n - u^{n-1}\right) - f_2^{n-1+\vartheta} \right\|_*^{p^*} \right)^{1/p^*} , \left(\sum_{n=1}^N \tau_n |f_2^{n-1+\vartheta}|^2 \right)^{1/2} \right)$$

and thus the weak* in $L^\infty(0,T;H)$ convergence of a subsequence $\{v_{\mathbb{I}_{k'}}\}$ towards a function $v \in L^\infty(0,T;H)$ as well as the weak in $\mathcal{X}^*$ convergence of the time derivatives $v'_{\mathbb{I}_{k'}}$ towards v' (here, we also employ density arguments and the definition of the weak time derivative).

We now prove $v = u$ from which we also obtain that the limit u is in $\mathcal{W}$. Since

$$v_{\mathbb{I}_k}(t) - u_{\mathbb{I}_k}(t) = (t - t_{n-1+\vartheta}) v'_{\mathbb{I}_k}(t) \text{ for } t \in (t_{n-1}, t_n] \ (n = 1, 2, \ldots, N) , \tag{4.2}$$

we obtain

$$\|v_{\mathbb{I}_k} - u_{\mathbb{I}_k}\|_{\mathcal{X}^*} \leq \tau_{\max}(\mathbb{I}_k) \, \|v'_{\mathbb{I}_k}\|_{\mathcal{X}^*}$$

and thus

$$v_{\mathbb{I}_k} - u_{\mathbb{I}_k} \to 0 \text{ in } \mathcal{X}^* \text{ as } k \to \infty . \tag{4.3}$$

Because $\mathcal{X}$ is dense in $L^1(0,T;H)$, the weak* in $L^\infty(0,T;H)$ limits of an appropriate common subsequence of $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ and $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ must coincide. □

Note that we do not have the boundedness and thus no weak convergence of $v_{\mathbb{I}_k}$ in $L^p(0,T;V)$ except under the assumptions of Lemma 3.4 (see also Section 6).

We are now ready to prove the main result of the paper in hand.

Theorem 4.2. *Let $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ be an admissible sequence of time grids and suppose that (4.1) as well as Assumption A with $\kappa = 0$ is fulfilled. If $\vartheta \in [1/2, 1]$ then the corresponding sequence $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of piecewise constant prolongations of the discrete solution to (1.4) converges weakly* in $L^\infty(0,T;H)$ and weakly in $L^p(0,T;V)$ towards the exact solution $u \in \mathcal{W}$ to (1.1). Moreover, the corresponding sequence $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of piecewise linear interpolants converges weakly* in $L^\infty(0,T;H)$ towards u and the sequence $\{v'_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ of time derivatives converges weakly in $\mathcal{X}^*$ towards u'.*

Proof. The numerical scheme (1.4) corresponding to the time grid $\mathbb{I}_k$ can be written as the differential equation

$$v'_{\mathbb{I}_k} + A_{\mathbb{I}_k} u_{\mathbb{I}_k} = \mathrm{R}_{\mathbb{I}_k} f , \tag{4.4}$$

where $A_{\mathbb{I}_k}(t) := A(t_{n-1+\vartheta}(\mathbb{I}_k))$ for $t \in (t_{n-1}(\mathbb{I}_k), t_n(\mathbb{I}_k)]$ $(n = 1, 2, \ldots, N_k)$ is the piecewise constant restriction of $\{A(t)\}_{t\in[0,T]}$ onto $\mathbb{I}_k$ and $(\mathrm{R}_{\mathbb{I}_k} f)(t) := \mathrm{R}^{n-1+\vartheta}_{\mathbb{I}_k} f$ for $t \in (t_{n-1}(\mathbb{I}_k), t_n(\mathbb{I}_k)]$ $(n = 1, 2, \ldots, N_k)$.

By standard arguments, we find the strong convergence

$$\mathrm{R}_{\mathbb{I}_k} f \to f \text{ in } \mathcal{X}^* \text{ as } k \to \infty . \tag{4.5}$$

Because of the growth condition for A and Lemma 3.3, we know that $\{A_{\mathbb{I}_k} u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ is bounded in $L^{p^*}(0,T;V^*)$. We thus have a subsequence, denoted by k', and an element $a \in L^{p^*}(0,T;V^*)$ such that

$$A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} \rightharpoonup a \text{ in } L^{p^*}(0,T;V^*) \text{ as } k' \to \infty .$$

The subsequence can be chosen in such a way that we also have the weak and weak* convergence from Lemma 4.1

With

$$v'_{\mathbb{I}_{k'}} \rightharpoonup u' \text{ in } \mathcal{X}^* \text{ as } k' \to \infty$$

(see Lemma 4.1), we then obtain

$$0 = v'_{\mathbb{I}_{k'}} + A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} - \mathrm{R}_{\mathbb{I}_{k'}} f \rightharpoonup u' + a - f \text{ in } \mathcal{X}^* \text{ as } k' \to \infty$$

and thus

$$u' + a = f \text{ in } \mathcal{X}^* . \tag{4.6}$$

It remains to show that $u \in \mathcal{W}$ fulfills the initial condition and that $a = AU$.

With $v_{\mathbb{I}_k}(0) = u^0(\mathbb{I}_k)$ and (4.1), we have

$$v_{\mathbb{I}_k}(0) \to u_0 \text{ in } H \text{ as } k \to \infty . \tag{4.7}$$

With $v_{\mathbb{I}_k}(T) = u^{N_k}(\mathbb{I}_k)$ and the first a priori estimate in Lemma 3.3, we can choose the subsequence such that

$$v_{\mathbb{I}_{k'}}(T) \rightharpoonup \xi \text{ in } H \text{ as } k' \to \infty \tag{4.8}$$

for some $\xi \in H$. Since $v_{\mathbb{I}_k} \in \mathcal{W}$, we can employ integration by parts which yields for all $v \in V$ and $\phi \in \mathcal{C}^1([0,T])$

$$\begin{aligned}
&(u(T), v)\phi(T) - (u(0), v)\phi(0) = \int_0^T \Big(\langle u'(t), v\rangle \phi(t) + \langle u(t), v\rangle \phi'(t)\Big) dt \\
&= \int_0^T \Big(\langle f(t) - a(t), v\rangle \phi(t) + \langle u(t), v\rangle \phi'(t)\Big) dt \\
&= \int_0^T \Big(\langle f(t) - \mathrm{R}_{\mathbb{I}_{k'}} f(t) + v'_{\mathbb{I}_{k'}}(t) + A_{\mathbb{I}_{k'}}(t) u_{\mathbb{I}_{k'}}(t) - a(t), v\rangle \phi(t) + \langle u(t), v\rangle \phi'(t)\Big) dt \\
&= \int_0^T \Big(\langle f(t) - \mathrm{R}_{\mathbb{I}_{k'}} f(t) + A_{\mathbb{I}_{k'}}(t) u_{\mathbb{I}_{k'}}(t) - a(t), v\rangle \phi(t) + \langle u(t) - v_{\mathbb{I}_{k'}}(t), v\rangle \phi'(t)\Big) dt \\
&\qquad\qquad + (v_{\mathbb{I}_{k'}}(T), v)\phi(T) - (v_{\mathbb{I}_{k'}}(0), v)\phi(0) .
\end{aligned}$$

Taking the limit on the right-hand side, we come up with

$$(u(T), v)\phi(T) - (u(0), v)\phi(0) = (\xi, v)\phi(T) - (u_0, v)\phi(0) .$$

Choosing $\phi(T) = 0$ and $\phi(0) = 0$, respectively, we find that

$$u(0) = u_0 , \quad u(T) = \xi \tag{4.9}$$

in H since $V \ni v$ is dense in H.

Employing the monotonicity of $A(t)$ $(t \in [0,T])$ (remember that $\kappa = 0$ but see Remark 4.3 below), we find (with $\langle \cdot, \cdot \rangle$ denoting the dual pairing between $\mathcal{X}^*$ and $\mathcal{X}$) for arbitrary $w \in L^p(0,T;V) \cap L^\infty(0,T;H)$

$$\begin{aligned}
\langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle &\geq \langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle - \langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} - A_{\mathbb{I}_{k'}} w, u_{\mathbb{I}_{k'}} - w \rangle \\
&= \langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, w \rangle + \langle A_{\mathbb{I}_{k'}} w, u_{\mathbb{I}_{k'}} - w \rangle .
\end{aligned} \tag{4.10}$$

We thus obtain from (4.4)

$$\begin{aligned} 0 &= \langle v'_{\mathbb{I}_{k'}} + A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} - \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \\ &= \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle - \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} - u_{\mathbb{I}_{k'}} \rangle + \langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle - \langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \\ &\geq \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle - \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} - u_{\mathbb{I}_{k'}} \rangle + \langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, w \rangle + \langle A_{\mathbb{I}_{k'}} w, u_{\mathbb{I}_{k'}} - w \rangle \\ &\qquad - \langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \,. \end{aligned} \tag{4.11}$$

We now study the terms on the right-hand side of (4.11).

Since $u, v_{\mathbb{I}_{k'}} \in \mathcal{W}$, we can apply again integration by parts and find

$$\langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle = \frac{1}{2} \left(|v_{\mathbb{I}_{k'}}(T)|^2 - |v_{\mathbb{I}_{k'}}(0)|^2 \right), \quad \langle u', u \rangle = \frac{1}{2} \left(|u(T)|^2 - |u(0)|^2 \right).$$

Because of (4.7), (4.8), (4.9), we thus obtain

$$\langle u', u \rangle \leq \liminf_{k' \to \infty} \langle v'_{\mathbb{I}_{k'}}, v_{\mathbb{I}_{k'}} \rangle \,.$$

With (4.2), we see that (omitting the subscript k' for a moment)

$$\begin{aligned} \langle v'_{\mathbb{I}}, v_{\mathbb{I}} - u_{\mathbb{I}} \rangle &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (t - t_{n-1+\vartheta}) \left| \frac{1}{\tau_n} \left(u^n - u^{n-1} \right) \right|^2 dt \\ &= -\frac{2\vartheta - 1}{2} \sum_{n=1}^{N} |u^n - u^{n-1}|^2 \leq 0 \end{aligned}$$

if $\vartheta \in [1/2, 1]$.

Because of the continuity of $t \mapsto A(t)v$ a.e. in $(0,T)$ for all $v \in V$ (see Assumption A), we have for almost all $t \in (0,T)$

$$A_{\mathbb{I}_k}(t) w(t) - A(t) w(t) \to 0 \text{ in } V^* \text{ as } k \to \infty \,.$$

On the other hand, we find from the growth condition that for almost all $t \in (0,T)$

$$\| A_{\mathbb{I}_k}(t) w(t) - A(t) w(t) \|_*^{p^*} \leq c\alpha(\|w\|_{L^\infty(0,T;H)})^{p^*} (1 + \|w(t)\|^p) \,,$$

and the right-hand side is integrable. Hence, Lebesgue's theorem shows that

$$A_{\mathbb{I}_k} w - Aw \to 0 \text{ in } L^{p^*}(0,T;V^*) \text{ as } k \to \infty \,.$$

Since $u_{\mathbb{I}_{k'}}$ converges weakly in $L^p(0,T;V)$ towards u and $A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}$ converges weakly in $L^{p^*}(0,T;V^*)$ towards a, we have

$$\langle A_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, w \rangle \to \langle a, w \rangle \,, \quad \langle A_{\mathbb{I}_{k'}} w, u_{\mathbb{I}_{k'}} - w \rangle \to \langle Aw, u - w \rangle \text{ as } k' \to \infty \,.$$

With (4.5) and the weak in $L^p(0,T;V)$ and weak* in $L^\infty(0,T;H)$ convergence of $u_{\mathbb{I}_{k'}}$ towards u, we finally have

$$\langle \mathrm{R}_{\mathbb{I}_{k'}} f, u_{\mathbb{I}_{k'}} \rangle \to \langle f, u \rangle \text{ as } k' \to \infty \,.$$

Altogether, we obtain from (4.11) together with (4.6)

$$0 \geq \langle u', u \rangle + \langle a, w \rangle + \langle Aw, u - w \rangle - \langle f, u \rangle = -\langle a, u \rangle + \langle a, w \rangle + \langle Aw, u - w \rangle$$

which yields

$$\langle a, u - w \rangle \geq \langle Aw, u - w \rangle \,.$$

With $w = u \pm sv$ ($v \in L^p(0,T;V) \cap L^\infty(0,T;H)$) and $s \to 0+$, the hemicontinuity of A proves, by density, $a = Au$ in $L^{p^*}(0,T;V^*)$. This completes the proof of u being a solution to (1.1).

By contradiction, we can show that the whole sequences $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ and $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ converge towards u since a solution to (1.1) is unique in $\mathcal{W}$. $\square$

Remark 4.3. By means of the transformation

$$\tilde{u}(t) := e^{-\kappa t}u(t)\,, \quad \tilde{f}(t) := e^{-\kappa t}f(t)\,, \quad \tilde{A}(t)v := e^{-\kappa t}(A(t)+\kappa I)e^{\kappa t}v \ (v \in V)\,,$$

problem (1.1) is equivalent to

$$\tilde{u}' + \tilde{A}\tilde{u} = \tilde{f} \quad \text{in } (0,T)\,, \quad \tilde{u}(0) = u_0\,.$$

However, the operators $\tilde{A}(t) : V \to V^*$ $(t \in [0,T])$ are monotone and coercive; more precisely, the family $\{\tilde{A}(t)\}_{t\in[0,T]}$ fulfills Assumption A with $\kappa = 0$ (see also [14, p. 211]). In this sense, Theorem 4.2 also applies to the more general case with $\kappa \neq 0$. Note, however, that the direct discretisation of (1.1) with $\kappa \neq 0$ (avoiding the transformation above) cannot be shown to be convergent as we would have the additional term $-\kappa\|u_{\mathbb{I}_{k'}}\|^2_{L^2(0,T;H)}$ in (4.10) and would need the strong convergence of $u_{\mathbb{I}_{k'}}$ towards u in $L^2(0,T;H)$ which is not at hand except under the additional and more restrictive assumptions of Lemma 3.4.

5. Stability and a priori error estimates

Lemma 5.1. *Let Assumption A be fulfilled and assume $\tau_{\max} < 1/(2\vartheta\kappa)$. The solutions $\{u^n\}_{n=1}^N$ and $\{v^n\}_{n=1}^N$ to (1.4) corresponding to the initial data $u^0 \in V$ and $v^0 \in V$ and right-hand side $\{f^{n-1+\vartheta}\}_{n=1}^N \subset H$ and $\{g^{n-1+\vartheta}\}_{n=1}^N \subset H$, respectively, then satisfy the estimate*

$$\max_{n=1,\dots,N} |u^n - v^n|^2 + (2\vartheta-1)\sum_{n=1}^N |(u^n-v^n)-(u^{n-1}-v^{n-1})|^2 \leq c\left(|u^0-v^0|^2 + \sum_{n=1}^N \tau_n|f^{n-1+\vartheta}-g^{n-1+\vartheta}|^2\right).$$

If $A(t)+\kappa I : V \to V^*$ $(t \in [0,T])$ is, in addition, uniformly monotone such that there is a constant $\mu_0 > 0$ and for all $t \in [0,T]$ and $v,w \in V$

$$\langle (A(t)+\kappa I)v - (A(t)+\kappa I)w, v-w\rangle \geq \mu_0\|v-w\|^p \tag{5.1}$$

then, for right-hand sides $\{f^{n-1+\vartheta}\}_{n=1}^N \subset V^*$ and $\{g^{n-1+\vartheta}\}_{n=1}^N \subset V^*$, the following estimate holds true:

$$\max_{n=1,\dots,N} |u^n - v^n|^2 + (2\vartheta-1)\sum_{n=1}^N |(u^n-v^n)-(u^{n-1}-v^{n-1})|^2$$
$$+\sum_{n=1}^N \tau_n\|u^{n-1+\vartheta}-v^{n-1+\vartheta}\|^p \leq c\left(|u^0-v^0|^2 + \sum_{n=1}^N \tau_n\|f^{n-1+\vartheta}-g^{n-1+\vartheta}\|_*^{p^*}\right).$$

If, moreover, the assumptions of Lemma 3.4 are fulfilled then

$$\max_{n=1,\dots,N} |u^n - v^n|^2 + (2\vartheta-1)\sum_{n=1}^N |(u^n-v^n)-(u^{n-1}-v^{n-1})|^2$$
$$+\sum_{n=1}^N \tau_n\|u^n-v^n\|^p \leq c\left(|u^0-v^0|^2 + \tau_1\|u^0-v^0\|^p + \sum_{n=1}^N \tau_n\|f^{n-1+\vartheta}-g^{n-1+\vartheta}\|_*^{p^*}\right).$$

Proof. We subtract the equations for u^n and v^n and test by $u^n - v^n$. The rest of the proof essentially follows the same lines as that of Lemma 3.3 and employs Lemma 3.2 as well as, for the last assertion, Lemma 3.4. $\square$

If $\vartheta = 1$ then again $u^0, v^0 \in H$ suffices in the above lemma. Note that (5.1) only makes sense for $p \geq 2$.

Estimates for the error $e^n := u(t_n) - u^n$ $(n = 1, 2, \dots, N)$ between the exact and the numerical solution easily follow from the above stability estimates based upon the error equation

$$\begin{aligned} &\frac{1}{\tau_n}(e^n - e^{n-1}) + A(t_{n-1+\vartheta})u(t_{n-1+\vartheta}) - A(t_{n-1+\vartheta})u^{n-1+\vartheta} = \rho^{n-1+\vartheta} \\ &\text{with } \rho^{n-1+\vartheta} := \frac{1}{\tau_n}\left(u(t_n) - u(t_{n-1})\right) - u'(t_{n-1+\vartheta}) + f(t_{n-1+\vartheta}) - f^{n-1+\vartheta}\,. \end{aligned} \tag{5.2}$$

For simplicity, we again consider only the natural restriction $f^{n-1+\vartheta} = \mathrm{R}^{n-1+\vartheta} f$ given by (3.2).

Theorem 5.2. *Let Assumption A be fulfilled and assume $\tau_{\max} < 1/(2\vartheta\kappa)$ as well as $\vartheta \in [1/2, 1]$. Moreover, let $u \in \mathcal{W}$ be the solution to (1.1) with $u_0 \in H$ and $f \in \mathcal{X}^*$. The solution $\{u^n\}_{n=1}^N$ to (1.4) with data $u^0 \in V$ and $f^{n-1+\vartheta} = \mathrm{R}^{n-1+\vartheta} f$ $(n = 1, 2, \dots, N)$ then fulfills the error estimate*

$$\begin{aligned} \max_{n=1,\dots,N} |u(t_n) - u^n|^2 &\leq c\left(|u^0 - u_0|^2 + \sum_{n=1}^N \tau_n^2 \int_{t_{n-1}}^{t_n} |(f' - u'')(t)|^2 dt\right) \\ &\leq c\left(|u^0 - u_0|^2 + \tau_{\max}^2 \|f' - u''\|^2_{L^2(0,T;H)}\right) \end{aligned}$$

if $f' - u'' \in L^2(0, T; H)$. If $\vartheta = 1/2$ and $f'' - u''' \in L^2(0, T; H)$ then

$$\begin{aligned} \max_{n=1,\dots,N} |u(t_n) - u^n|^2 &\leq c\left(|u^0 - u_0|^2 + \sum_{n=1}^N \tau_n^4 \int_{t_{n-1}}^{t_n} |(f'' - u''')(t)|^2 dt\right) \\ &\leq c\left(|u^0 - u_0|^2 + \tau_{\max}^4 \|f'' - u'''\|^2_{L^2(0,T;H)}\right). \end{aligned}$$

Let, in addition, (5.1) be fulfilled. If $f' - u'' \in L^{p^*}(0, T; V^*)$ then

$$\begin{aligned} &\max_{n=1,\dots,N} |u(t_n) - u^n|^2 + \sum_{n=1}^N \tau_n \|u(t_{n-1+\vartheta}) - u^{n-1+\vartheta}\|^p \\ &\leq c\left(|u^0 - v^0|^2 + \sum_{n=1}^N \tau_n^{p/(p-1)} \int_{t_{n-1}}^{t_n} \|(f' - u'')(t)\|_*^{p^*} dt\right) \\ &\leq c\left(|u^0 - v^0|^2 + \tau_{\max}^{p/(p-1)} \|f' - u''\|^{p^*}_{L^{p^*}(0,T;V^*)} dt\right). \end{aligned}$$

If also the assumptions of Lemma 3.4 are fulfilled then one can replace

$$\sum_{n=1}^N \tau_n \|u(t_{n-1+\vartheta}) - u^{n-1+\vartheta}\|^p \ \textit{by}\ \sum_{n=1}^N \tau_n \|u(t_n) - u^n\|^p$$

in the foregoing estimate. If $\vartheta = 1/2$ *and* $f'' - u''' \in L^{p^*}(0,T;V^*)$ *then*

$$\begin{aligned}
&\max_{n=1,\dots,N} |u(t_n) - u^n|^2 + \sum_{n=1}^N \tau_n \|u(t_{n-1+\vartheta}) - u^{n-1+\vartheta}\|^p \\
&\le c\left(|u^0 - v^0|^2 + \sum_{n=1}^N \tau_n^{2p/(p-1)} \int_{t_{n-1}}^{t_n} \|(f'' - u''')(t)\|_*^{p^*} dt \right) \\
&\le c\left(|u^0 - v^0|^2 + \tau_{\max}^{2p/(p-1)} \|f'' - u'''\|_{L^{p^*}(0,T;V^*)}^{p^*} dt \right).
\end{aligned}$$

Proof. We commence with estimates of the consistency error $\rho^{n-1+\vartheta}$ $(n = 1, 2, \dots, N)$. Let $f' - u'' \in L^2(0,T;H)$. Integration by parts yields the representation

$$\rho^{n-1+\vartheta} = \frac{1}{\tau_n}\int_{t_{n-1}}^{t_{n-1+\vartheta}} (t - t_{n-1})(f' - u'')(t)dt - \frac{1}{\tau_n}\int_{t_{n-1+\vartheta}}^{t_n} (t_n - t)(f' - u'')(t)dt\,.$$

With Hölder's inequality, we now come up with the estimate

$$\sum_{n=1}^N \tau_n |\rho^{n-1+\vartheta}|^2 \le c\sum_{n=1}^N \tau_n^2 \int_{t_{n-1}}^{t_n} |(f' - u'')(t)|^2 dt \le c\tau_{\max}^2 \|f' - u''\|_{L^2(0,T;H)}^2\,.$$

If $f' - u'' \in L^{p^*}(0,T;V^*)$, we analogously find

$$\begin{aligned}
\sum_{n=1}^N \tau_n \|\rho^{n-1+\vartheta}\|_*^{p^*} &\le c\sum_{n=1}^N \tau_n^{p/(p-1)} \int_{t_{n-1}}^{t_n} \|(f' - u'')(t)\|_*^{p^*} dt \\
&\le c\tau_{\max}^{p/(p-1)} \|f' - u''\|_{L^{p^*}(0,T;V^*)}^{p^*}\,.
\end{aligned}$$

If $\vartheta = 1/2$, the consistency error admits the representation

$$\begin{aligned}
\rho^{n-1+\vartheta} =& -\frac{1}{2\tau_n}\int_{t_{n-1}}^{t_{n-1/2}} (t - t_{n-1})^2 (f'' - u''')(t)dt \\
&- \frac{1}{2\tau_n}\int_{t_{n-1/2}}^{t_n} (t_n - t)^2 (f'' - u''')(t)dt\,,
\end{aligned}$$

from which we arrive at

$$\sum_{n=1}^N \tau_n |\rho^{n-1+\vartheta}|^2 \le c\sum_{n=1}^N \tau_n^4 \int_{t_{n-1}}^{t_n} |(f'' - u''')(t)|^2 dt \le c\tau_{\max}^4 \|f'' - u'''\|_{L^2(0,T;H)}^2$$

and

$$\begin{aligned}
\sum_{n=1}^N \tau_n \|\rho^{n-1+\vartheta}\|_*^{p^*} &\le c\sum_{n=1}^N \tau_n^{2p/(p-1)} \int_{t_{n-1}}^{t_n} \|(f'' - u''')(t)\|_*^{p^*} dt \\
&\le c\tau_{\max}^{2p/(p-1)} \|f'' - u'''\|_{L^{p^*}(0,T;V^*)}^{p^*}
\end{aligned}$$

if $f'' - u''' \in L^2(0,T;H)$ and $f'' - u''' \in L^{p^*}(0,T;V^*)$, respectively.

The rest of the proof consists in applying the stability estimates of Lemma 5.1 to the error equation (5.2). □

The first part of the theorem above provides local error estimates of optimal first order (resp. second order if $\vartheta = 1/2$ and if the exact solution possesses more regularity) in terms of the discrete counterpart of the $L^\infty(0,T;H)$-norm if the initial approximation is of appropriate order. The second part provides –under weaker assumptions on the right-hand side and the regularity of the exact solution– local error estimates of order $p/2(p-1)$ (resp. $p/(p-1)$ if $\vartheta = 1/2$) in terms of the discrete $L^\infty(0,T;H)$-norm and of order $1/(p-1)$ (resp. $2/(p-1)$ if $\vartheta = 1/2$) in terms of the discrete $L^p(0,T;V)$-norm, respectively. Again, it suffices to assume $u^0 \in H$ if $\vartheta = 1$.

6. Strongly continuous perturbation

In what follows, we study problem (1.2). Suppose that Assumption A is fulfilled. With respect to the perturbation of the monotone main part, we make

Assumption B. *$\{B(t)\}_{t\in[0,T]}$ is a family of operators $B(t) : V \to V^*$ such that for all $v \in V$ the mapping $t \mapsto B(t)v : [0,T] \to V^*$ is continuous for almost all $t \in [0,T]$. There are constants $\kappa_B, \lambda_B \geq 0$ such that for all $t \in [0,T]$ and $v \in V$*

$$\langle B(t)v, v\rangle \geq -\frac{\mu}{4}\|v\|^p - \kappa_B|v|^2 - \lambda_B\,. \tag{6.1}$$

There exists $\delta \in (0, p-1]$ and for any $R > 0$ there exists $\alpha_B = \alpha_B(R) > 0$, $\beta = \beta(R) > 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $|v|, |w| \leq R$

$$\|B(t)v\|_* \leq \alpha_B(R)\left(1 + \|v\|^{p-1}\right),$$

$$\|B(t)v - B(t)w\|_* \leq \beta(R)\max(\|v\|, \|w\|)^{p-1-\delta}|v-w|^{\delta/p}\,. \tag{6.2}$$

Note that (6.1) is fulfilled if e.g.

$$\|B(t)v\|_* \leq c\left(1 + \|v\|^{p-1-\delta}|v|^{2\delta/p}\right), \quad t \in [0,T],\ v \in V,$$

for some $c \geq 0$ and $\delta \in (0, p-1]$ which can be seen by Young's inequality. Moreover, if V is compactly embedded in H (which we shall always assume if $B \neq 0$) then (6.2) implies that $B(t) : V \to V^*$ is strongly continuous.

Finally, the corresponding Nemytskii operator B maps $L^p(0,T;V) \cap L^\infty(0,T;H)$ into $L^{p^*}(0,T;V^*)$, is bounded and, by the theorem of Lions-Aubin, strongly continuous as a mapping of $\mathcal{W}$ into $L^{p^*}(0,T;V^*)$.

The numerical scheme we wish to study is again (1.4) but with $A(t_{n-1+\vartheta})u^{n-1+\vartheta}$ being replaced by $A(t_{n-1+\vartheta})u^{n-1+\vartheta} + B(t_{n-1+\vartheta})u^{n-1+\vartheta}$.

Theorem 6.1. *Suppose that Assumption A and Assumption B are fulfilled and that $V \overset{c}{\hookrightarrow} H$. The ϑ-scheme applied to (1.2) with $u^0 \in V$ (resp. $u^0 \in H$ if $\vartheta = 1$) and $\{f^{n-1+\vartheta}\}_{n=1}^N \subset V^*$ then possesses at least one solution $\{u^n\}_{n=1}^N \subset V$ if $\tau_{\max} < 1/(\vartheta(\kappa + \kappa_B))$. Let, in addition, $f^{n-1+\vartheta} = f_1^{n-1+\vartheta} + f_2^{n-1+\vartheta}$ with $f_1^{n-1+\vartheta} \in V^*$, $f_2^{n-1+\vartheta} \in H$ $(n = 1, 2, \ldots, N)$ and $\tau_{\max} < 1/(2\vartheta(\kappa + \kappa_B))$. Then any solution fulfills the following a priori estimates:*

$$\max_{n=1,\ldots,N}|u^n|^2 + (2\vartheta - 1)\sum_{n=1}^N |u^n - u^{n-1}|^2 + \sum_{n=1}^N \tau_n\|u^{n-1+\vartheta}\|^p$$
$$\leq c\left(|u^0|^2 + \sum_{n=1}^N \tau_n\left(\|f_1^{n-1+\vartheta}\|_*^{p^*} + |f_2^{n-1+\vartheta}|^2\right) + (\lambda + \lambda_B)T\right) =: M\,,$$

$$\sum_{n=1}^{N} \tau_n \left\| \frac{1}{\tau_n} \left(u^n - u^{n-1}\right) - f_2^{n-1+\vartheta} \right\|_*^{p^*} +$$

$$\sum_{n=1}^{N} \tau_n \|A(t_{n-1+\vartheta}) u^{n-1+\vartheta}\|_*^{p^*} + \sum_{n=1}^{N} \tau_n \|B(t_{n-1+\vartheta}) u^{n-1+\vartheta}\|_*^{p^*} \le M' ,$$

where M' is a function in M that is bounded on bounded subsets. Finally, Lemma 3.4 remains true.

Proof. The existence follows from Brézis' theorem on pseudomonotone operators (see e.g. [45, Thm. 27.A]). The a priori estimates follow as in Lemma 3.3 employing (6.1) and the growth condition for $B(\cdot)$. □

Because of the a priori estimates above, Lemma 4.1 still remains valid. However, we will also need a result of strong convergence in order to handle the non-monotone perturbation.

Theorem 6.2. *Let $\{\mathbb{I}_k\}_{k\in\mathbb{N}}$ be an admissible sequence of time grids such that $\sup_{k\in\mathbb{N}} \tau_{\max}(\mathbb{I}_k) < 1/(2\vartheta\kappa_B)$ and (3.1) for some $r \ge 1$ are fulfilled. Suppose that (4.1) as well as Assumption A with $\kappa = 0$ and Assumption B are fulfilled and that $V \overset{c}{\hookrightarrow} H$. Moreover, assume that $\{\tau_1(\mathbb{I}_k)\|u^0(\mathbb{I}_k)\|^p\}_{k\in\mathbb{N}}$ is bounded. If $\vartheta \in (1/2, 1]$ then there is a subsequence $\{\mathbb{I}_{k'}\}$ of time grids such that the corresponding sequence $\{u_{\mathbb{I}_{k'}}\}$ of piecewise constant prolongations and the corresponding sequence $\{v_{\mathbb{I}_{k'}}\}$ of piecewise linear interpolants of the discrete solution to the ϑ-scheme applied to (1.2) converges strongly in any $L^r(0,T;H)$ ($r \ge 1$), weakly* in $L^\infty(0,T;H)$ and weakly in $L^p(0,T;V)$ towards an exact solution $u \in \mathcal{W}$ to (1.2). Moreover, the subsequence $\{v'_{\mathbb{I}_{k'}}\}$ of time derivatives of the piecewise linear interpolants converges weakly in $\mathcal{X}^*$ towards u'.*

Proof. The proof differs from that of Theorem 4.2 only in the consideration of the additional term $B_{\mathbb{I}_k} u_{\mathbb{I}_k}$ as we have

$$v'_{\mathbb{I}_k} + A_{\mathbb{I}_k} u_{\mathbb{I}_k} + B_{\mathbb{I}_k} u_{\mathbb{I}_k} = \mathrm{R}_{\mathbb{I}_k} f$$

instead of (4.4). Here, $B_{\mathbb{I}_k}$ denotes the piecewise constant restriction of $\{B(t)\}_{t\in[0,T]}$ onto $\mathbb{I}_k$.

Because of Lemma 3.4, which applies here due to the assumptions made, we can extract a subsequence such that convergence as in Lemma 4.1 takes place and, in addition,

$$v_{\mathbb{I}_{k'}} \rightharpoonup u \text{ in } L^p(0,T;V) , \quad v_{\mathbb{I}_{k'}} \to u \text{ in } L^r(0,T;H) \; \forall r \ge 1 \text{ as } k' \to \infty . \tag{6.3}$$

The weak convergence result follows from the boundedness of $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ in $L^p(0,T;V)$ which is provided by Lemma 3.4. The strong convergence result is a consequence of the Lions-Aubin theorem since $\{v_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{W}$. We shall now prove that also

$$u_{\mathbb{I}_{k'}} \to u \text{ in } L^r(0,T;H) \, \forall r \ge 1 \text{ as } k' \to \infty . \tag{6.4}$$

Since $\{u_{\mathbb{I}_k}\}_{k\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$ it suffices to show strong convergence in $L^2(0,T;H)$. From Lemma 4.1, we already know that $u_{\mathbb{I}_{k'}} \overset{*}{\rightharpoonup} u$ in $L^\infty(0,T;H)$ and thus

$$u_{\mathbb{I}_{k'}} \rightharpoonup u \text{ in } L^2(0,T;H) \text{ as } k' \to \infty . \tag{6.5}$$

So, it suffices to show $\|u_{\mathbb{I}_{k'}}\|_{L^2(0,T;H)} \to \|u\|_{L^2(0,T;H)}$ as $L^2(0,T;H)$ is a Hilbert space. We easily find

(6.6)
$$\begin{aligned}
&\Big| \|u_{\mathbb{I}_{k'}}\|^2_{L^2(0,T;H)} - \|u\|^2_{L^2(0,T;H)} \Big| = \Big| \big(u_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}}\big)_{L^2(0,T;H)} - (u,u)_{L^2(0,T;H)} \Big| \\
&= \Big| \big(u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}}\big)_{L^2(0,T;H)} + \big(v_{\mathbb{I}_{k'}} - u, u_{\mathbb{I}_{k'}}\big)_{L^2(0,T;H)} + \big(u_{\mathbb{I}_{k'}} - u, u\big)_{L^2(0,T;H)} \Big| \\
&\le \|u_{\mathbb{I}_{k'}} - v_{\mathbb{I}_{k'}}\|_{\mathcal{X}^*} \|u_{\mathbb{I}_{k'}}\|_{\mathcal{X}} + \|v_{\mathbb{I}_{k'}} - u\|_{L^2(0,T;H)} \|u_{\mathbb{I}_{k'}}\|_{L^2(0,T;H)} \\
&\qquad + \Big| \big(u_{\mathbb{I}_{k'}} - u, u\big)_{L^2(0,T;H)} \Big| .
\end{aligned}$$

The relations (4.3), (6.3), (6.5) show that the right-hand side in (6.6) converges towards zero as $k' \to \infty$. We thus have shown (6.4).

We are now going to prove

$$B_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} \to Bu \text{ in } L^{p^*}(0,T;V^*) \text{ as } k' \to \infty . \tag{6.7}$$

From Assumption B, we infer that for all $t \in [0,T]$ (remember here $u \in \mathcal{W} \hookrightarrow \mathcal{C}([0,T];H)$)

$$B_{\mathbb{I}_{k'}}(t)u(t) - B(t)u(t) \to 0 \text{ in } V^* \text{ as } k' \to \infty .$$

Due to the growth condition, $t \mapsto \|B_{\mathbb{I}_k}(t)u(t) - B(t)u(t)\|_*^{p^*}$ is also majorized by an integrable function. Hence, Lebesgue's theorem yields

$$B_{\mathbb{I}_{k'}} u \to Bu \text{ in } L^{p^*}(0,T;V^*) \text{ as } k' \to \infty . \tag{6.8}$$

Moreover, (6.2) and Hölder's inequality give

$$\|B_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}} - B_{\mathbb{I}_{k'}} u\|_{L^{p^*}(0,T;V^*)} \le \beta(M_1) M_2^{p-1-\delta} \|u_{\mathbb{I}_{k'}} - u\|^{\delta/p}_{L^1(0,T;H)} , \tag{6.9}$$

where

$$\begin{aligned}
M_1 &:= \max\left(\|u\|_{L^\infty(0,T;H)}, \sup_{k\in\mathbb{N}} \|u_{\mathbb{I}_k}\|_{L^\infty(0,T;H)} \right) , \\
M_2 &:= \max\left(\|u\|_{L^p(0,T;V)}, \sup_{k\in\mathbb{N}} \|u_{\mathbb{I}_k}\|_{L^p(0,T;V)} \right) .
\end{aligned}$$

The relations (6.8) and (6.9) with (6.4) imply (6.7).

The rest of the proof can be carried out similarly as the proof of Theorem 4.2 employing in particular that

$$\langle B_{\mathbb{I}_{k'}} u_{\mathbb{I}_{k'}}, u_{\mathbb{I}_{k'}} \rangle \to \langle Bu, u \rangle \text{ as } k' \to \infty .$$

Since we do not have uniqueness at hand for (1.2), we cannot make a statement for the whole sequence of approximations. □

One can also derive stability and a priori error estimates (analogous to those in [13]). This, however, requires further restrictions on B. We only focus on error estimates and corresponding stability estimates uniform with respect to the time grid.

Theorem 6.3. *The assertions of Lemma 5.1 and Theorem 5.2, respectively, remain true if, in addition to the assumptions there, Assumption B is fulfilled and if for any $R > 0$ there is a constant $c_B(R) \ge 0$ such that for all $t \in [0,T]$ and $v, w \in V$ with $|v|, |w| \le R$*

$$\langle B(t)v - B(t)w, v - w \rangle \ge -c_B(R)|v - w|^2 \tag{6.10}$$

or, in case (5.1) is satisfied and $p = 2$,

$$\langle B(t)v - B(t)w, v - w\rangle \geq -\frac{\mu_0}{4}\|v - w\|^2 - c_B(R)|v - w|^2\,. \tag{6.11}$$

Proof. The proof follows the same arguments as that of Lemma 5.1 and Theorem 5.2, respectively. ☐

Let us, finally, remark that for (6.10) it is sufficient to assume that $B(t)$ $(t \in [0,T])$ maps V into H with

$$|B(t)v - B(t)w| \leq c_B(R)|v - w|\,,$$

whereas (6.11) for $p = 2$ follows from $B(t) : V \to H$ $(t \in [0,T])$ with

$$|B(t)v - B(t)w| \leq \sqrt{\mu_0 c_B(R)}\,\|v - w\|$$

as well as from $B(t) : V \to V^*$ $(t \in [0,T])$ with

$$\|B(t)v - B(t)w\|_* \leq \left(\frac{\mu_0}{2(2-\delta)}\right)^{(2-\delta)/2}\left(\frac{2c_B(R)}{\delta}\right)^{\delta/2}\|v - w\|^{1-\delta}|v - w|^{\delta}$$

for some $\delta \in (0,1]$. This can be seen by applying Young's inequality.

In case (5.1) is satisfied with $p > 2$, one may come up with the assumption

$$\langle B(t)v - B(t)w, v - w\rangle \geq -\frac{\mu_0}{4}\|v - w\|^p - c_B(R)|v - w|^2 \tag{6.12}$$

analogous to (6.11). However, it is easy to show that for $p > 2$ (6.12) implies (6.10).

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